

In-Class Activity 3 Solutions

Question 1 Derive the cost function $C(q)$ for a firm with production function

$$8k^{1/2}l^{1/4}$$

Assume that capital costs 4 per unit and labor costs 5 per unit.

The cost-minimization problem for a given output q^* is

$$\text{minimize } 4k + 5l \quad \text{subject to } 8k^{1/2}l^{1/4} = q^*$$

The Lagrangean is

$$\mathcal{L} = 4k + 5l - \lambda(8k^{1/2}l^{1/4} - q^*)$$

The optimal conditions are

$$\frac{\partial \mathcal{L}}{\partial k} = 4 - \lambda \left(\frac{8}{2} k^{-1/2} l^{1/4} \right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial l} = 5 - \lambda \left(\frac{8}{4} k^{1/2} l^{-3/4} \right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 8k^{1/2}l^{1/4} - q^* = 0$$

From the first two equations, we have

$$\frac{4k^{-1/2}l^{1/4}}{2k^{1/2}l^{-3/4}} = \frac{4}{5} \implies \frac{l}{k} = \frac{2}{5} \implies l = \frac{2}{5}k$$

Then, from the third equation

$$8k^{1/2} \left(\frac{2}{5} \right)^{1/4} k^{1/4} = q^* \implies k = \left(\frac{1}{16} \right) \left(\frac{5}{2} \right)^{1/3} (q^*)^{4/3}$$

$$l = \frac{2}{5}k = \left(\frac{1}{16} \right) \left(\frac{2}{5} \right)^{2/3} (q^*)^{4/3}$$

Then, the minimum cost of producing q^* is

$$4 \left(\left(\frac{1}{16} \right) \left(\frac{5}{2} \right)^{1/3} (q^*)^{4/3} \right) + 5 \left(\left(\frac{1}{16} \right) \left(\frac{2}{5} \right)^{2/3} (q^*)^{4/3} \right) = \left(\frac{15}{16} \right) \left(\frac{2}{5} \right)^{2/3} (q^*)^{4/3}$$

Thus the firm's cost function is $C(q) = Zq^{4/3}$, where

$$Z = \left(\frac{15}{16} \right) \left(\frac{2}{5} \right)^{2/3}$$

Question 2: Suppose the cost function derived in the previous question is for a small firm operating in a perfectly competitive market. Find the expression for the supply function of the firm having an upper limit of production capacity at L .

When the market price is fixed at p , the firm's profit from producing q is

$$\Pi_p(q) = pq - C(q) = pq - Zq^{4/3}$$

For each p , the profit-maximizing value of q is determined by the first-order condition

$$\begin{aligned}\Pi'_p(q) = 0 &\implies p - C'(q) = p - \frac{4}{3}Zq^{1/3} = 0 \\ &\implies p = \frac{4}{3}Zq^{1/3}\end{aligned}$$

The above gives the inverse supply function $p^S(q)$. Writing q in terms of p , we get the supply function q^S :

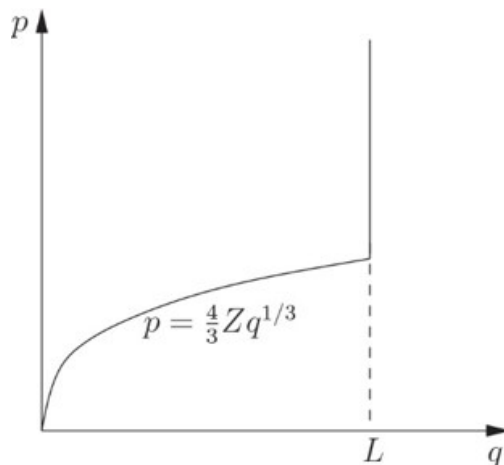
$$q = \frac{27p^3}{64Z^3} = \frac{16}{5}p^3$$

Note that when the firm is producing at its profit maximizing level q , the actual profit is

$$\Pi(q) = qC'(q) - C(q) = q\left(\frac{4}{3}Zq^{1/3}\right) - Zq^{4/3} = \frac{Z}{3}q^{4/3}$$

This is zero when $q = 0$ and positive for $q > 0$. So, the startup point is $q_s = 0$.

If L is the upper limit on the firm's production capacity, and $0 \leq q \leq L$, then a point (q, p) is in the supply set if $q = (16/5)p^3$. For larger values of p , when $L < (16/5)p^3$ or $p > (5L/16)^{1/3}$, the firm should produce L . So, for these values of p the points (L, p) are in the supply set.



Question 3: Dave's utility function for apples and bananas is

$$u(x_1, x_2) = x_1^2 x_2,$$

where x_1 is the number of apples and x_2 the number of bananas. Each apple costs \$0.15 and each banana costs \$0.10. Use the method of Lagrange multipliers to determine how many of each Dave will buy to maximize his utility if he has \$1.50 to spend. (*Assume that fractions of apples and bananas can be bought.*)

The consumer's problem is

$$\text{maximize } x_1^2 x_2 \quad \text{subject to } 0.15x_1 + 0.1x_2 = 1.5$$

The Lagrangean is

$$\mathcal{L} = x_1^2 x_2 - \lambda(0.15x_1 + 0.1x_2 - 1.5)$$

The optimal conditions are

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 x_2 - 0.15\lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = x_1^2 - 0.1\lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0.15x_1 + 0.1x_2 - 1.5 = 0$$

From the first two equations, we have

$$\frac{2x_1 x_2}{x_1^2} = \frac{0.15}{0.1} \implies x_2 = \frac{3}{4}x_1$$

Then, from the third equation

$$0.15x_1 + 0.1\left(\frac{3}{4}x_1\right) = 1.5$$

$$\implies x_1^* = 20/3 \quad \implies x_2^* = \frac{3}{4}x_1^* = 5$$

Dave should therefore buy 20/3 apples and 5 bananas.