

EE3109 Applied Electromagnetics
2024/25 Semester B
Assignment 1A

1. Write Maxwell's Equations in differential form. Write within brackets the unit for each variable (eg. $\mathbf{E}$ [V/m]).

Solution:

Gauss's Law:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (\mathbf{E} \text{ in [V/m]}, \rho \text{ in [C/m}^3\text{]}, \epsilon_0 \text{ in [F/m]})$$

Gauss's Law for Magnetism:

$$\nabla \cdot \mathbf{B} = 0 \quad (\mathbf{B} \text{ in [T]})$$

Faraday's Law of Induction:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad ([\text{V/m}], [\text{T}], [\text{s}])$$

Ampere-Maxwell Law:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad ([\text{T}], [\text{A/m}^2], [\text{H/m}], [\text{F/m}], [\text{s}])$$

Units: $\mathbf{E}$ in [V/m], ρ in [C/m³], ϵ_0 in [F/m], $\mathbf{B}$ in [T], t in [s], $\mathbf{J}$ in [A/m²], μ_0 in [H/m]

2. A plane wave has magnetic flux density $\mathbf{B}^-(z, t) = B_0 \cos(\omega t + kz + \phi) \hat{\mathbf{x}}$ [T]
- In what direction does it travel, and what is the speed of this plane wave?
 - What is the corresponding electric field?
 - What is the electric field at the origin?
 - What is the electric field at the point $(x, y, z) = (10\text{m}, -10\text{m}, 0\text{m})$?

Solution:

- (a) Given $\mathbf{B}^-(z, t) = B_0 \cos(\omega t + kz + \phi) \hat{\mathbf{x}}$ [T], the phase can be written as $\theta = \omega t + kz + \phi$. Phase velocity is the rate at which the constant phase point travels i.e $v_p = dz/dt$ when θ is constant. Re-evaluate the phase expression to get:

$$z = -\frac{\omega}{k}t + \frac{\theta - \phi}{k}$$
$$v_p = dz/dt = -\frac{\omega}{k} \quad [\text{m/s}]$$

Conclusion: The given wave travels in the negative z-direction with the speed of ω/k .

- (b) Using Maxwell's equations, specifically the relation between the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ in a plane wave, we find the corresponding electric field using phasor and plane wave relationships:

$$\mathbf{B} = B_0 e^{jkz} e^{j\phi} \hat{\mathbf{x}};$$
$$\mathbf{E} = \eta_0 \mathbf{H} \times \hat{\mathbf{k}} = \frac{\eta_0}{\mu_0} \mathbf{B} \times \hat{\mathbf{k}} = \frac{\eta_0 B_0}{\mu_0} e^{jkz} e^{j\phi} (\hat{\mathbf{x}} \times (-\hat{\mathbf{z}})) = \frac{\eta_0 B_0}{\mu_0} e^{jkz} e^{j\phi} \hat{\mathbf{y}};$$

Converting into the time-domain, we have:

$$\mathbf{E}(z, t) = \frac{\eta_0 B_0}{\mu_0} \cos(\omega t + kz + \phi) \hat{\mathbf{y}} \quad [\text{V/m}]$$

(c) At the origin $(x, y, z) = (0, 0, 0)$, we have:

$$\mathbf{E}(0, t) = \frac{\eta_0 B_0}{\mu_0} \cos(\omega t + \phi) \hat{\mathbf{y}} \text{ [V/m]}$$

(d) Since the plane wave travels in the $-z$ -direction, the fields do not change across the xy -plane, so at the point $(x, y, z) = (10\text{m}, -10\text{m}, 0\text{m})$, we also have the same electric field as we have in (c).

3. The electric field of a plane wave is $\mathbf{E}(\mathbf{r}, t) = 15 \cos(5 \times 10^9 \pi t + mx + ny - 70^\circ) (2\hat{\mathbf{x}} - \hat{\mathbf{y}}) \text{ [mV/m]}$. The wave travels in a lossless material with $\epsilon_r = 1$ and $\mu_r = 9$. Find:
- Its frequency
 - Its wavelength
 - Its propagation direction
 - The constants m and n
 - The corresponding $\mathbf{H}(\mathbf{r}, t)$.
 - The amplitude of the $\mathbf{H}$ field at the origin, at time $t = 0$.

