

MA1200 Calculus and Basic Linear Algebra I

Lecture Note 5

Exponential Function and Logarithmic Function

Exponential Function

An exponential function $f(x)$ is a function of the following form:

$$f(x) = a^x$$

where a is a real constant which $a > 0$ and $a \neq 1$.

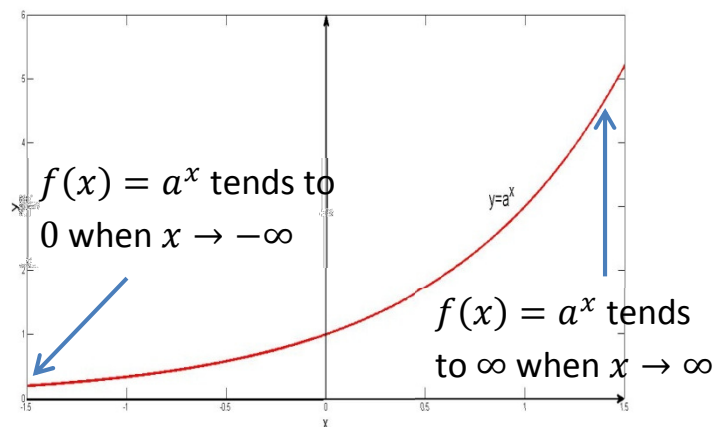
Remark:

- Here, the domain of $f(x)$ is $\mathbb{R}$. (i.e. you can input any real number x into the function $f(x) = a^x$.)
- Here, we do not allow $a \leq 0$ since the function is then undefined for some values of x . For example, one cannot define the values of $0^{-2} = \frac{1}{0^2} = \frac{1}{0}$ and $(-2)^{\frac{1}{2}} = \sqrt{-2}$.
- When $a = 1$, the function simply becomes $f(x) = 1^x = 1$ which is a constant function.

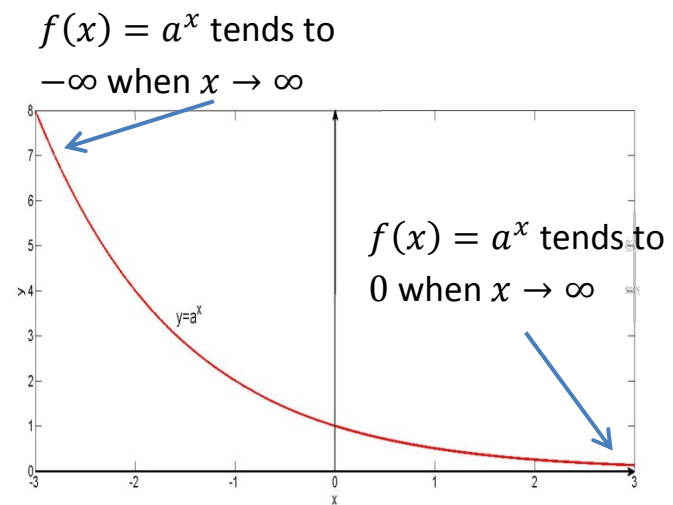
Graph of Exponential Function

From the graphs, we observe that

- The range of the function $f(x)$ is always $(0, \infty)$ (i.e. $0 < f(x) < \infty$).
- For $0 < a < 1$, the function $f(x) = a^x$ is *strictly decreasing*.
For $a > 1$, the function $f(x) = a^x$ is *strictly increasing*.



$$f(x) = a^x$$
$$a > 1$$



$$f(x) = a^x$$
$$0 < a < 1$$

Basic Operation of Exponential Functions

$$(1) a^x \cdot a^y = a^{x+y}, \quad (2) \frac{a^x}{a^y} = a^x \cdot a^{-y} = a^{x-y}.$$

$$(3) (a^x)^y = a^{xy}$$

$$(4) a^x \cdot b^x = (ab)^x$$

Example 1

Compute $\frac{3^{2x+1}5^{2x-1}}{15^{2x}}$ and $\frac{100^{2x+1}}{25^{2x}}$.

