

Midterm #1

Solutions:

Course: EE3121 *Differential Equations for Electrical Engineering*
Date:

Question 1

Solution. (1) Step 1: Separate variable

$$\frac{dy}{y^2 + 3y + 2} = dt$$

Step 2: Integrate

$$\begin{aligned} \int_{y(0)}^{y(t)} \frac{1}{y^2 + 3y + 2} dy &= \int_0^t dt \\ \Rightarrow \int_0^{y(t)} \frac{1}{y+1} - \frac{1}{y+2} dy &= t \end{aligned}$$

After integration

$$\begin{aligned} \ln(y+1) - \ln(y+2) + \ln 2 &= t \\ \Rightarrow \ln \frac{y+1}{y+2} \cdot 2 &= t \\ \Rightarrow \frac{y+1}{y+2} \cdot 2 &= e^t \end{aligned}$$

Calculate $y(t)$

$$y(t) = \frac{e^t - 1}{1 - \frac{1}{2}e^t}$$

(2) From (1) we have

$$y(t) = \frac{e^t - 1}{1 - \frac{1}{2}e^t}$$

The solution is valid for $1 - \frac{1}{2}e^t > 0$, therefore,

$$t < \ln 2$$

(3) Let $\frac{dy}{dt} = y^2 + 3y + 2 = f(y)$, and set a bound $y \leq b$,

$$M = b^2 + 3b + 2$$

$$\frac{\partial f(y)}{\partial y} = 2y + 3 \leq 2b + 3$$

Set $a > \frac{b}{M}$,

$$t_E = \min(a, \frac{b}{M}) = \frac{b}{b^2 + 3b + 2}$$

$$\frac{dt_E}{db} = 0$$

$$\Rightarrow b = \sqrt{2}$$

$$t_E = \frac{\sqrt{2}}{3\sqrt{2} + 4}$$

Question 2

Solution. $y_1(t)$: Set $\varepsilon = 0$,

$$\frac{dy}{dt} + \frac{2t}{1+t^2}y = 0$$

$$y_1(t) = y_0 \exp\left(-\int_{t_0}^t p(\tau)d\tau\right) + \int_{t_0}^t \exp\left(-\int_{\tau}^t p(\tau)d\tau\right) g(\tau)d\tau$$

$$\int_{t_0}^t p(\tau)d\tau = \int_0^t \frac{2\tau}{1+\tau^2}d\tau = \ln(\tau^2 + 1)\Big|_0^t = \ln(t^2 + 1)$$

Therefore,

$$y_1(t) = \exp\left(-\ln(t^2 + 1)\right) = \frac{1}{t^2 + 1}$$

$y_2(t)$:

$$\frac{dy_2}{dt} + \frac{2t}{1+t^2}y_2 = -\varepsilon y_1^2 = -\frac{\varepsilon}{t^2 + 1}$$

$$\begin{aligned} y_2(t) &= \frac{1}{t^2+1} + \int_0^t \exp\left(-\int_{\tau}^t \frac{2\tau}{\tau^2+1}d\tau\right) \left(-\varepsilon \cdot \frac{1}{\tau^2+1}\right)^2 d\tau = \frac{1}{t^2+1} + \int_0^t \exp\left(\ln\left(\frac{\tau^2+1}{t^2+1}\right)\right) \left(-\varepsilon \cdot \frac{1}{\tau^2+1}\right)^2 d\tau \\ &= \frac{1}{t^2+1} + \int_0^t \frac{\tau^2+1}{t^2+1} \cdot \left(-\varepsilon \cdot \frac{1}{\tau^2+1}\right)^2 d\tau = \frac{1}{t^2+1} - \frac{\varepsilon}{t^2+1} \int_0^t \frac{1}{\tau^2+1} d\tau = \frac{1}{t^2+1} - \frac{\varepsilon}{t^2+1} \arctan t \end{aligned} \quad (1)$$