Solution:

(a)

$$f = \frac{\omega}{2\pi} = \frac{5 \times 10^9 \pi}{2\pi} = 2.5 \times 10^9 = 2.5 \text{ [GHz]}$$

(b)

$$\lambda = \frac{v_p}{f} = \frac{c}{f \sqrt{\mu_r \epsilon_r}} = \frac{3 \times 10^8}{(2.5 \times 10^9) \sqrt{(9)(1)}} = \frac{1}{25} = 0.04 \text{ [m]}$$

(c) Firstly, $-\mathbf{k} \cdot \mathbf{r} = mx + ny \Rightarrow \mathbf{k} = -(m\hat{\mathbf{x}} + n\hat{\mathbf{y}})$

$\mathbf{k}$ is perpendicular to $\mathbf{E}$,

so $0 = \mathbf{k} \cdot \mathbf{E} = -(m\hat{\mathbf{x}} + n\hat{\mathbf{y}}) \cdot 15 \cos(5 \times 10^9 \pi t + mx + ny - 70^\circ) (2\hat{\mathbf{x}} - \hat{\mathbf{y}})$.

Simplifying gives: $2m - n = 0 \Rightarrow 2m = n \Rightarrow \mathbf{k} = -(m\hat{\mathbf{x}} + 2m\hat{\mathbf{y}})$

So, the (normalized) propagation direction is either $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$ or $\left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0\right)$. (two possible solutions)

(d) $k = |\mathbf{k}| = \sqrt{m^2 + n^2 + 0^2} = \pm\sqrt{5}m \Rightarrow m = \pm \frac{k}{\sqrt{5}} = \pm \frac{50\pi}{\sqrt{5}} = \pm 70.25 \text{ [m}^{-1}\text{]}; n = 2m = \pm 140.5 \text{ [m}^{-1}\text{]}$

Combining gives $\mathbf{k} = -(m, n, 0) = \pm(70.25, 140.5, 0) \text{ [m}^{-1}\text{]}$ (again, two possible solutions).

(e) We will choose the solution $\mathbf{k} = (70.25, 140.5, 0) \text{ [m}^{-1}\text{]}$.

Convert $\mathbf{E}(\mathbf{r}, t)$ into phasor form: $\mathbf{E} = 15 \exp(-70.25x - 140.5y - 70^\circ) (2\hat{\mathbf{x}} - \hat{\mathbf{y}}) \text{ [A/m]}$.

Using the relationship for plane waves,

$$\begin{aligned} \mathbf{H} &= \frac{1}{\eta} \hat{\mathbf{k}} \times \mathbf{E} = \frac{1}{\eta_0 \sqrt{\mu_r / \epsilon_r}} 15 \exp(-70.25x - 140.5y - 70^\circ) \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right) \times (2, -1, 0) \\ &= \frac{1}{(120\pi)(3)} (15) \exp(-70.25x - 140.5y - 70^\circ) \left(-\frac{1}{\sqrt{5}} - \frac{4}{\sqrt{5}}\right) \hat{\mathbf{z}} \\ &= -0.02966 \exp(-70.25x - 140.5y - 70^\circ) \hat{\mathbf{z}} \text{ [A/m]} \\ &= 29.66 \exp(-70.25x - 140.5y + 110^\circ) \hat{\mathbf{z}} \text{ [mA/m]} \end{aligned}$$

Converting into time domain gives $\mathbf{H}(\mathbf{r}, t) = 29.66 \cos(5 \times 10^9 \pi t - 70.25x - 140.5y + 110^\circ) \hat{\mathbf{z}} \text{ [mA/m]}$.

