

L09: Terminated Transmission Lines

EE3109 Applied Electromagnetics

Lecture Notes

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Jan – Apr 2023

Outline

- Course Matter
- Terminated TLs
- Multiple Reflections in TLs
- Power Flow in TLs

Tentative Course Schedule

Week #	Week of	Course Coverage
1	Jan 9	Course Introduction, Math Review
2	Jan 16	Introduction to Maxwell Equations, Materials and Boundaries
	Jan 23	<i>Happy Chinese New Year!</i>
3	Jan 30	Wave Equations, Plane Waves, Propagation in Lossy Media
4	Feb 6	Polarizations, <i>Asst. 1a Due</i>
5	Feb 13	EM Potential, Polarizations Cont'd, Potentials <i>Asst. 1b Due</i>
6	Feb 20	<i>Test 1</i> , Reflection and Transmission
7	Feb 27	Reflection and Transmission Cont'd
8	Mar 6	Power and Velocity
9	Mar 13	Waveguides and Transmission Lines (TL), VIP in a TL, Terminated and Lossy TLs, <i>Asst. 2 Due, Lab 1 Begins</i>
10	Mar 20	Impedance in a TL, Smith Charts
11	Mar 27	Smith Charts Cont'd, Impedance Matching, Quarter-wave Transformers, <i>Lab 2 Begins</i>
12	Apr 3	<i>Test 2</i> , Introduction to Antennas
13	Apr 10	Dipole Antennas, <i>Asst. 3 Due</i>

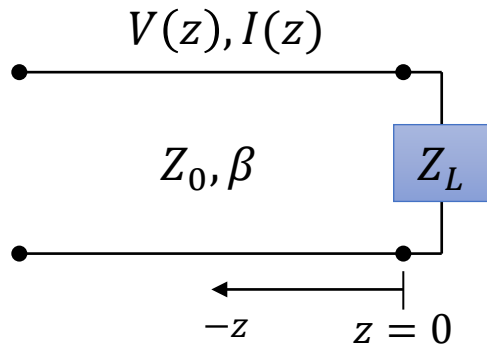
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Review and Preview

- Last class we introduced the concept of the transmission line (TL), and examined basic wave propagation in the TL.
- This class, we will continue look at what happens at terminated TLs

Terminated Lossless TL



$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z}$$

- $V(z)$ and $I(z)$ are solutions to 2nd-order linear ODEs, and feature two unknowns: V_0^+ and V_0^- .
- We can reduce one unknown if we know the load impedance.
- Method: i.e. use $\left. \frac{V(z)}{I(z)} \right|_{z=0} = Z(z=0) = Z_L$

Reflection Coefficient

$$Z_L = Z(z = 0) = Z_0 \left. \frac{V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}}{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}} \right|_{z=0} = Z_0 \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}$$

Define:

$$\Gamma_L = \frac{V_0^-}{V_0^+} \equiv \text{reflection coefficient}$$

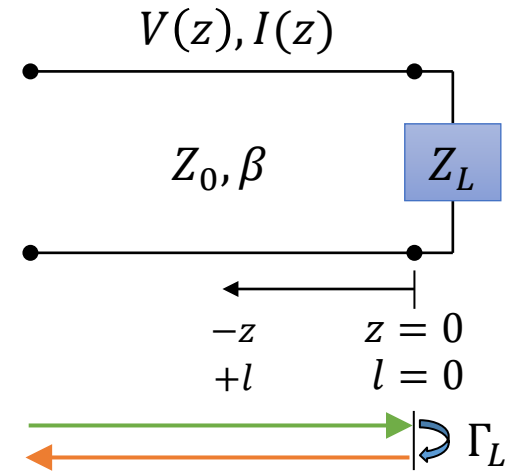
Substitute and rearrange to get:

$$Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \Leftrightarrow \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$V(z)$ and $I(z)$ become:

$$V(z) = V_0^+ e^{-j\beta z} (1 + \Gamma_L e^{+j2\beta z})$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} (1 - \Gamma_L e^{+j2\beta z})$$



