

EE3109 Applied Electromagnetics
2024/25 Semester B
Assignment 3 Solution

1. A low loss transmission line has the following properties: $R' \ll j\omega L'$; $G' \ll j\omega C'$. Find expressions for α, β, v_p and Z_0 for the low loss transmission line; show that these expressions approach the lossless counterparts when R' and $G' \rightarrow 0$.

Solution:

Characteristic Impedance:

$$Z_0 = \sqrt{\frac{(R' + j\omega L')}{(G' + j\omega C')}} = \sqrt{\frac{j\omega L' \left[\left(\frac{R'}{j\omega L'} \right) + 1 \right]}{j\omega C' \left[\left(\frac{G'}{j\omega C'} \right) + 1 \right]}} \cong \sqrt{\frac{j\omega L'}{j\omega C'}} = \sqrt{\frac{L'}{C'}}.$$

The highlighted terms are small because of the low loss property, and can hence be neglected.

Propagation Constant:

$$\begin{aligned} \gamma &= \sqrt{(R' + j\omega L')(G' + j\omega C')} = \sqrt{R'G' + (j\omega L')(G') + (R')(j\omega C') + (j\omega L')(j\omega C')} \\ &= j\omega\sqrt{L'C'} \sqrt{\left(\frac{R'}{j\omega L'} \right) \left(\frac{G'}{j\omega C'} \right) + \left(\frac{R'}{j\omega L'} \right) + \left(\frac{G'}{j\omega C'} \right) + 1} \end{aligned}$$

All the bracketed items are small because of the low loss property. The first term (highlighted) is second-order small, which will be neglected in a first order approximation. Using the Taylor series $\sqrt{1+x} = 1 + \frac{x}{2}$ for $x \ll 1$, we have:

$$\begin{aligned} \gamma &\cong j\omega\sqrt{L'C'} \sqrt{\left(\frac{R'}{j\omega L'} \right) + \left(\frac{G'}{j\omega C'} \right) + 1} \cong j\omega\sqrt{L'C'} \left(1 + \frac{1}{2} \left(\frac{R'}{j\omega L'} + \frac{G'}{j\omega C'} \right) \right) \\ &= j\omega\sqrt{L'C'} + \frac{\sqrt{L'C'}}{2} \left(\frac{R'}{L'} + \frac{G'}{C'} \right) = j\omega\sqrt{L'C'} + \frac{1}{2} \left(\frac{R'}{Z_0} + G'Z_0 \right). \end{aligned}$$

(The last step uses Z_0 derived above.)

$$\begin{aligned} \gamma = \alpha + j\beta &\Rightarrow \alpha = \frac{1}{2} \left(\frac{R'}{Z_0} + G'Z_0 \right); \beta = \omega\sqrt{L'C'}; \\ v_p &= \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}. \end{aligned}$$

From the derivations, we see that Z_0, β and v_p are the same as for a lossless transmission line. α will also vanish as R' and $G' \rightarrow 0$, as expected for a lossless transmission line.

2. A lossless air transmission line has a characteristic impedance of 50Ω and a length of 30mm. The transmission line is terminated with a 100Ω resistor in parallel with a 10 nH inductor. An RF wave of 15 GHz travels along the transmission line.

- Find Z_L and Γ_L along the transmission line.
- Find α and β along the transmission line,
- Write the voltage phasor as a function of distance along the line. Substitute known numerical values into the equation and simplify as much as possible.

Now the input of the transmission line is connected to a microwave source, modelled with an ideal 5V voltage source in series with a generator resistance of $R_g = 75\Omega$.

- Write the voltage phasor as a function of distance along the line. Substitute known numerical values into the equation and simplify as much as possible.

Solution:

- The load is a resistor in parallel with an inductor:

$$Z_L = \frac{R_L(j\omega L_L)}{R_L + (j\omega L_L)} = \frac{100(j300\pi)}{100 + j300\pi} = 98.87 + j10.49 [\Omega]$$

The corresponding reflection coefficient is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{98.87 + j10.49 - 50}{98.87 + j10.49 + 50} = 0.3317 + j0.0471$$

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$$\alpha = 0 \text{ (lossless TL); } \beta = \frac{\omega}{v_p} = \frac{2\pi(1.5 \times 10^{10})}{3 \times 10^8} = 100\pi = 314.2 \text{ [rad/m]}$$

