
EE3121

Differential Equations for Engineers

Tutorial 5-1



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Tutorial

Solve the following second-order linear ODEs:

(a)

$$\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 41y = 0, \quad y(0) = 6, \quad \dot{y}(0) = -2.$$

(b) (Use variation of parameters method)

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 3t, \quad y(0) = 0, \quad \dot{y}(0) = 0.$$

(c) (Use undetermined coefficients method)

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 8e^{-4t} \sin 2t, \quad y(0) = 0, \quad \dot{y}(0) = 0.$$

Tutorial

(a)

$$\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 41y = 0, \quad y(0) = 6, \quad \dot{y}(0) = -2$$

Solution:

Step 1: Try the solution e^{rt}

Step 2: Characteristic polynomial $r^2 - 8r + 41 = 0$

$$r_{1,2} = \frac{8 \pm \sqrt{8^2 - 4 \cdot 41}}{2} = 4 \pm 5i = \lambda + \omega i$$

$$y(t) = c_1 e^{\lambda t} \cos \omega t + c_2 e^{\lambda t} \sin \omega t = c_1 e^{4t} \cos 5t + c_2 e^{4t} \sin 5t$$

Tutorial

$$\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 41y = 0, \quad y(0) = 6, \quad \dot{y}(0) = -2$$
$$y(t) = c_1 e^{4t} \cos 5t + c_2 e^{4t} \sin 5t$$

Step 3: Solve for c_1 and c_2 by initial conditions

$$\frac{dy}{dt} = c_1 (4e^{4t} \cos 5t - c_1 e^{4t} 5 \sin 5t) + c_2 (4e^{4t} \sin 5t + e^{4t} 5 \cos 5t)$$

$$\begin{cases} y(0) = c_1 = 6, \\ \left. \frac{dy}{dt} \right|_{t=0} = 4c_1 + 5c_2 = -2 \end{cases}$$

Fill in the remaining steps.

Tutorial

(b) (Use variation of parameters method)

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 3t, \quad y(0) = 0, \quad \dot{y}(0) = 0.$$

Solution:

Step 1: Find two fundamental solutions

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 0$$

Characteristic polynomial

$$r^2 + 6r + 8 = 0 \Rightarrow (r + 2)(r + 4) = 0 \Rightarrow r_1 = -2, r_2 = -4$$

Fundamental solutions $y_1(t) = e^{-2t}$, $y_2(t) = e^{-4t}$

Tutorial

Overdamped system with fundamental solutions

$$y_1(t) = e^{-2t}, \quad y_2(t) = e^{-4t}$$

Step 2: Find the kernel function

$$K(t - \tau) = \frac{e^{-\lambda_1(t-\tau)} - e^{-\lambda_2(t-\tau)}}{\lambda_2 - \lambda_1} = \frac{e^{-2(t-\tau)} - e^{-4(t-\tau)}}{4 - 2}$$

Step 3: Calculate the convolution integral

$$\begin{aligned} y(t) &= \int_0^t K(t - \tau) g(\tau) d\tau = \frac{1}{2} \int_0^t \left(e^{-2(t-\tau)} - e^{-4(t-\tau)} \right) 3\tau d\tau \\ &= \frac{3}{2} e^{-2t} \int_0^t e^{2\tau} \tau d\tau - \frac{3}{2} e^{-4t} \int_0^t e^{4\tau} \tau d\tau \end{aligned}$$

Hint: Use integration by parts.

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(c) (Use undetermined coefficients method)

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 8e^{-4t} \sin 2t, \quad y(0) = 0, \quad \dot{y}(0) = 0.$$

Solution:

Step 1: Trial solution $y(t) = (C \sin 2t + D \cos 2t) e^{-4t}$

$$\begin{aligned} \frac{dy}{dt} &= e^{-4t} (2C \cos 2t - 2D \sin 2t) - 4e^{-4t} (C \sin 2t + D \cos 2t) \\ &= 2e^{-4t} [-(2C + D) \sin 2t + (C - 2D) \cos 2t] \end{aligned}$$

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$$\frac{dy}{dt} = 2e^{-4t} [-(2C + D) \sin 2t + (C - 2D) \cos 2t]$$

$$\begin{aligned} \frac{d^2y}{dt^2} &= -8e^{-4t} [-(2C + D) \sin 2t + (C - 2D) \cos 2t] + \\ &\quad 2e^{-4t} [-2(2C + D) \cos 2t - 2(C - 2D) \sin 2t] \\ &= 4e^{-4t} [(3C + 4D) \sin 2t - (4C - 3D) \cos 2t] \end{aligned}$$

Step 2: Substitute into the LHS of ODE

$$\begin{aligned} \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y &= 4e^{-4t} [[(3C + 4D) - 2(2C + D) + C] \sin 2t - \\ &\quad [- (4C - 3D) + 2(C - 2D) + D] \cos 2t] \\ &= 8e^{-4t} (D \sin 2t - C \cos 2t) \end{aligned}$$

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$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 8e^{-4t} (D \sin 2t - C \cos 2t)$$

Compare to the RHS of ODE

$$8e^{-4t} (D \sin 2t - C \cos 2t) = 8e^{-4t} \sin 2t \Rightarrow \begin{cases} C = 0 \\ D = 1 \end{cases}$$

The steady-state solution is $y_s(t) = e^{-4t} \cos 2t$

Step 3: Find the fundamental solution $y_0(t)$ such that the solution $y(t) = y_0(t) + y_s(t)$ satisfies initial conditions $y(0) = 0$, $\dot{y}(0) = 0$.

Characteristic polynomial $r^2 + 4r + 4 = 0 \Rightarrow r_{1,2} = -2$

$$y_0(t) = (c_1 + c_2 t) e^{-2t}, \quad y(t) = (c_1 + c_2 t) e^{-2t} + e^{-4t} \cos 2t$$

Fill in the remaining steps.

END