+ve'ly travelling (incident) wave
-ve'ly travelling (reflected) wave

At the load ($z = 0$):

$$V(z = 0) = V_0^+ (1 + \Gamma_L)$$

$$I(z = 0) = \frac{V_0^+}{Z_0} (1 - \Gamma_L)$$

Reflection Coefficient along the TL

Look at these equations again:

$$V(z) = V_0^+ e^{-j\beta z} (1 + \Gamma_L e^{+j2\beta z})$$

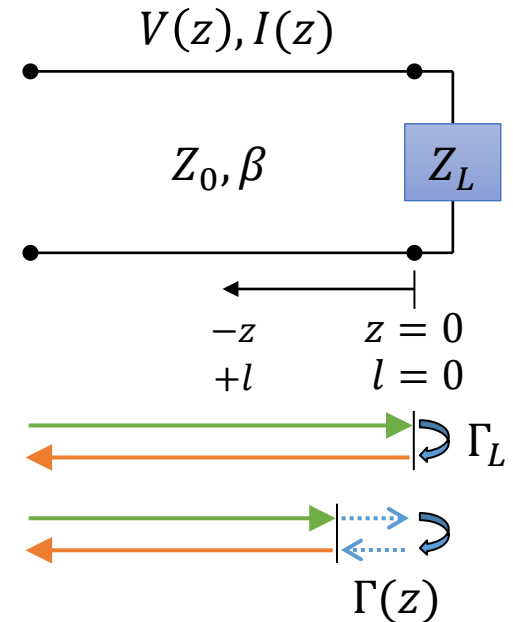
$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} (1 - \Gamma_L e^{+j2\beta z})$$

Define:

$$\Gamma(z) = \Gamma_L e^{+j2\beta z}, \text{ s. t.}$$

$$V(z) = V_0^+ e^{-j\beta z} (1 + \Gamma(z))$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} (1 - \Gamma(z))$$



+ve'ly travelling (incident) wave
-ve'ly travelling (reflected) wave

$\Gamma(z)$ is the reflection coefficient at TL location z looking toward the load.

Impedance along the TL

Similarly, impedance can be defined as a function of TL location:

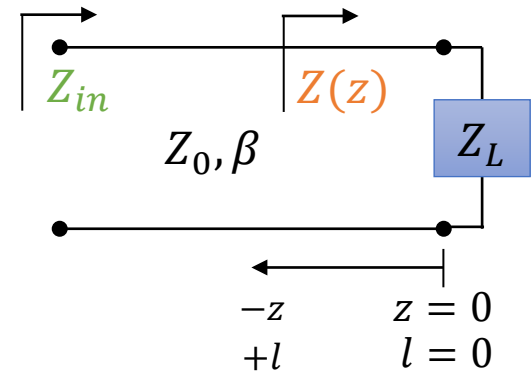
$$Z(z) = Z_0 \frac{1 + \Gamma(z)}{1 - \Gamma(z)} = Z_0 \frac{1 + \Gamma_L e^{+j2\beta z}}{1 - \Gamma_L e^{+j2\beta z}}$$

For a transmission line of length l , the input impedance is:

$$Z_{in} = Z(l) = Z_0 \frac{1 + \Gamma_L e^{-j2\beta l}}{1 - \Gamma_L e^{-j2\beta l}}$$

⋮

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}$$



Impedance along the TL

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}$$

Special Cases:

1. Open Circuit ($Z_L = \infty$):
 $Z_{in} = -jZ_0 \cot \beta l$

2. Short Circuit ($Z_L = 0$):
 $Z_{in} = jZ_0 \tan \beta l$

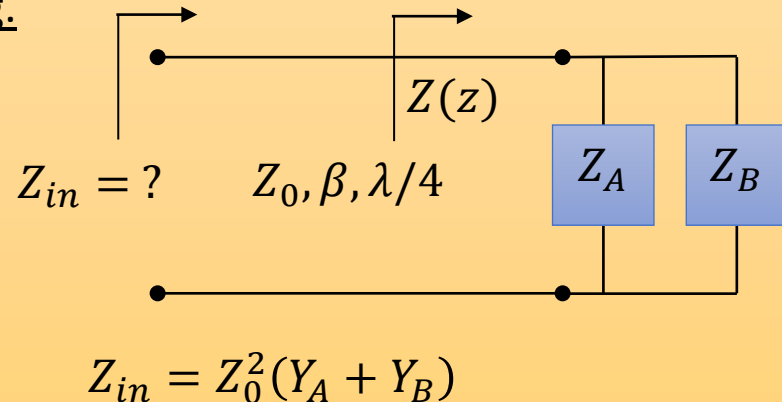
3. Cap. or Ind. ($Z_L = jX_L$):
 $Z_{in} = jX_{in}$