☺Solution:

$$\frac{3^{2x+1}5^{2x-1}}{15^{2x}} = \frac{3^{2x+1} \cdot 5^{2x-1}}{3^{2x} \cdot 5^{2x}} = 3^{2x+1-2x} \cdot 5^{2x-1-2x} = 3 \cdot 5^{-1} = \frac{3}{5},$$

$$\frac{100^{2x+1}}{25^{2x}} = \frac{100 \cdot 100^{2x}}{25^{2x}} = \frac{100 \cdot (25^{2x} \cdot 4^{2x})}{25^{2x}} = 100 \cdot 4^{2x}.$$

Special exponential function e^x

The number e "Euler's number"

Consider the expression $\left(1 + \frac{1}{n}\right)^n$ where n is real number, this expression has a special property that $\left(1 + \frac{1}{n}\right)^n$ tends to a finite number (≈ 2.71828) when n gets larger ($n \rightarrow \infty$):

n	100	1,000	10,000	100,000	1,000,000	10,000,000
$\left(1 + \frac{1}{n}\right)^n$	2.70481	2.71692	2.71815	2.71827	2.71828	2.71828

value of $\left(1 + \frac{1}{n}\right)^n$ when $n = \infty$

This finite number is denoted by e $\left(= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n\right)$, is called *natural base*. The corresponding exponential function $f(x) = e^x$ is called *natural exponential function* (or simply exponential function).

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$$

Importance of the exponential function e^x

- Practically, exponential function in many modeling such as population growth in biology and continuously compound interest in finance.
- In calculus, the exponential function e^x has a special property that

$$\frac{d}{dx} e^x = e^x, \quad \int e^x dx = e^x + C.$$

This property makes the exponential function useful in theoretical aspect (say in the study of differential equation).

- In algebra, the exponential function e^x is an important element in the Euler's formula:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

where $i = \sqrt{-1}$. (You will learn more in MA1201)

Logarithmic Functions

Given a real number a (with $a > 0$ and $a \neq 1$), the *logarithmic function with base a* , denoted by $y = f(x) = \log_a x$, is the number such that

$$a^y = x.$$

From the definition, one can see that the function $\log_a x$ is the *inverse function* of a^x . So we expect that

$$\log_a(a^x) = x \text{ and } a^{\log_a x} = x.$$

Two commonly used logarithmic functions

- When $a = 10$, the function $\log_{10} x$ (or simply $\log x$) is called *common logarithm* which is commonly used in actual computation (say solving equation involving exponential function).
- When $a = e$, the function $\log_e x$ is called *natural logarithm* which is commonly used in calculus (e.g. differentiation and integration). We usually denote this function by $\ln x$.

$$e^{\ln(x)} = x$$
$$\ln(e^x) = x$$

Example 2

Compute $\log_5 25$, $\log_3 \sqrt[3]{9}$, $\log_2 0.5$ and $\ln 1$ using definition.

☺Solution:

$$y = \log_5 25 \Rightarrow 5^y = 25 \Rightarrow 5^y = 5^2 \Rightarrow y = \log_5 25 = 2.$$

$$y = \log_3 \sqrt[3]{9} \Rightarrow 3^y = \sqrt[3]{9} \Rightarrow 3^y = 9^{\frac{1}{3}} \Rightarrow 3^y = 3^{\frac{2}{3}} \Rightarrow y = \log_3 \sqrt[3]{9} = \frac{2}{3}.$$

$$y = \log_2 0.5 \Rightarrow 2^y = 0.5 \Rightarrow 2^y = \frac{1}{2} \Rightarrow 2^y = 2^{-1} \Rightarrow y = \log_2 0.5 = -1.$$

$$y = \ln 1 \Rightarrow e^y = 1 \Rightarrow e^y = e^0 \Rightarrow y = \ln 1 = 0.$$

