

MA1201 Calculus and Basic Linear Algebra II

Chapter 5

Complex Number

What is a Complex Number?

- To start with, we consider the following quadratic equations

$$x^2 + 2x + 2 = 0.$$

- One can use quadratic formula to find

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2}.$$

- One cannot use any real number to represent $\sqrt{-4}$ since the square of any real number is always non-negative and there is no real number u such that

$$u^2 = -4.$$

- One would like to extend the current number system (real number) so that it can include the square root of a negative number.
- To start with, we define i to be the number satisfying the equation

$$i^2 = -1 \quad \text{or} \quad i = \sqrt{-1}.$$

Definition (Complex Number)

We say a number z is complex number if and only if it can be expressed in the following form:

$$z = a + bi \quad (\text{or} \quad z = a + b\sqrt{-1})$$

where a and b are both real numbers.

We also denote the set of these numbers to be $\mathbb{C}$.

Different from the classical number system you have learnt before, a complex number is the sum of a real number a (real part) and multiple of $i = \sqrt{-1}$ (imaginary part).

$$z = a + bi$$

Real Part Imaginary Part

$a = \operatorname{Re} z$ $b = \operatorname{Im} z$

Equality of two complex numbers

We say two complex numbers are equal $a + bi = c + di$ if and only if

$$a = c \quad \text{and} \quad b = d.$$

In other word, two complex numbers are equal only when *both* real part and imaginary part are the same.

Some Examples of Complex Numbers

- $\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$ is a complex number.
- $4 - 3i$ is a complex number.
- $3i = 0 + 3i$ is also a complex number.

(Remark: We call this complex number to be *purely imaginary* since its real part is zero)

- Any real number (e.g.: $8 = 8 + 0i$) is also a complex number.

(Remark: We said this complex number to be *purely real*).

In other words, the complex number is an extension of real number $\mathbb{R}$ by introducing a new imaginary number $i = \sqrt{-1}$.

Basic Operation of Complex Number

For consistency, the operation of complex number must be same as that of real numbers since the complex numbers includes all real numbers.

Let $z_1 = a + bi$ and $z_2 = c + di$ be two complex numbers, we then have

Operation of Complex Number (Addition, subtraction and multiplication)

$$(1) \quad z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i.$$

$$(2) \quad z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i.$$

$$\begin{aligned} (3) \quad z_1 z_2 &= (a + bi)(c + di) = a(c + di) + bi(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci + bd \underbrace{(-1)}_{i^2 = -1} \\ &= (ac - bd) + (ad + bc)i. \end{aligned}$$

Question: How to do division $\frac{z_1}{z_2} = \frac{a+bi}{c+di}$?

Objective: Express the fraction into a complex number, i.e.

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di} = \frac{a + b\sqrt{-1}}{c + d\sqrt{-1}} = \dots = x + y\sqrt{-1} = \underbrace{x + yi}_{\text{Target!}}$$

We can eliminate the $\sqrt{-1}$ in the denominator by rationalization: multiplying both numerator and denominator by a common factor $c - di$.

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a + bi}{c + di} = \frac{a + bi}{c + di} \left(\frac{c - di}{c - di} \right) = \frac{ac + bci - adi - bdi^2}{c^2 + cdi - cdi - d^2i^2} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2} \right) i. \end{aligned}$$

Remark: There is no need for you to remember this formula. Just need to know how it is derived.

Example 1

Express the following in the form of $a + bi$, where a and b are real numbers.

$$(a) i^{2013}, \quad (b) (4 + i)(2 - 3i), \quad (c) \frac{3 + i}{1 - i}$$

☺Solution of (a)

$$i^{2013} = i^{2012}i = (i^2)^{1006}i = (-1)^{1006}i = i$$

☺Solution of (b)

$$(4 + i)(2 - 3i) = 8 - 10i - 3i^2 = 8 - 10i - 3(-1) = 11 - 10i.$$

☺Solution of (c)

$$\frac{3 + i}{1 - i} = \frac{3 + i}{1 - i} \left(\frac{1 + i}{1 + i} \right) = \frac{3 + 4i + i^2}{1 - i^2} = \frac{3 + 4i + (-1)}{1 - (-1)} = \frac{2 + 4i}{2} = 1 + 2i.$$