4. Matched Circuit ($Z_L = Z_0$):
 $Z_{in} = Z_0$

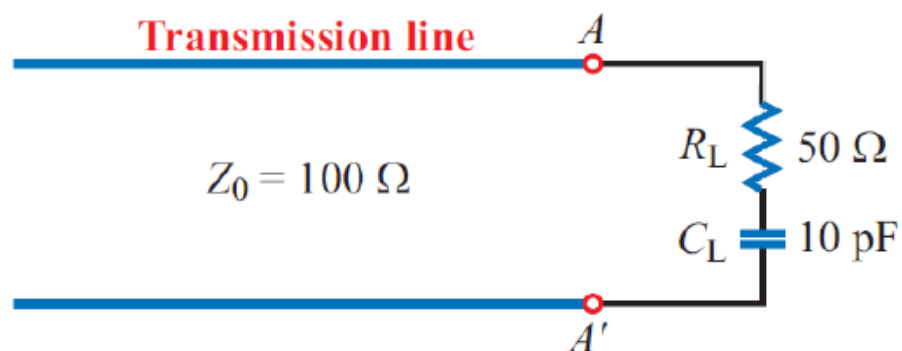
5. $\frac{1}{4}$ -Wave Transformer ($l = \lambda/4$):
 $Z_{in} = Z_0^2 / Z_L$

6. $\frac{1}{2}$ -Wave Transformer ($l = \lambda/2$):
 $Z_{in} = Z_L$

Eg.



Example 1



A $100\text{-}\Omega$ transmission line is connected to a load consisting of a $50\text{-}\Omega$ resistor in series with a 10-pF capacitor. Find the reflection coefficient at the load for a 100-MHz signal.

Solution: The following quantities are given (Fig. 2-13):

$$R_L = 50\ \Omega, \quad C_L = 10\ \text{pF} = 10^{-11}\ \text{F},$$

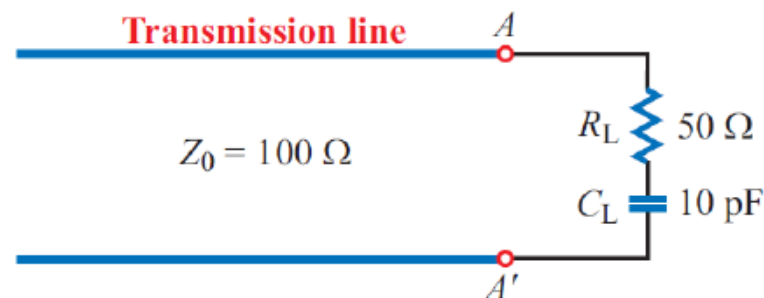
$$Z_0 = 100\ \Omega, \quad f = 100\ \text{MHz} = 10^8\ \text{Hz}.$$

$$Z_L = R_L + \frac{1}{j\omega C_L} = 50 + \frac{1}{j(2\pi \times 10^8)(10^{-11})} = 50 - j159\ [\Omega]$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{-50 - j159}{150 - j159} = 0.76\angle -60.7^\circ$$

$$\text{Or } |\Gamma| = 0.76, \quad \theta_r = -60.7^\circ$$

Example 1b



For the situation of the previous question:

$$Z_L = 50 - j159 \, \Omega; \Gamma_L = 0.76 \angle -60.7^\circ; Z_0 = 100 \, \Omega; f = 100 \, \text{MHz}$$

Find $V(z)$, $I(z)$ along the transmission line, if the transmission line is an air line that is 10 m long.