- The voltage along the line is:

$$\begin{aligned} V(z) &= V_0^+ e^{-j\beta z} (1 + \Gamma_L e^{-j2\beta z}) \\ &= V_0^+ e^{-j100\pi z} (1 + (0.3317 + j0.0741)e^{-j200\pi z}) \text{ [V]} \end{aligned}$$

- The voltage along the line is:

$$V(z) = \frac{V_g Z_0}{R_g + Z_0} \left(\frac{e^{-j\beta l} e^{-j\beta z}}{1 - \Gamma_L \Gamma_G e^{-j2\beta l}} \right) (1 + \Gamma_L e^{j2\beta z})$$

We now find the known parameters:

$$\beta l = (100\pi)(30 \times 10^{-3}) = 3\pi \Rightarrow e^{-j\beta l} = e^{-j\pi} = -1$$

$$\Gamma_G = \frac{Z_G - Z_0}{Z_G + Z_0} = \frac{75 - 50}{75 + 50} = 0.2$$

$$\begin{aligned} \frac{V_g Z_0}{R_g + Z_0} \left(\frac{e^{-j\beta l}}{1 - \Gamma_L \Gamma_G e^{-j2\beta l}} \right) &= \left(\frac{5(50)}{125} \right) \left(\frac{-1}{1 - (0.3317 + j0.0471)(0.2)(1)} \right) \\ &= -2.142 - j0.022 \end{aligned}$$

Substituting these into the equation for voltage gives

$$V(z) = (-2.142 - j0.022)e^{-j100\pi z} (1 + (0.3317 + j0.0471)e^{j200\pi z}) \text{ [V]}$$

3. Consider the following circuit, which consists of a 0.4λ air transmission line with a characteristic impedance of $Z_0 = 50 \Omega$, connecting a generator with $V_g = 0.5 \text{ V}$, $Z_g = 50 \Omega$ to the load $Z_L = 30 - j10 \Omega$. The generator is tuned to operate at 2.4 GHz .
- Write the current as a function of position along the transmission line.
 - With the aid of a Smith Chart, design an L-Section which matches the impedance Z_L to the generator impedance of $Z_g = 50 \Omega$.
 - Replace the shunt component with a short-circuited transmission line with the aid of a Smith Chart. Determine the electrical length of this transmission line.
 - Due to a change in design, the generator impedance is now changed to $Z_g = 70 \Omega$. Add a component (L, C or transmission line section) to transform the input impedance of your match network in (b) to achieve matching to the new generator impedance.

Solution:

- (a) We find the necessary parameters then substitute into the function for $I(z)$.

$$\beta = \frac{\omega}{V_p} = \frac{2.4 \times 10^9 \times 2\pi}{3 \times 10^8} = 16\pi; \quad \beta l = \left(\frac{2\pi}{\lambda}\right)(0.4\lambda) = 0.8\pi = 2.513 \text{ [rad/m]};$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{-20 - j10}{-80 - j10} = -\frac{3}{13} - j\frac{2}{13}$$

$$= -0.2308 - j0.1538 = 0.2773 \angle -146.3^\circ;$$

$$\Gamma_G = 0 \because Z_G = Z_0; \quad \frac{V_g}{R_g + Z_0} = \frac{0.5}{50 + 50} = 0.005 \text{ [A]} = 5 \text{ [mA]}.$$

$$I(z) = \frac{V_g}{R_g + Z_0} \left(\frac{e^{-j\beta l} e^{-j\beta z}}{1 - \Gamma_L \Gamma_G e^{-j2\beta l}} \right) (1 - \Gamma_L e^{+2j\beta z})$$

$$= 5(e^{-j0.8\pi} e^{-j16\pi z})(1 + (0.2308 + j0.1538)e^{+j32\pi z}) \text{ [mA]}.$$

- (b) See attached S.C. (black).

Key parameters: $z_L = 0.6 - j0.2$; $z_A = 0.6 + j0.5$; $y_A = 1 - j0.82$; $jx = j0.7$; $jb = j0.82$. Series $L = 2.32 \text{ [nH]}$ and shunt $C = 1.088 \text{ [pF]}$ required.

Alternative answer:

Key parameters: $z_L = 0.6 - j0.2$; $z_A = 0.6 - j0.5$; $y_A = 1 + j0.82$; $jx = -j0.3$; $jb = -j0.82$. Series $C = 4.42 \text{ [pF]}$ and shunt $L = 4.044 \text{ [nH]}$ required.