Example 2 (Power of complex number)

Compute $(1 + 2i)^4$ and $(1 - 3i)^{-3}$

☺Solution

Using Binomial Theorem, we get

$$\begin{aligned}(1 + 2i)^4 &= C_0^4(1)^4 + C_1^4(1)^3(2i) + C_2^4(1)^2(2i)^2 + C_3^4(1)(2i)^3 + C_4^4(2i)^4 \\ &= 1 + 8i + 24i^2 + 32i^3 + 16i^4 = 1 + 8i + 24(-1) + 32(-i) + 16(1) \\ &= -7 - 24i.\end{aligned}$$

$$\begin{aligned}(1 - 3i)^{-3} &= \frac{1}{(1 - 3i)^3} = \frac{1}{C_0^3(1)^3 + C_1^3(1)^2(-3i) + C_2^3(1)(-3i)^2 + C_3^3(-3i)^3} \\ &= \frac{1}{1 - 9i + 27i^2 - 27i^3} = \frac{1}{1 - 9i + 27(-1) - 27(-i)} = \frac{1}{-26 + 18i} \\ &= \frac{1}{-26 + 18i} \left(\frac{-26 - 18i}{-26 - 18i} \right) = \frac{-26 - 18i}{676 - 324i^2} = \frac{-26 - 18i}{676 + 324} = \frac{-13 - 9i}{500}.\end{aligned}$$

Example 3 (Square Root of a complex number)

Similar to the case for real number, we define the *square root* ($\sqrt{z}$) of a complex number $z = x + yi$ as a complex number $a + bi$ satisfying

$$(a + bi)^2 = z = x + yi.$$

Using this definition, find the value(s) of $\sqrt{3 + 4i}$.

😊Solution:

According to the definition, we have to find a complex number $a + bi$ satisfying

$$(a + bi)^2 = 3 + 4i.$$

To find the unknowns a and b , we expand the equation and get

$$a^2 + 2abi + b^2 \underbrace{i^2}_{-1} = 3 + 4i \Rightarrow (a^2 - b^2) + 2abi = 3 + 4i.$$

Comparing the real part and the imaginary part, we have

$$a^2 - b^2 = 3, \quad 2ab = 4.$$

From the second equation, we have $b = \frac{2}{a}$.

Substitute this into the first equation, we have

$$a^2 - \left(\frac{2}{a}\right)^2 = 3 \Rightarrow a^4 - 3a^2 - 4 = 0$$

$$\Rightarrow (a^2 - 4)(a^2 + 1) = 0$$

$$\Rightarrow (a - 2)(a + 2)(a^2 + 1) = 0$$

$$\Rightarrow a - 2 = 0 \text{ or } a + 2 = 0 \text{ or } a^2 + 1 = 0 \text{ (rejected as } a \text{ is real)}$$

$$\Rightarrow a = 2 \text{ (} b = 1 \text{) or } a = -2 \text{ (} b = -1 \text{).}$$

Therefore, we conclude that

$$\sqrt{3 + 4i} = a + bi = 2 + i \text{ or } -2 - i.$$

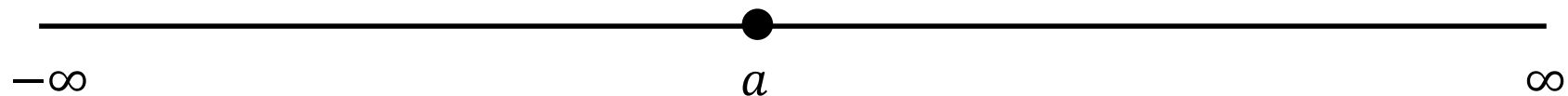
Similar to the case of real number, the square root of a complex number has more than one value.

Question: How to compute $(a + bi)^n$ or $\sqrt[n]{a + bi}$ in general?