Solution:

First we find the relevant constants in this TL question:

$$\omega = 2\pi f = 2\pi \times 10^8 \, \text{Hz}; v_p = c = 3 \times 10^8 \, [\text{m/s}]; \beta = \frac{\omega}{v_p} = \frac{2\pi}{3} \, [\text{m}^{-1}]$$

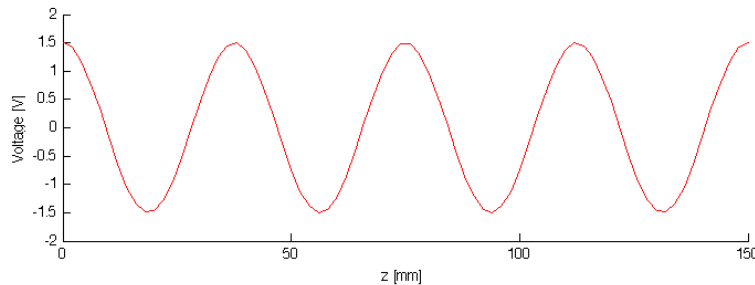
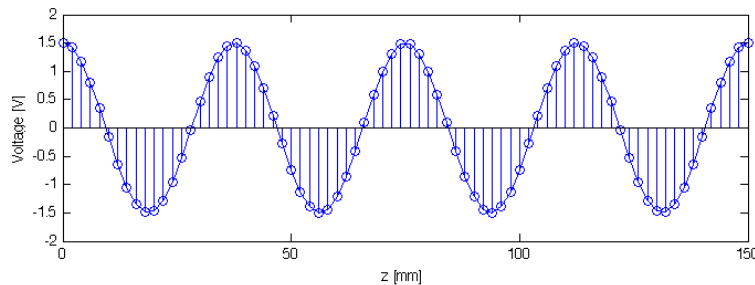
Now we write the voltage and current:

$$V(z) = V_0^+ e^{-j\beta z} (1 + \Gamma_L e^{+j2\beta z})$$
$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} (1 - \Gamma_L e^{+j2\beta z})$$

Substituting in Z_0 , β and Γ_L , we arrive at $V(z)$ and $I(z)$ as a function of V_0^+ and z .

On $V(z)$, $I(z)$ and S

For $\Gamma_L = 0.5$, we plot $V(z)$ and $I(z)$ along the transmission line w.r.t. time:



$$V(z) = V_0^+ e^{-j\beta z} (1 + \Gamma_L e^{+j2\beta z})$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} (1 - \Gamma_L e^{+j2\beta z})$$

Remarks:

1. $V(z)$ is maximized when $I(z)$ is minimized, and vice versa.
2. V_{max} (and I_{max}) occurs every $\lambda/2$
3. Standing Wave Ratio:

$$S = \frac{V_{max}}{V_{min}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} ;$$

$$\Rightarrow |\Gamma_L| = \frac{S - 1}{S + 1}$$

VSWR in the Real World



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PE51089
Mag Mount Antenna
890-960 &
1710-1880 MHz
3.0 dBi

Product Features

- 890-960 / 1710-1880 MHz
- 50W power handling
- High Gain, 8dBi Omni Radiation Pattern
- Mag Mount Antenna with Cable
- SMA Male, SMA Female, N Female & TNC Male

Specifications ⁽¹⁾

Parameter	Value	Units	Description
Band1 Frequency	890-960	MHz	
Band2 Frequency	1710-1880	MHz	
Impedance	50	Ohms	
Gain	3	dBi	Max Gain
Polarization	Vert		Linear Vertical
Max Input Pwr	50	W	Watts(Max)
Maximum VSWR	<=2.0	ratio	Between Min and Max Freq Ranges
Length	230	mm	9.0 inches
Cable Length	5	m	16Ft
Weight	90	g	Grams
Lightning Protection			DC Ground
Temperature Range	-40+60	Deg C	



Ordering Information

Some Questions for YOU....