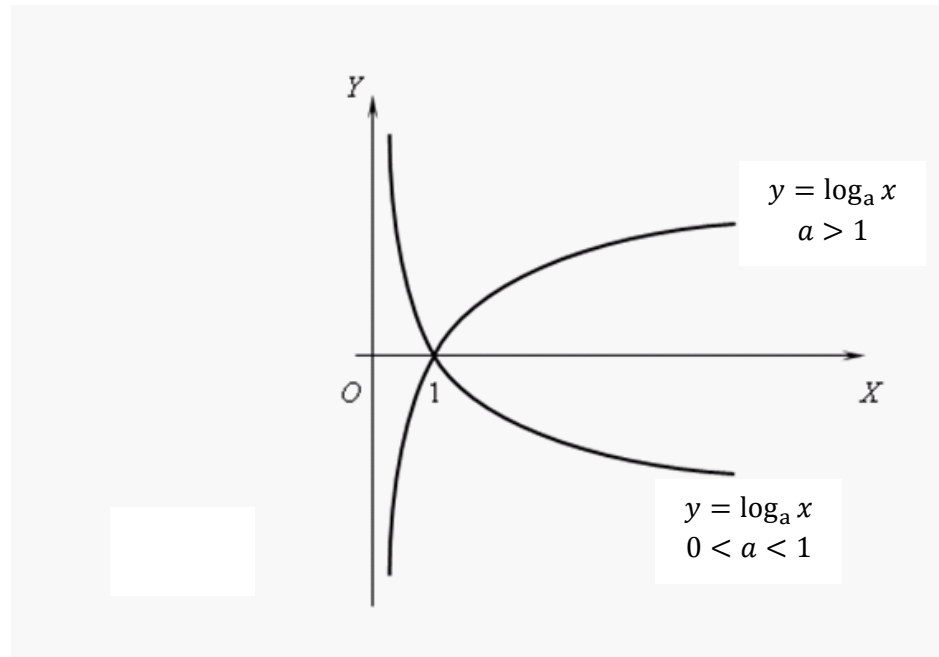
Example 3

Can we find the value of $\log(-2)$?

☺Solution:

$$y = \log(-2) \Rightarrow \underbrace{10^y}_{\text{positive}} = \underbrace{-2}_{\text{negative}}. \text{ So } y = \log(-2) \text{ does not exist.}$$

Graph of logarithmic functions and its properties



- The domain of $f(x) = \log_a x$ is $(0, \infty)$ (i.e. $0 < x < \infty$).
- The range of $f(x) = \log_a x$ is $\mathbb{R}$.
- For $0 < a < 1$, the function $f(x) = \log_a x$ is decreasing.
For $a > 1$, the function $f(x) = \log_a x$ is increasing.

Operation of Logarithmic functions

(The proof can be found in Appendix A)

1. $\log_a a^x = x$ and $a^{\log_a x} = x$ (Properties of inverse functions)

2. $\log_a(M^n) = n \log_a M$ *(any $n \in \mathbb{R}$)*

3. $\log_a(MN) = \log_a M + \log_a N$

4. $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$

5. $\log_a M = \frac{\log M}{\log a}$ *"Change of basis" (to base 10)*

Example 4

$$2 \log 8 + 3 \log 5 - 3 \log 2 = \log 8^2 + \log 5^3 - \log 2^3$$

$$= \log \frac{8^2 \times 5^3}{2^3} \stackrel{8=2^3}{=} \log \frac{2^6 \times 5^3}{2^3} = \log 10^3 = 3 \log 10 \stackrel{\log 10=1}{=} 3.$$

General change of basis: (From a to b .)

$$5.1) \log_a(M) = \frac{\log_b(M)}{\log_b(a)}$$

Proof:

$$\log_a M = \log_a \left(b^{\log_b M} \right) = (\log_a b) \cdot \log_b M \stackrel{(*)}{=} \frac{\log_b M}{\log_b a}$$

(*) : $\log_a(b) = \frac{1}{\log_b(a)}$ ($\Leftrightarrow \log_a(b) \log_b(a) = 1$) holds, since

$$a^{\log_a(b) \log_b(a)} = \left(a^{\log_a(b)} \right)^{\log_b(a)} = b^{\log_b(a)} = a^1$$

Example 5

(a) Solve the equation $3 \cdot 5^{2x-1} + 2 = 23$

(b) Solve the equation $2^{2x+2} - 17 \cdot 2^x + 4 = 0$.