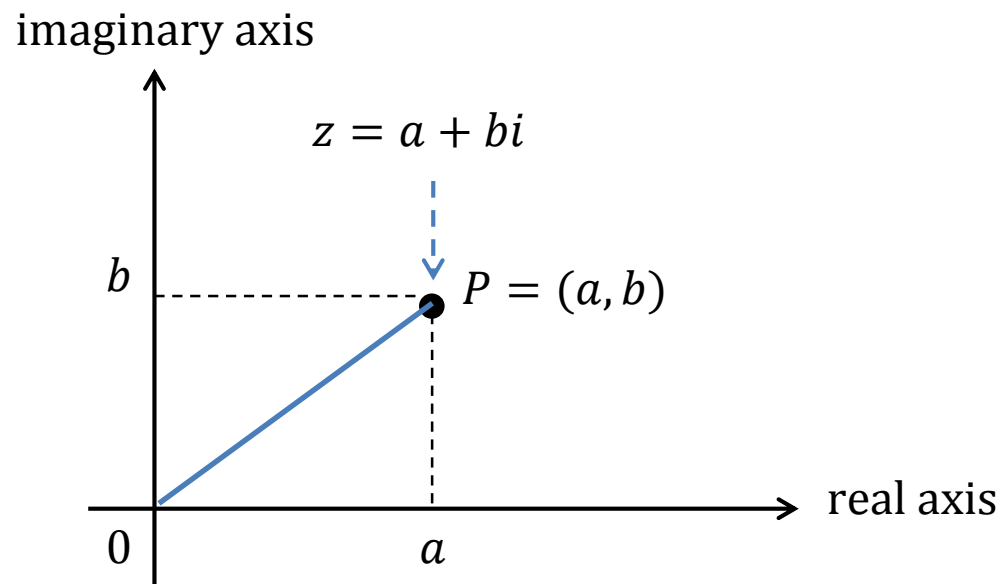
- Although we can use the previous method to do the computation, it is not efficient when n is large (say $n = 10, 100$ etc.). One has to seek other alternatives in order to reduce the computation cost.
- In order to develop a better method, one has to represent a complex number $(a + bi)$ in other ways: **polar form** $z = r(\cos \theta + i \sin \theta)$ and Euler (or exponential) form.

Geometric Representation of a Complex Number

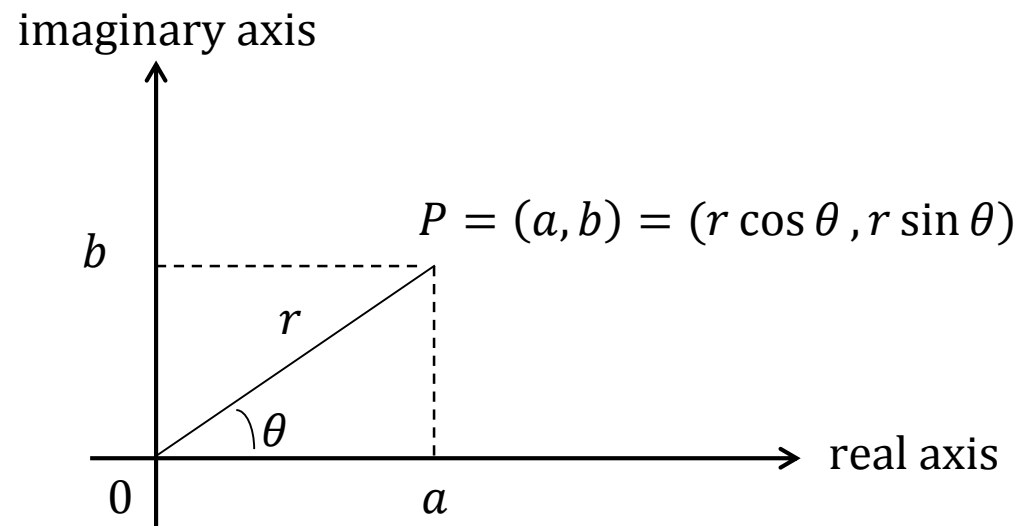
- Recall that every real number can be represented by a point of a straight line (real line from $-\infty$ to $+\infty$).