(If we choose the other $\mathbf{k}$, we will find a corresponding $\mathbf{H}(\mathbf{r}, t)$ with its direction reversed.)

(f) $\mathbf{H}(x = 0, y = 0, t = 0) = 29.66 \cos(110^\circ) \hat{\mathbf{z}} \text{ [mA/m]} = -10.14 \hat{\mathbf{z}} \text{ [mA/m]}$

4. Consider an electric field phasor in the form of a Gaussian beam:

$$\mathbf{E}(x, y, z, t) = E_0 \exp\left(-\frac{(x^2 + y^2)}{w_0^2}\right) \exp(-jkz) \hat{\mathbf{x}}.$$

- (a) Find the corresponding magnetic field phasor using Maxwell's equations.
 (b) Show that, close to the center of the beam (i.e. for $x^2 + y^2 \ll w_0^2$), the Gaussian beam can be approximated by a plane wave.

Solution:

- (a) We find $\mathbf{H}$ from $\mathbf{E}$ using the phasor form of the Faraday's Law

$$\mathbf{H} = \frac{1}{-j\omega\mu_0} \nabla \times \mathbf{E} = \frac{1}{-j\omega\mu_0} \left(\frac{\partial E_x}{\partial z} \hat{\mathbf{y}} - \frac{\partial E_x}{\partial y} \hat{\mathbf{z}} \right)$$

(the other components vanish b/c $E_y = E_z = 0$.)

We now calculate the derivatives:

$$\frac{\partial E_x}{\partial z} = E_0 \exp\left(-\frac{(x^2 + y^2)}{w_0^2}\right) \left[\frac{\partial}{\partial z} \exp(-jkz) \right] = E_0 \exp\left(-\frac{(x^2 + y^2)}{w_0^2}\right) [-jk \exp(-jkz)] = -jkE_x$$

The green part is treated as a multiplicative constant, because we are taking a derivative with respect to z it doesn't vary with z .

We do a similar thing with the other derivative:

$$\frac{\partial E_x}{\partial y} = E_0 \exp(-jkz) \left[\frac{\partial}{\partial y} \exp\left(-\frac{(x^2 + y^2)}{w_0^2}\right) \right] = E_0 \exp(-jkz) \left[-\frac{2y}{w_0^2} \exp\left(-\frac{(x^2 + y^2)}{w_0^2}\right) \right] = -\frac{2y}{w_0^2} E_x$$

Substituting into the equation for the magnetic field, we have:

$$\begin{aligned} \mathbf{H} &= \frac{1}{-j\omega\mu_0} \left(-jkE_x \hat{\mathbf{y}} + \frac{2y}{w_0^2} E_x \hat{\mathbf{z}} \right) = E_x \left(\frac{1}{\eta_0} \hat{\mathbf{y}} - j \frac{2y}{\omega\mu_0 w_0^2} \hat{\mathbf{z}} \right) \\ &= E_0 \exp\left(-\frac{(x^2 + y^2)}{w_0^2}\right) \exp(-jkz) \left(\frac{1}{\eta_0} \hat{\mathbf{y}} - j \frac{2y}{\omega\mu_0 w_0^2} \hat{\mathbf{z}} \right) \end{aligned}$$

- (b) For $x^2 + y^2 \ll w_0^2$, the exponential term can be approximated as:

$$\exp\left(-\frac{(x^2 + y^2)}{w_0^2}\right) \approx 1$$

Thus, the electric field phasor simplifies to:

$$\mathbf{E}(x, y, z, t) = E_0 \exp(-jkz) \hat{\mathbf{x}}$$

Additionally, since $y^2 < x^2 + y^2 \ll w_0^2 \Rightarrow |y| \ll w_0$, the z -component of the magnetic field is negligible.

Hence the magnetic field phasor simplifies to:

$$\mathbf{H}(x, y, z, t) = \frac{E_0}{\eta} \exp(-jkz) \hat{\mathbf{y}}$$

Therefore, close to the center of the beam, the Gaussian beam can be approximated by a plane wave.