(Hint: Note that $2^{2x+2} = 2^2 \cdot 2^{2x} = 4 \cdot 2^{2x}$)

☺Solution:

(a) $3 \cdot 5^{2x-1} + 2 = 23 \Rightarrow 5^{2x-1} = 7 \Rightarrow 2x-1 \approx \log_5(7) = \frac{\log(7)}{\log(5)}$
 $\Rightarrow x = \frac{1}{2} \left(\frac{\log 7}{\log 5} + 1 \right) \approx 1.1045.$

(b) Let $y = 2^x$, then

$$2^{2x+2} - 17 \cdot 2^x + 4 = 0 \Rightarrow 4 \cdot (2^x)^2 - 17 \cdot 2^x + 4 = 0$$

$$\Rightarrow 4y^2 - 17y + 4 = 0 \Rightarrow (4y - 1)(y - 4) = 0$$

$$\Rightarrow y = \frac{1}{4} \text{ or } y = 4$$

$$\Rightarrow 2^x = \frac{1}{4} = 2^{-2} \text{ or } 2^x = 4 = 2^2$$

$$\Rightarrow x = -2 \text{ or } x = 2.$$

Example 6

(a) Solve $9^{x+1} = 12^{x-1}$.

(b) $\log(x+1) + \log(2x-1) = \log 14$

☺Solution

(a) By taking logarithm on both sides, we have

$$9^{x+1} = 12^{x-1} \Rightarrow \log 9^{x+1} = \log 12^{x-1}$$

$$\Rightarrow (x+1)\log 9 = (x-1)\log 12 \Rightarrow (\log 12 - \log 9)x = \log 9 + \log 12$$

$$\Rightarrow x = \frac{\log 9 + \log 12}{\log 12 - \log 9}.$$

(b) Note that

$$\log(x+1) + \log(2x-1) = \log 14 \Rightarrow \log[(x+1)(2x-1)] = \log 14$$

$$\Rightarrow (x+1)(2x-1) = 14 \Rightarrow 2x^2 + x - 15 = 0$$

$$\Rightarrow (x+3)(2x-5) = 0$$

$$\Rightarrow x = -3 \text{ (rejected)} \text{ or } x = \frac{5}{2}.$$

(*)

(*) $x = -3$ would lead to negative argument in $\log(x+1)$.

Not defined!

Appendix A – Proof of the “Operation of Logarithmic Function”

Property 1 $\log_a(a^x) = x$

It follows from the fact that $\log_a x$ is the inverse function of a^x .

Property 2 $\log_a(M^n) = n \cdot \log_a(M)$

We let $A = \log_a M \Leftrightarrow M = a^A$.

Let $y = \log_a M^n$, then by the definition of logarithmic function, we have

$$a^y = M^n \Rightarrow a^y = a^{An} \Rightarrow y = nA \Rightarrow \log_a M^n = n \log_a M.$$

Property 3 & 4 $\log_a(MN) = \log_a(M) + \log_a(N)$ & $\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$

3) We let $A = \log_a M \Leftrightarrow M = a^A$ and $B = \log_a N \Leftrightarrow N = a^B$.

Let $y = \log_a MN$, then by the definition of logarithmic function, we have

$$a^y = MN \Rightarrow a^y = a^A a^B \Rightarrow a^y = a^{A+B} \Rightarrow y = A + B$$

$$\Rightarrow \log_a MN = \log_a M + \log_a N.$$

4) Let $z = \log_a \frac{M}{N}$, then we have

$$a^z = \frac{M}{N} \Rightarrow a^z = \frac{a^A}{a^B} \Rightarrow a^z = a^{A-B} \Rightarrow z = A - B$$