Question 3

Solution. Equilibrium points:

$$\frac{dy}{dt} = 0$$

$$y^4 - 20y^2 + 64 = (y^2 - 4)(y^2 - 16) = 0$$

Equilibrium points are

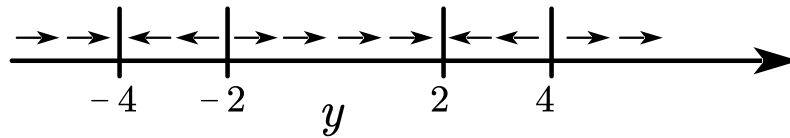
$$y_{1,2} = \pm 2, \quad y_{3,4} = \pm 4$$

When $y < -4$

$$\frac{dy}{dt} > 0$$

When $-4 < y < -2$

$$\frac{dy}{dt} < 0$$



When $-2 < y < 2$

$$\frac{dy}{dt} > 0$$

When $2 < y < 4$

$$\frac{dy}{dt} < 0$$

When $y > 4$

$$\frac{dy}{dt} > 0$$

Therefore, $-4, 2$ are stable points, $-2, 4$ are unstable points.

Question 4

Solution. (1) According to Kirchhoff Circuit Laws,

$$I_0 = I_R + I_C + I_L$$

$$I_C = C \frac{dV_C}{dt}, V_L = L \frac{dI_L}{dt}$$

$$I_0 = C \frac{dV_C}{dt} + \frac{V_C}{R} + \int_0^t \frac{V_C}{L} dt$$

$$C \frac{d^2 V_C}{dt^2} + \frac{1}{R} \frac{dV_C}{dt} + \frac{V_C}{L} = 0, \quad V_C(0) = 0, \quad \dot{V}_C(0) = \frac{I_0}{C}$$

(2) From (1),

$$\frac{1}{10} \frac{d^2 V_C}{dt^2} + \frac{1}{2} \frac{dV_C}{dt} + \frac{3}{5} V_C = 0, \quad V_C(0) = 0, \quad \dot{V}_C(0) = 10I_0$$

Characteristic Polynomial:

$$\frac{1}{10} r^2 + \frac{1}{2} r + \frac{3}{5} = 0$$

$$\Rightarrow r_1 = -2, \quad r_2 = -3$$

therefore,

$$V_C(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

$$\frac{dV_C(t)}{dt} = -2c_1 e^{-2t} - 3c_2 e^{-3t}$$

From the initial conditions,

$$\Rightarrow \begin{cases} V_C(0) = c_1 + c_2 = 0 \\ \frac{dV_C(0)}{dt} = -2c_1 - 3c_2 = 10I_0 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = 10I_0 \\ c_2 = -10I_0 \end{cases}$$

$$V_C(t) = 10I_0e^{-2t} - 10I_0e^{-3t}$$

(3) When $I_0 = e^{-t}$,

$$C \frac{dV_C}{dt} + \frac{V_C}{R} + \int_0^t \frac{V_C}{L} dt = e^{-t}$$

Take derivative

$$\frac{1}{10} \frac{d^2V_C}{dt^2} + \frac{1}{2} \frac{dV_C}{dt} + \frac{3}{5} V_C = -e^{-t}, V_C(0) = 0, \dot{V}_C(0) = 10$$

Step 1: Trial solution:

$$V_C(t) = ce^{-t}$$

Step 2: Subsitute the trial solution into the differential equation:

$$\frac{dV_C(t)}{dt} = -ce^{-t}, \quad \frac{d^2V_C(t)}{dt^2} = ce^{-t}$$

Step 3: Evaluate the D.E.

$$\frac{1}{10} ce^{-t} - \frac{1}{2} ce^{-t} + \frac{3}{5} ce^{-t} = -e^{-t}$$

$$\Rightarrow \frac{1}{10}c - \frac{1}{2}c + \frac{3}{5}c = -1$$

$$\Rightarrow c = -5$$

$$V_C(t) = -5e^{-t}$$

because $V_C(0) \neq 0$

$$V_C(t) = c_1e^{-2t} + c_2e^{-3t} - 5e^{-t}$$

$$\Rightarrow \begin{cases} V_C(0) = c_1 + c_2 - 5 = 0 \\ \dot{V}_C(0) = -2c_1 - 3c_2 + 5 = 10 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = 20 \\ c_2 = -15 \end{cases}$$

$$V_C(t) = -15e^{-3t} + 20e^{-2t} - 5e^{-t}$$