- Using similar logic, one can represent a complex number $a + bi$ by a point P (with coordinate (x, y) in a 2-D plane (also called an **Argand diagram**, a complex plane):



In MA1200, we knew that every point in 2-D can be expressed using **polar coordinates** (r, θ) .



- $r = \sqrt{a^2 + b^2} > 0$ represents the distance between O and the point (a, b) . It is called **modulus** of $z = a + bi$ and is denoted by $|z|$
- θ (in radian) represents the angle between the line OP and the positive real axis. It is called **argument** of z and is denoted by $\arg z$. Usually, θ is taken to be within $-\pi < \theta \leq \pi$. Such θ is called **principal value** of $\arg z$.

Polar form of a complex number

Recall that the parameters r , θ , a and b are related by

$$a = r \cos \theta \quad \text{and} \quad b = r \sin \theta.$$

Then the complex number $z = a + bi$ can be rewritten as

$$z = r \cos \theta + (r \sin \theta)i = r(\cos \theta + i \sin \theta) \dots \dots (*)$$

This is called polar form of the complex number.

Remark:

One has to be careful that the complex number is in Polar form only when it is expressed in the form of (*).

Example of polar form

1. $z_0 = 2 \left(\cos \frac{\theta}{3} + i \sin \frac{\theta}{3} \right)$ is in polar form with $|z_0| = 2$ and $\arg z_0 = \frac{\theta}{3}$.

Example of non-polar form

2. $z_1 = \cos \frac{\theta}{3} - i \sin \frac{\theta}{3}$ is NOT in polar form since there is a “–” between $\cos$? And $\sin$?
3. $z_2 = \cos 2\theta + i \sin \theta$ is NOT in polar form since the angles inside the cosine and sine function are not the same
4. $z_3 = -2 \left(\cos \frac{\theta}{3} + i \sin \frac{\theta}{3} \right)$ is NOT in polar form since -2 is negative.
5. $z_4 = \sin \theta + i \cos \theta$ is NOT in polar form since the real part is not of the form $\cos$??.
6. $z_5 = i(\cos \theta + i \sin \theta)$ is NOT in polar form since $r = i$ (1^{st} term) is not a real number.

How to express a complex number $a + bi$ in polar form?

Step 1: Given $z = a + bi$, one can find r (modulus of z) by

$$r = |z| = \sqrt{a^2 + b^2}.$$

Step 2: Next, we need to find the argument of z ($\arg z$). The whole process is divided into two main steps:

- Draw the diagram (xy -plane) and locate the position of $z = a + bi$. Identify the appropriate value of θ ($-\pi < \theta \leq \pi$).
- Use this diagram to obtain the value of θ using geometric method.

Example 4

Express the complex number $z_1 = 1 + \sqrt{3}i$ and $z_2 = 2\sqrt{3} - 2i$ into polar form.

☺Solution

Note that $r_1 = |z_1| = \sqrt{1^2 + (\sqrt{3})^2} = 2$

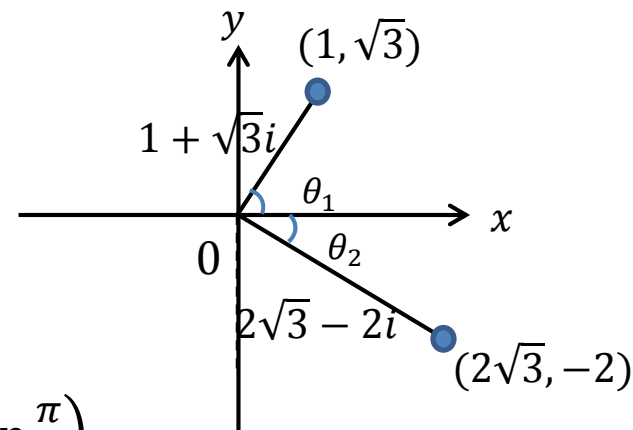
Then $\theta_1 = \tan^{-1} \frac{\sqrt{3}}{1} = \frac{\pi}{3}$.