$$\Rightarrow \log_a \frac{M}{N} = \log_a M - \log_a N.$$

Property 5

$$\log_a(M) = \frac{\log(M)}{\log(a)}$$

We let $A = \log M$ and $B = \log a$ and we let $y = \log_a M$, then we have

$$a^y = M \Rightarrow a^y = 10^A \Rightarrow \log a^y = \log 10^A$$

From
Property 2

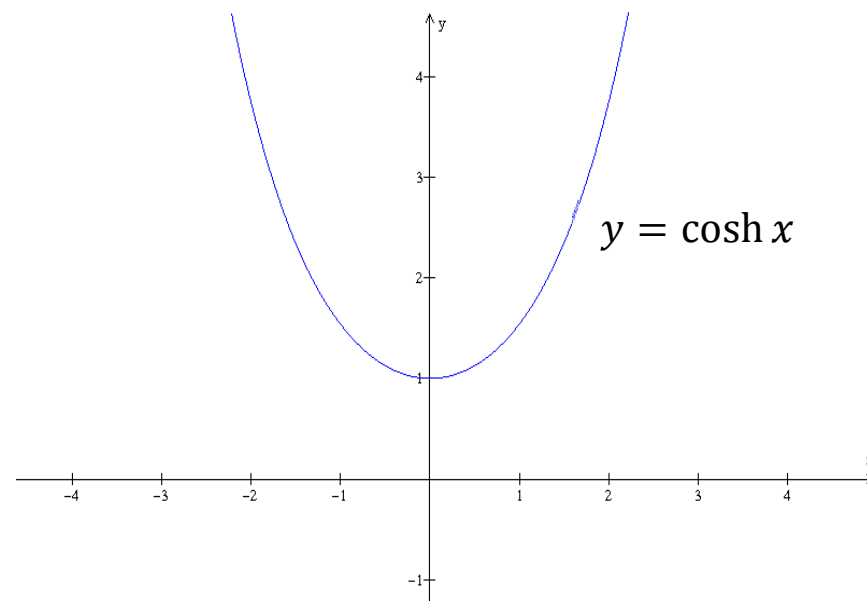
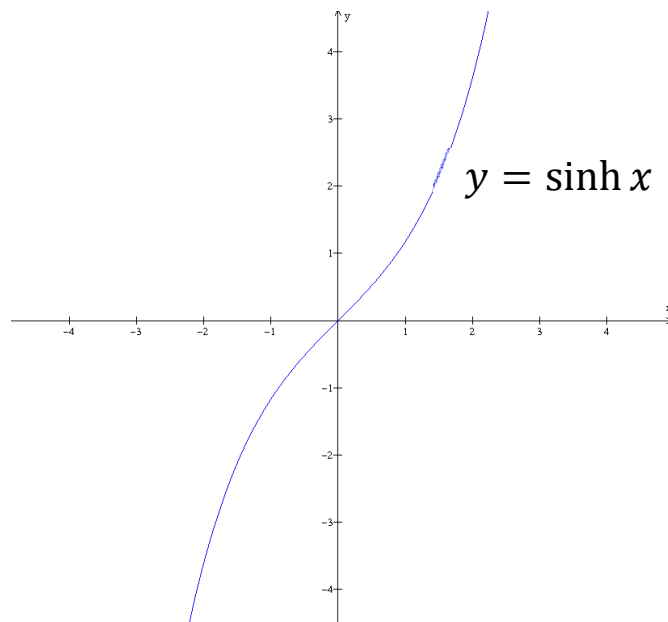
$$\Rightarrow y \log a = A \Rightarrow y = \frac{A}{\log a} \Rightarrow \log_a M = \frac{A}{B}.$$

Hyperbolic Sine and Hyperbolic Cosine Functions

The hyperbolic sine and hyperbolic cosine function are defined as the algebraic combination of some exponential functions:

$$\cosh x = \frac{1}{2}(e^x + e^{-x}), \quad \sinh x = \frac{1}{2}(e^x - e^{-x}).$$

The graphs of these functions are shown below:



Example 7

Solve the equation

$$\sinh x = 2.$$

☺Solution:

