

Name: _____ SID: _____

MA2001 (CA1)

Quiz 2

SOLUTION

Semester A, 2023-24

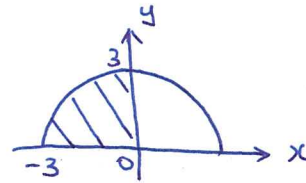
Answer all questions. Write your answers in the space provided.

1. Evaluate the following double integral by the change of variables using polar coordinates:

$$\int_{-3}^0 \left[\int_0^{\sqrt{9-x^2}} (x^2 + y^2)^{\frac{3}{2}} dy \right] dx$$

[10 marks]

$$R: \begin{cases} -3 \leq x \leq 0 \\ 0 \leq y \leq \sqrt{9-x^2} \end{cases}$$



Using polar coordinates,

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \text{where} \quad \begin{cases} 0 \leq r \leq 3 \\ \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$$

$$\int_{-3}^0 \left[\int_0^{\sqrt{9-x^2}} (x^2 + y^2)^{\frac{3}{2}} dy \right] dx$$

$$= \int_0^3 \left[\int_{\frac{\pi}{2}}^{\pi} r^3 \cdot r \, d\theta \right] dr$$

$$= (\pi - \frac{\pi}{2}) \int_0^3 r^4 \, dr$$

$$= \frac{\pi}{2} \left[\frac{r^5}{5} \right]_0^3$$

$$= \frac{243\pi}{10}$$

2. Convert the triple integral

$$\iiint_V (x^2 + y^2 + z^2) dx dy dz$$

to an iterated integral using spherical coordinates with order $d\rho d\phi d\theta$, where V is the bounded region which lies inside the cylinder $x^2 + y^2 = 4$, outside the cone $z = \sqrt{3x^2 + 3y^2}$ and above the xy -plane.

In other words, write your answer in the form

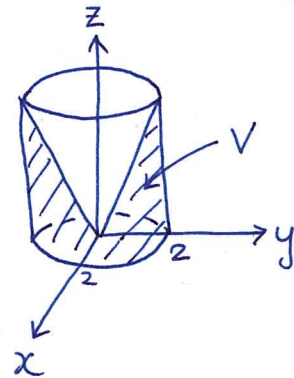
$$\int_{\boxed{?}}^{\boxed{?}} \left[\int_{\boxed{?}}^{\boxed{?}} \left(\int_{\boxed{?}}^{\boxed{?}} \boxed{???} d\rho \right) d\phi \right] d\theta.$$

You do not need to evaluate the integral.

[10 marks]

Using spherical coordinates :

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases}$$



$$z = \sqrt{3x^2 + 3y^2}$$

$$\Rightarrow \rho \cos \phi = \sqrt{3\rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta)} = \sqrt{3} \rho \sin \phi$$

$$\Rightarrow \tan \phi = \frac{1}{\sqrt{3}}$$

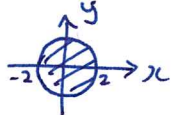
$$\Rightarrow \phi = \frac{\pi}{6}$$

$$\therefore \frac{\pi}{6} \leq \phi \leq \frac{\pi}{2}$$

$$x^2 + y^2 = 4 \Rightarrow \rho^2 \sin^2 \phi = 4 \Rightarrow \rho = 2 \csc \phi \text{ or } \rho = -2 \csc \phi \text{ (rej.)}$$

$$\therefore 0 \leq \rho \leq 2 \csc \phi$$

Projection of V onto xy -plane:



$$\therefore 0 \leq \theta \leq 2\pi$$

$$|J| = \rho^2 \sin \phi$$

$$x^2 + y^2 + z^2 = \rho^2$$

$$\iiint_V (x^2 + y^2 + z^2) dx dy dz = \int_0^{2\pi} \left[\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\int_0^{2 \csc \phi} \rho^4 \sin \phi d\rho \right) d\phi \right] d\theta$$

- End -