- (c) See attached S.C. (blue). Stub length: $l = 0.359 \lambda$.

Alternative answer: Stub length: $l = 0.391\lambda - 0.25\lambda = 0.141\lambda$.

- (d) Use a TL as a $\lambda/4$ -transformer to transform from $Z_{in} = 50 \Omega$ to $Z_{in,new} = 70 \Omega$:

$$Z_{0,TL} = \sqrt{Z_{in} Z_{in,new}} = \sqrt{(50)(70)} = 59.16 \text{ [}\Omega\text{]}; \quad l_{TL} = 0.25 \lambda$$

L-Section Matching: $Z_L = 30\Omega - j10\Omega$; $Z_0 = 50\Omega$.

$$Z_L' = 0.6 - j0.2$$

From S.C.:

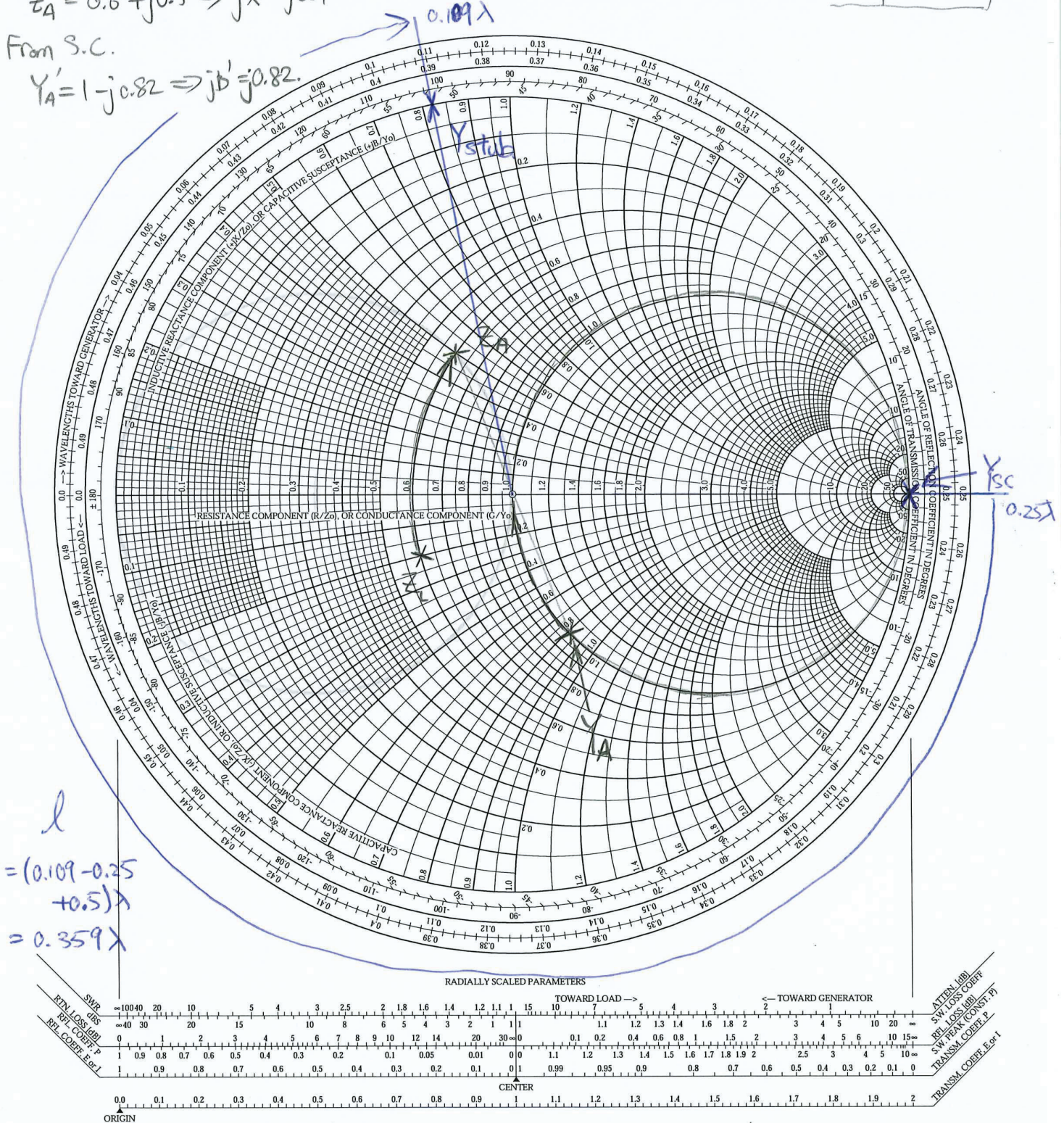
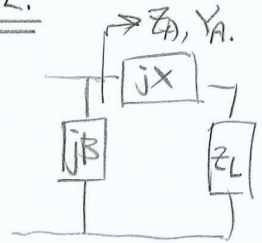
$$Z_A' = 0.6 + j0.5 \Rightarrow jX' = j0.7$$