Using the definition of $\sinh x$, we have

$$\sinh x = 2 \Rightarrow \frac{1}{2}(e^x - e^{-x}) = 2 \Rightarrow e^x - e^{-x} = 4$$

$$\Rightarrow e^{2x} - 1 = 4e^x \Rightarrow e^{2x} - 4e^x - 1 = 0.$$

We let $y = e^x$, then the equation becomes

$$y^2 - 4y - 1 = 0 \Rightarrow y = \frac{4 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5}$$

$$\Rightarrow e^x = 1 + \sqrt{5} \text{ or } 1 - \sqrt{5} \text{ (rej.)}$$

$$\Rightarrow x = \ln(1 + \sqrt{5}).$$

Example 8

(a) Using the definition of $\sinh x$ and $\cosh x$, show that

(i) $\cosh^2 x - \sinh^2 x = 1$

(ii) $\cosh^2 x + \sinh^2 x = \cosh 2x$

(b) Hence, or otherwise, solve the equation $3 \sinh x + \cosh 2x = 0$.

☺Solution of (a)

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left[\frac{1}{2}(e^x + e^{-x}) \right]^2 - \left[\frac{1}{2}(e^x - e^{-x}) \right]^2 \\ &= \frac{1}{4} \left[e^{2x} + \underbrace{2e^x e^{-x}}_{=1} + e^{-2x} - \left(e^{2x} - \underbrace{2e^x e^{-x}}_{=1} + e^{-2x} \right) \right] = \frac{1}{4} (4) = 1\end{aligned}$$

$$\begin{aligned}\cosh^2 x + \sinh^2 x &= \left[\frac{1}{2}(e^x + e^{-x}) \right]^2 + \left[\frac{1}{2}(e^x - e^{-x}) \right]^2 \\ &= \frac{1}{4} \left[e^{2x} + \underbrace{2e^x e^{-x}}_{=1} + e^{-2x} + \left(e^{2x} - \underbrace{2e^x e^{-x}}_{=1} + e^{-2x} \right) \right] = \frac{1}{2} (e^{2x} + e^{-2x}) \\ &= \cosh 2x\end{aligned}$$

☺Solution of (b)

Note that

$$3 \sinh x + \cosh 2x = 0 \Rightarrow 3 \sinh x + (\cosh^2 x + \sinh^2 x) = 0$$

$$\Rightarrow 3 \sinh x + (1 + \sinh^2 x) + \sinh^2 x = 0$$

$$\Rightarrow 2 \sinh^2 x + 3 \sinh x + 1 = 0.$$

Let $y = \sinh x$, then the equation becomes

$$2y^2 + 3y + 1 = 0 \Rightarrow (2y + 1)(y + 1) = 0 \Rightarrow y = -\frac{1}{2} \text{ or } y = -1.$$

$$\Rightarrow \sinh x = -\frac{1}{2} \text{ or } \sinh x = -1.$$

$$\Rightarrow \frac{1}{2}(e^x - e^{-x}) = -\frac{1}{2} \text{ and } \frac{1}{2}(e^x - e^{-x}) = -1.$$

Using similar techniques as in Example 7, we get

$$x = \ln \frac{-1 + \sqrt{5}}{2} \text{ or } x = \ln(-1 + \sqrt{2}).$$

Summary of Ch. 3, 4 & 5

Chapter 4: Polynomials & Rational Functions

1.) Quadratic polynomials:

• Standard form

$$f(x) = a(x - h)^2 + k$$

Shape of the graph
 $a > 0 \Rightarrow$ U-shape curve
 $a < 0 \Rightarrow$ inverted U-shape

The value of x which the function attains its maximum/minimum.
[x-coordinate of vertex]

maximum (if $a < 0$) or minimum (if $a > 0$) of the function
[y-coordinate of vertex]

• Solution to $x^2 + bx + c = 0$ is

$$x_{1,2} = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

("Root formula")

$\hookrightarrow$ This gives factorization

$$x^2 + bx + c = (x - x_1)(x - x_2)$$

(solution ["root"] might not exist, or $x_1 = x_2$.)