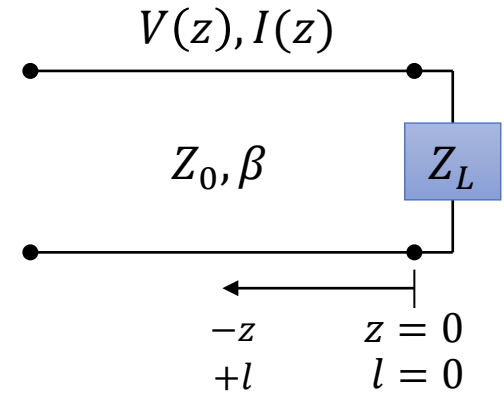
1. Find Z_{in} for the previous loaded transmission line.
2. How do you find the nearest V_{max} / I_{max} from the load?
3. What would S be, and how would the plot of $V(z)$ and $I(z)$ look, if the load is:
 - (a) An open circuit?
 - (b) A short circuit?
 - (c) A reactive element?
 - (d) A matched termination?

Finding distance to the nearest $V_{\max}$ and $V_{\min}$ from the load

Let $\Gamma_L = |\Gamma_L|e^{j(\theta_L)}$. Then,

$$V(z) = V_0^+ e^{-j\beta z} (1 + |\Gamma_L| e^{j(2\beta z + \theta_L)})$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} (1 - |\Gamma_L| e^{j(2\beta z + \theta_L)})$$



$V_{\max}$ occurs when the phase of the second term is an **even** multiple of π , i.e. $2\beta z + \theta_L = 2m\pi$ (Equation A) where m is an integer, b/c in this case the incident and reflected voltage waves will **constructively** interfere.

$V_{\min}$ occurs when the phase of the second term is an **odd** multiple of π , i.e. $2\beta z + \theta_L = (2m + 1)\pi$ (Equation B) where m is an integer, b/c in this case the incident and reflected voltage waves will **destructively** interfere.

If we know Z_L , we can find θ_L . Using θ_L and equations (A) and (B) above, you can find the smallest $|z|$ that will satisfy the equations above. These will give you the distances to the nearest voltage maximum and minimum.

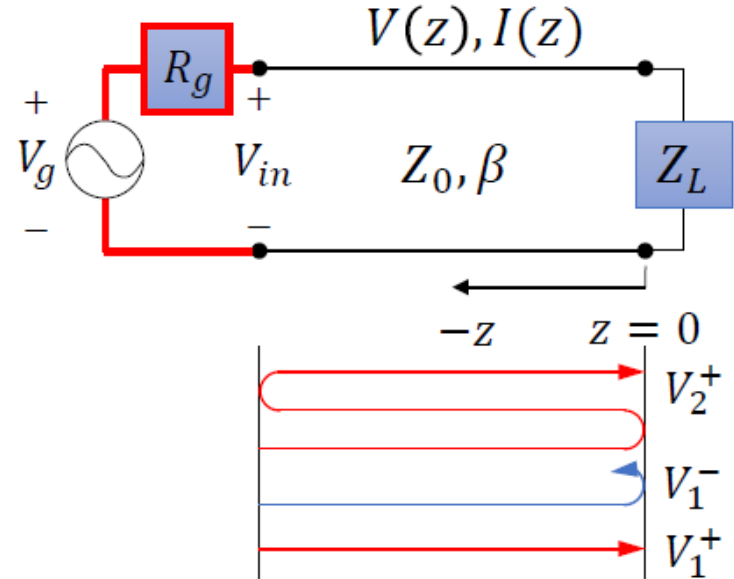
Remember:

- (i) z , as defined in class, is a negative number: eg. 3 mm from the load will be $z = -3$ [mm].
- (ii) If the voltage is at a maximum value, then the current is at a minimum value, and vice versa.

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Terminated Lossless TL with a Generator Impedance



$$V(z) = V_0^+ e^{-j\beta z} (1 + \Gamma_L e^{+j2\beta z})$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} (1 - \Gamma_L e^{+j2\beta z})$$