From S.C.

$$Y_A' = 1 - j0.82 \Rightarrow jB' = j0.82$$

The Complete Smith Chart

Black Magic Design



$$l = (0.109 - 0.25 + 0.5)\lambda = 0.359\lambda$$

Positive $x' \Rightarrow$ Use inductor:

$$jX'Z_0 = j\omega L \Rightarrow L = \frac{X'Z_0}{\omega} = 2.32 \text{ nH}$$

Positive $b' \Rightarrow$ Use capacitor:

$$jB'Y_0 = j\omega C \Rightarrow C = \frac{B'}{\omega Z_0} = 1.088 \text{ pF}$$

4. The goal of this question is to design a co-axial transmission line, with an inner and outer conductor radii of a and b respectively.
- Derive the per-unit-length capacitance and inductance of the co-axial transmission line. (Most of this is done for you in the course notes.)
 - Find the characteristic impedance and phase velocity of the lossless co-axial transmission line.
 - For a dielectric material with $\epsilon_r = 2.2$, design a co-axial TL with $Z_0 = 75 \Omega$. Choose the outer conductor radius to be the same as that for an RG-59 cable.

Solution:

- (a) See the notes for Lecture 8, Slides 13-14, for derivations on L and C over a unit length l . The per unit length capacitance and inductance are

$$L' = \frac{L}{l} = \frac{\frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right)}{l} = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) \quad \text{and} \quad C' = \frac{C}{l} = \frac{2\pi\epsilon l \left[\ln\left(\frac{b}{a}\right)\right]^{-1}}{l} = 2\pi\epsilon \left[\ln\left(\frac{b}{a}\right)\right]^{-1}$$

- (b) The characteristic impedance and phase velocity are

$$Z_0 = \sqrt{\frac{L'}{C'}} = \sqrt{\frac{\frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right)}{2\pi\epsilon \left[\ln\left(\frac{b}{a}\right)\right]^{-1}}} = \frac{1}{2\pi} \sqrt{\frac{\mu_0}{\epsilon}} \ln\left(\frac{b}{a}\right) = \frac{\eta_0}{2\pi\sqrt{\epsilon_r}} \ln\left(\frac{b}{a}\right) ;$$

$$V_p = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{\left(\frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right)\right) \left(2\pi\epsilon \left[\ln\left(\frac{b}{a}\right)\right]^{-1}\right)}} = \frac{1}{\sqrt{\mu_0\epsilon}}$$

- (c) We find from internet sources (e.g. Wikipedia) that the RG-59 has a diameter of 6.1mm, hence $b = 3.05$ mm.

Rearranging the equation for Z_0 gives

$$\ln\left(\frac{b}{a}\right) = 2\pi\sqrt{\epsilon_r} \frac{Z_0}{\eta_0} = 2\pi\sqrt{2.2} \frac{75}{120\pi} = 1.854 \Rightarrow \frac{b}{a} = e^{1.854} = 6.386 \Rightarrow a = \frac{b}{6.386} = 0.4776 \text{ [mm]} .$$

5. **(10% BONUS)** Prove that one can find the admittance point on the Smith Chart at a location diametrically opposite (same distance from the origin, but in the opposite direction) to the impedance point. Hint: remember the Smith Chart's relationship with the reflection coefficient.

Solution:

We map the point $y' = 1/z'$ onto the Γ -plane as follows:

$$\Gamma_y = \frac{Y - Y_0}{Y + Y_0} = \frac{y' - 1}{y' + 1} = \frac{\frac{1}{z'} - 1}{\frac{1}{z'} + 1} = \frac{1 - z'}{1 + z'} = -\frac{z' - 1}{z' + 1} = -\Gamma_z$$

Hence y' and z' appear on diametrically opposite points on the Smith Chart.