Thus, $z_1 = r_1(\cos \theta_1 + i \sin \theta_1) = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$.

Note that $r_2 = |z_2| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = 4$

Then $\theta_2 = -\tan^{-1} \frac{2}{2\sqrt{3}} = -\frac{\pi}{6}$.

Thus, $z_2 = r_2(\cos \theta_2 + i \sin \theta_2) = 4 \left(\cos \left(-\frac{\pi}{6} \right) + i \sin \left(-\frac{\pi}{6} \right) \right)$.



Example 5

Express the complex number $z = -1 - i$ into polar form.

☺Solution:

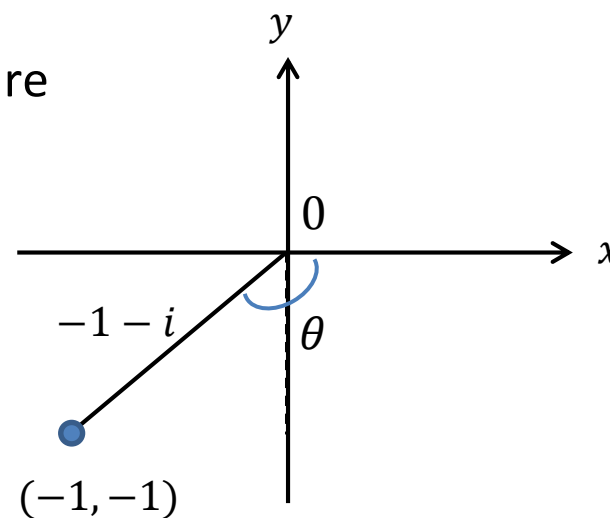
Note that $r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$.

To find the argument $\arg z$, we consider the figure on the right, it is easy to observe that

$$\begin{aligned}\theta &= -\left(\pi - \tan^{-1}\left(\frac{1}{1}\right)\right) \\ &= -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}\end{aligned}$$

(Note: $\pi(\text{rad}) = 180^\circ$)

Hence, $z = \sqrt{2} \left[\cos\left(-\frac{3\pi}{4}\right) + i \sin\left(-\frac{3\pi}{4}\right) \right]$.



Example 6

Express the complex number $z = -1 + \sqrt{3}i$ into polar form.

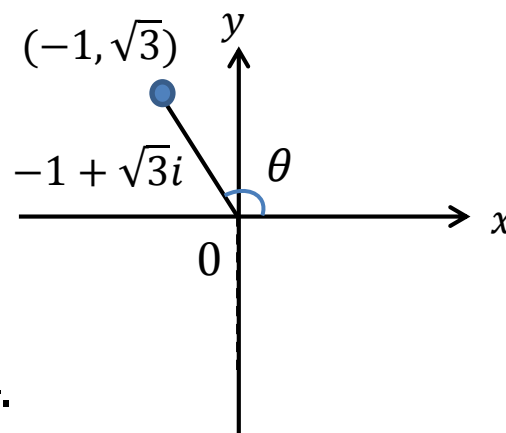
☺Solution:

Using similar method, we have

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

and

$$\theta = \pi - \tan^{-1} \frac{\sqrt{3}}{1} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$



Therefore the polar form of this complex is given by

$$z = -1 + \sqrt{3}i = 2 \left(\cos \left(\frac{2\pi}{3} \right) + i \sin \left(\frac{2\pi}{3} \right) \right).$$

Example 7

Express the complex number $z = \sin \theta + i \cos \theta$ into polar form where $0 < \theta < \pi$.

What are $|z|$ and $\arg z$?

(Be careful: It is not in polar form yet!!!)