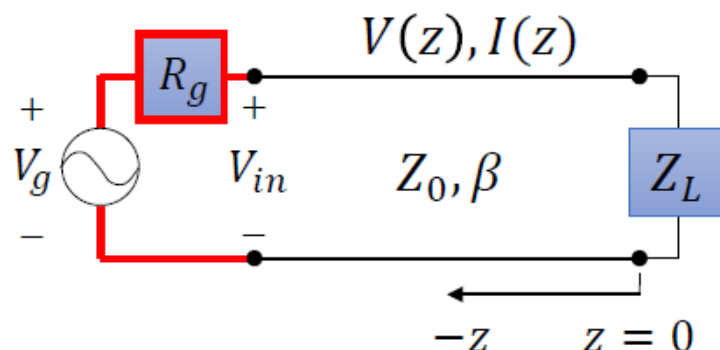
Transient Perspective

1. When the circuit is closed, and initial wave V_1^+ travels from the generator to the load, without first observing the load $\Rightarrow V_1^+ = Z_0 I_1^+ \Rightarrow V_1^+ = \frac{Z_0}{R_g + Z_0} (e^{-j\beta l})$.
2. More wave components are created after reflections from the load and generator:
 $V_1^- = \Gamma_L V_1^+$; $V_2^+ = \Gamma_G e^{-j2\beta l} V_1^- = \Gamma_L \Gamma_G e^{-j2\beta l} V_1^+$; $V_2^- = (\Gamma_L)^2 \Gamma_G e^{-j2\beta l} V_1^+ \dots$
3. The positively travelling waves superimpose to form V_0^+ :

$$V_0^+ = V_1^+ + V_2^+ + V_3^+ + \dots = V_1^+ \sum_{n=0}^{\infty} (\Gamma_L \Gamma_G e^{-j2\beta l})^n = \frac{V_1^+}{1 - \Gamma_L \Gamma_G e^{-j2\beta l}}$$

$$\Rightarrow V_0^+ = V_g \frac{Z_0}{R_g + Z_0} \left(\frac{e^{-j\beta l}}{1 - \Gamma_L \Gamma_G e^{-j2\beta l}} \right)$$

Terminated Lossless TL with a Generator impedance



$$V(z) = \frac{V_g Z_0}{R_g + Z_0} \left(\frac{e^{-j\beta l} e^{-j\beta z}}{1 - \Gamma_L \Gamma_G e^{-j2\beta l}} \right) (1 + \Gamma_L e^{+j2\beta z})$$
$$I(z) = \frac{V_g}{R_g + Z_0} \left(\frac{e^{-j\beta l} e^{-j\beta z}}{1 - \Gamma_L \Gamma_G e^{-j2\beta l}} \right) (1 - \Gamma_L e^{+j2\beta z})$$

Now we can write the voltage and current along the line given the source voltage (or current).

Power Along a Terminated TL

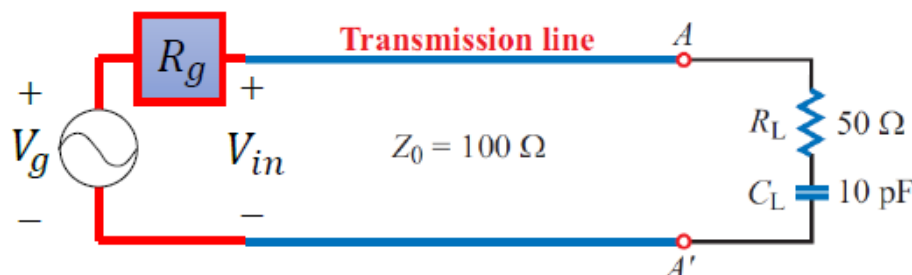
Power along a TL: $P(z, t) = V(z, t)I(z, t)$

Time Averaged Power:

$$\begin{aligned} P(z) &= \frac{1}{2} \operatorname{Re}\{V(z)I(z)^*\} \\ &= \frac{|V_0^+|^2}{2Z_0} \operatorname{Re}\{(e^{-j\beta z} + \Gamma_L e^{j\beta z})(e^{j\beta z} - \Gamma_L^* e^{-j\beta z})\} \\ &= \frac{|V_0^+|^2}{2Z_0} \operatorname{Re}\{1 - \Gamma_L^* e^{-j2\beta z} + \Gamma_L e^{j2\beta z} - |\Gamma_L|^2\} \\ &= \underbrace{\frac{|V_0^+|^2}{2Z_0}}_{\text{Incident Power}} - \underbrace{|\Gamma_L|^2 \frac{|V_0^+|^2}{2Z_0}}_{\text{Reflected Power}} \end{aligned}$$

Return Loss: $RL = -10 \log \left(\frac{P_r}{P_i} \right) = -20 \log |\Gamma_L|$

Example 2



A $100\ \Omega$ transmission line is connected to a load as shown. If $V_g = 5\text{ V}$; $R_g = 75\ \Omega$ $f = 100\text{ MHz}$, and the line is 0.3 wavelengths long, find: (a) Z_{in} ; (b) The voltage and current along the TL; (c) The power delivered to the load.