2.) Factorizing Polynomials:

• Example:

$$\begin{aligned}f(x) &= 2x^3 - 5x^2 - 4x + 3 \\&= (x - 3)(2x^2 + x - 1) \\&= (x - 3)(2x - 1)(x + 1).\end{aligned}$$

Guess first root.

Long division.

Root formula

• Long division:

$$\begin{aligned}f(x) &= \underline{2x^3} - 5x^2 - 4x + 3 \\&= 2x^2(x - 3) + 6x^2 - 5x^2 - 4x + 3 \\&= 2x^2(x - 3) + \underline{x^2} - 4x + 3 \\&= 2x^2(x - 3) + x(x - 3) + 3x - 4x + 3 \\&= 2x^2(x - 3) + x(x - 3) \quad \underbrace{-x + 3}_{= -(x - 3)} \\&= (2x^2 + x - 1) \cdot (x - 3)\end{aligned}$$

$$\begin{array}{r}2x^2 + x - 1 \\x - 3 \overline{) 2x^3 - 5x^2 - 4x + 3} \\ \underline{2x^3 - 6x^2} \\ x^2 - 4x \\ \underline{x^2 - 3x} \\ -x + 3 \\ \underline{-x + 3} \\ 0\end{array}$$

3. Rational Functions:

$$r(x) = \frac{f(x)}{g(x)} \quad \begin{array}{l} \swarrow \text{polynomial} \\ \swarrow \text{polynomial} \end{array}$$

• Example:

$$f(x) = \frac{x+1}{\underbrace{x^4 - x^3 + x^2 - x}_{\text{hard to use}}} \stackrel{(*)}{=} \frac{x+1}{x(x-1)(x^2+1)} = \frac{1}{x-1} - \frac{1}{x} + \frac{1}{x^2+1} \quad \underbrace{\hspace{10em}}_{\text{easy to use}}$$

Decomposition
into "partial fractions"

• Method of partial fractions: (only for proper rational functions)

(1) Factorize denominator g .

(2) Make correct ansatz! E.g.:

$$\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

(3) Solve for unknowns $A, B, \dots$!

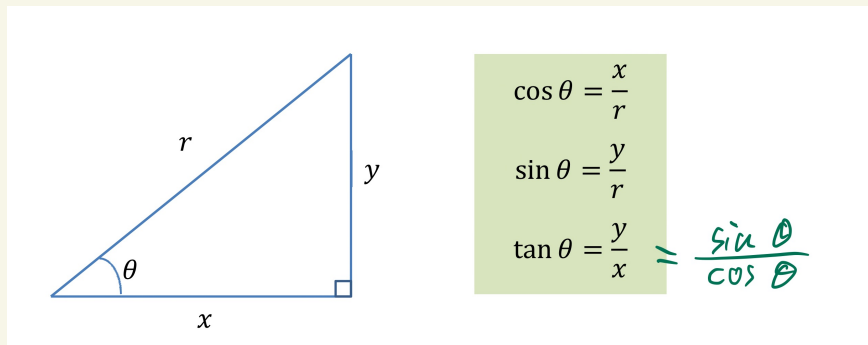
• Ansatz for step (2):

Scenario	Proposed decomposition
If $g(x)$ can be factorized into a product of distinct linear factor: $r(x) = \frac{f(x)}{(a_1x + b_1)(a_2x + b_2) \dots (a_nx + b_n)}$	$r(x) = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_n}{a_nx + b_n}.$
If there is some repeated linear factors in the factorization of $g(x)$: $r(x) = \frac{f(x)}{(ax + b)^n(cx + d)}$	$r(x) = \underbrace{\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \dots + \frac{A_n}{(ax + b)^n}}_{\text{for } (ax+b)^n} + \underbrace{\frac{B}{cx + d}}_{\text{for } (cx+d)}$
If there is quadratic factor (cannot be factorized, e.g. $x^2 + 4$) in the factorization of $g(x)$: $r(x) = \frac{f(x)}{(ax^2 + bx + c)(dx + e)}$	$r(x) = \frac{\overbrace{Ax + B}^{\text{degree 1}}}{\underbrace{ax^2 + bx + c}_{\text{degree 2}}} + \frac{C}{dx + e}$
If there is degree n polynomial in the factorization of $g(x)$: $r(x) = \frac{f(x)}{(a_nx^n + \dots + a_1x + a_0)(bx + c)}$	$r(x) = \frac{\overbrace{A_{n-1}x^{n-1} + \dots + A_1x + A_0}^{\text{degree } n-1}}{\underbrace{a_nx^n + \dots + a_1x + a_0}_{\text{degree } n}} + \frac{B}{bx + c}$