☺Solution:

Note that $\sin \theta > 0$ for $0 < \theta < \pi$, the point lies in either 1st/4th quadrant.

$$r = \sqrt{\sin^2 \theta + \cos^2 \theta} = \sqrt{1} = 1 \quad \text{and}$$

$$\phi = \tan^{-1} \left(\frac{\cos \theta}{\sin \theta} \right) = \tan^{-1} \left(\frac{1}{\tan \theta} \right) = \tan^{-1} \left(\tan \left(\frac{\pi}{2} - \theta \right) \right) = \frac{\pi}{2} - \theta.$$

Therefore the polar form of z is

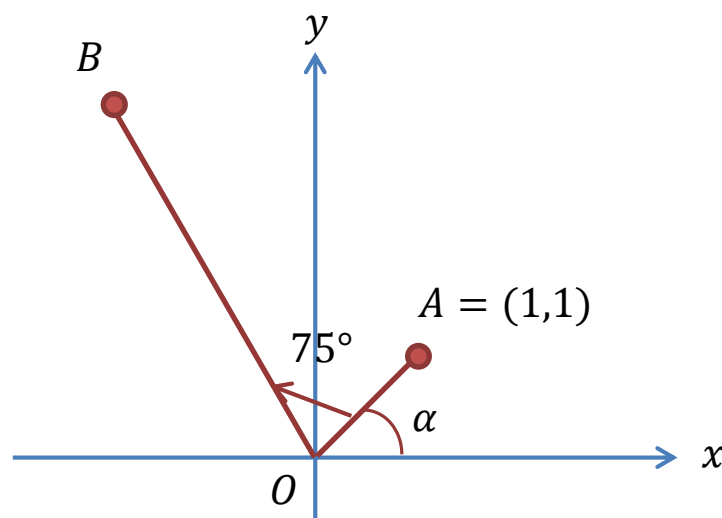
$$\sin \theta + i \cos \theta = 1 \left(\cos \left(\frac{\pi}{2} - \theta \right) + i \sin \left(\frac{\pi}{2} - \theta \right) \right).$$

Here $|z| = r = 1$ and $\arg z = \frac{\pi}{2} - \theta$.

Example 8 (A bit harder)

Let $z_A = 1 + i$ be a complex number and A be the point in argand diagram representing the complex number z_A . Suppose the line OA is being rotated by 75° along anti-clockwise direction and is then stretched in length by 3 times (fixing the endpoint origin O), we let OB (where O, B are the endpoints) be the resulting line.

- (a) Find the corresponding complex number z_B representing the point B .
- (b) Find the coordinates of B .



☺Solution

- (a) To find the complex number z_B , we need to find the *modulus* and *argument* of z_B .

Using the above figure, we get

- Modulus of z_B = Length of OB

$$= 3 \times \text{Length of } OA = 3 \times \sqrt{1^2 + 1^2} = 3\sqrt{2}.$$
- Argument of $z_B = \alpha + 75^\circ = \tan^{-1} \frac{1}{1} + 75^\circ$

$$= 45^\circ + 75^\circ = 120^\circ = \frac{2\pi}{3}.$$

Thus the complex number z_B is given by

$$z_B = 3\sqrt{2} \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) = -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{6}}{2}i.$$

- (b) Since B is a point representing the complex number z_B , hence the coordinates of B is $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{6}}{2}\right)$.

Multiplication and division of complex numbers in polar form

In fact, the polar form can greatly reduce the computational cost in doing multiplication and division of complex numbers.

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ be two complex numbers expressed in polar form. The product of z_1 and z_2 is given by

$$\begin{aligned}
 z_1 z_2 &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\
 &= r_1 r_2 (\cos \theta_1 \cos \theta_2 + i \sin \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i^2 \sin \theta_1 \sin \theta_2) \\
 &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\
 &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)).
 \end{aligned}$$

The last inequality follows from compound angle formula:

$$\begin{cases} \cos(A + B) = \cos A \cos B - \sin A \sin B \\ \sin(A + B) = \sin A \cos B + \cos A \sin B \end{cases}$$