Solution

$$(a) \ Z_L = R_L + \frac{1}{j\omega C_L} = 50 - j[(2\pi \times 10^8)(10^{-11})]^{-1} = 50 - j159.2\ [\Omega]$$

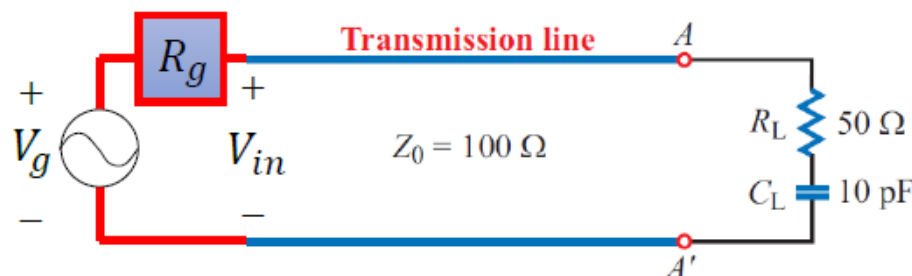
$$\beta l = \left(\frac{2\pi}{\lambda}\right)(0.3\lambda) = 0.6\pi \Rightarrow \tan \beta l = \tan 0.6\pi = -3.078$$

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} = (100) \frac{(50 - j159.2) + j(100)(-3.078)}{(100) + j(50 - j159.2)(-3.078)} = 29.79 + j108.0\ [\Omega]$$

$$(b) \ \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 - j159.2 - 100}{50 - j159.2 + 100} = 0.3730 - j0.6655;$$

$$\Gamma_G = \frac{Z_G - Z_0}{Z_G + Z_0} = \dots = -0.1429$$

Example 2 Cont'd



(b) Cont'd

$$\begin{aligned}
 I(z) &= \frac{V_g}{R_g + Z_0} \left(\frac{e^{-j\beta l} e^{-j\beta z}}{1 - \Gamma_L \Gamma_G e^{-j2\beta l}} \right) (1 - \Gamma_L e^{+j2\beta z}) \\
 &= \frac{5}{75 + 100} \left(\frac{e^{-j0.6\pi} e^{-j\beta z}}{1 - (-0.0533 + 0.0951)e^{-j1.2\pi}} \right) (1 - (0.3730 - j0.6655)e^{+j2\beta z}) \\
 &= (-0.01145 + j0.02561)e^{-j\beta z} (1 - (0.3730 - j0.6655)e^{+j2\beta z}) \text{ [A]}
 \end{aligned}$$

$$\begin{aligned}
 V(z) &= \frac{V_g Z_0}{R_g + Z_0} \left(\frac{e^{-j\beta l} e^{-j\beta z}}{1 - \Gamma_L \Gamma_G e^{-j2\beta l}} \right) (1 + \Gamma_L e^{+j2\beta z}) \\
 &= (-1.145 + j2.561)e^{-j\beta z} (1 + (0.3730 - j0.6655)e^{+j2\beta z}) \text{ [V]}
 \end{aligned}$$

(c)

$$P_L = \frac{|V_0^+|^2}{2Z_0} (1 - |\Gamma_L|^2) = \frac{|-1.145 + j2.561|^2}{2(100)} (1 - |0.373 - j0.6655|^2) = 16.43 \text{ [mW]}$$

Summary

- In this class we examined what happens when a TL is terminated by a load. We introduced the concepts of
 - Impedance at the load and along the TL
 - Reflection coefficient at the load and along the TL
 - Voltage Standing Wave Ratio
 - Multiple Reflections in a TL
 - Power along the TL
 - Return Loss