Chapter 4

Trigonometric Functions

• Definitions



• Periodicity:

	$x = 90^\circ - \theta$	$x = 90^\circ + \theta$	$x = 180^\circ - \theta$	$x = 180^\circ + \theta$
$\sin(x)$	$\cos \theta$	$\cos \theta$	$\sin \theta$	$-\sin \theta$
$\cos(x)$	$\sin \theta$	$-\sin \theta$	$\cos \theta$	$-\cos \theta$
$\tan(x)$	$\frac{1}{\tan \theta}$	$-\frac{1}{\tan \theta}$	$-\tan \theta$	$\tan \theta$

	$x = 270^\circ - \theta$	$x = 270^\circ + \theta$	$x = 360^\circ - \theta$ (or $-\theta$)	$x = 360^\circ + \theta$
$\sin(x)$	$-\cos \theta$	$-\cos \theta$	$-\sin \theta$	$\sin \theta$
$\cos(x)$	$-\sin \theta$	$\sin \theta$	$\cos \theta$	$\cos \theta$
$\tan(x)$	$\frac{1}{\tan \theta}$	$-\frac{1}{\tan \theta}$	$-\tan \theta$	$\tan \theta$

• general solutions:

(i) The general solution of $\sin x = k$ (where $-1 \leq k \leq 1$) is given by

$$x = 180^\circ n + (-1)^n \sin^{-1} k, \quad n \text{ is integer.}$$

(ii) The general solution of $\cos x = k$ (where $-1 \leq k \leq 1$) is given by

$$x = 360^\circ n \pm \cos^{-1} k, \quad n \text{ is integer.}$$

(iii) The general solution of $\tan x = k$ (where $k \in \mathbb{R}$) is given by

$$x = 180^\circ n + \tan^{-1} k, \quad n \text{ is integer}$$

• Pythagoras' Theorem:

$$\sin^2(x) + \cos^2(x) = 1$$

• Compound Angle Thm:

($\Rightarrow$ Sum-to-Product
& Product-to-Sum
Formulas)

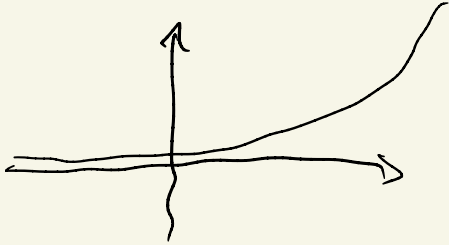
$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Chapter 5:

1) Exponential Functions:



$$f(x) = a^x \quad (x \in \mathbb{R})$$

$$a > 0, \quad a \neq 1$$

• "Natural" exponential $e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n$

• Formulas:

$$(1) \quad a^x \cdot a^y = a^{x+y}, \quad (2) \quad \frac{a^x}{a^y} = a^x \cdot a^{-y} = a^{x-y}.$$

$$(3) \quad (a^x)^y = a^{xy}$$

$$(4) \quad a^x \cdot b^x = (ab)^x$$

2) Logarithms:

$$1. \log_a a^x = x \text{ and } a^{\log_a x} = x$$

$$2. \log_a M^n = n \log_a M$$

$$3. \log_a MN = \log_a M + \log_a N$$

$$4. \log_a \frac{M}{N} = \log_a M - \log_a N$$

$$5. \log_a M = \frac{\log_b M}{\log_b a}.$$

(Definition as inverse of a^x)

} kind of opposite to formulas
for exponential (previous slide)

("Change of base")