On the other hand, we consider the quotient $\frac{z_1}{z_2}$ and get

$$\begin{aligned}
 \frac{z_1}{z_2} &= \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2} \left(\frac{\cos \theta_1 + i \sin \theta_1}{\cos \theta_2 + i \sin \theta_2} \right) \\
 &= \frac{r_1}{r_2} \left(\frac{\cos \theta_1 + i \sin \theta_1}{\cos \theta_2 + i \sin \theta_2} \right) \left(\frac{\cos \theta_2 - i \sin \theta_2}{\cos \theta_2 - i \sin \theta_2} \right) \\
 &= \frac{r_1}{r_2} \left(\frac{\cos \theta_1 \cos \theta_2 + i \sin \theta_1 \cos \theta_2 - i \cos \theta_1 \sin \theta_2 - i^2 \sin \theta_1 \sin \theta_2}{\underbrace{\cos^2 \theta_2 + \sin^2 \theta_2}_{=1}} \right) \\
 &= \frac{r_1}{r_2} [(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)] \\
 &= \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)).
 \end{aligned}$$

Again, the last inequality is due to the compound angle formula.

We summarize the result in the following theorem:

Theorem: (Multiplication and Division of complex numbers in polar forms)

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$, then

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)),$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)).$$

Corollary: (Properties of modulus and argument of complex numbers)

$$|z_1 z_2| = r_1 r_2 = |z_1| |z_2|, \quad \left| \frac{z_1}{z_2} \right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|},$$

$$\arg z_1 z_2 = \theta_1 + \theta_2 = \arg z_1 + \arg z_2,$$

$$\arg \frac{z_1}{z_2} = \theta_1 - \theta_2 = \arg z_1 - \arg z_2.$$

Example 9

Compute $(1 + i)^3$ and $(1 + i)^{-2}$

☺Solution:

It is not good for you to do computation by expanding. It may be better for you to transform the complex number $1 + i$ in polar form first.

Note that $r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \tan^{-1} \frac{1}{1} = \tan^{-1} 1 = \frac{\pi}{4}$, so

$$1 + i = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right).$$

Then

$$\begin{aligned} (1 + i)^3 &= \left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^3 \\ &= \left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right] \left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right] \left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right] \\ &= 2\sqrt{2} \left(\cos \frac{2\pi}{4} + i \sin \frac{2\pi}{4} \right) \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = 2\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \end{aligned}$$

$$= 2\sqrt{2} \left(-\frac{\sqrt{2}}{2} + i \left(\frac{\sqrt{2}}{2} \right) \right) = -2 + 2i.$$

$$\begin{aligned} (1+i)^{-2} &= \frac{1}{(1+i)^2} = \frac{1}{\left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right] \left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]} \\ &= \frac{1}{2 \left(\cos \frac{2\pi}{4} + i \sin \frac{2\pi}{4} \right)} = \frac{\cos 0 + i \sin 0}{2 \left(\cos \frac{2\pi}{4} + i \sin \frac{2\pi}{4} \right)} \\ &= \frac{1}{2} \left[\cos \left(0 - \frac{\pi}{2} \right) + i \sin \left(0 - \frac{\pi}{2} \right) \right] = \frac{1}{2} \left(\cos \left(-\frac{\pi}{2} \right) + i \sin \left(-\frac{\pi}{2} \right) \right) = \frac{1}{2} i(-1) \\ &= -\frac{1}{2} i. \end{aligned}$$

Example 10

Compute

$$\frac{1 + \sqrt{3}i}{(3 - \sqrt{3}i)^2}.$$

☺Solution

Note that (left as exercise)

$$1 + \sqrt{3}i = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right), \quad 3 - \sqrt{3}i = 2\sqrt{3} \left(\cos \left(-\frac{\pi}{6} \right) + i \sin \left(-\frac{\pi}{6} \right) \right).$$

Then

$$\begin{aligned} (3 - \sqrt{3}i)^2 &= \left[2\sqrt{3} \left(\cos \left(-\frac{\pi}{6} \right) + i \sin \left(-\frac{\pi}{6} \right) \right) \right] \left[2\sqrt{3} \left(\cos \left(-\frac{\pi}{6} \right) + i \sin \left(-\frac{\pi}{6} \right) \right) \right] \\ &= 12 \left[\cos \left(-\frac{\pi}{3} \right) + i \sin \left(-\frac{\pi}{3} \right) \right]. \end{aligned}$$

$$\begin{aligned}& \frac{1 + \sqrt{3}i}{(3 - \sqrt{3}i)^2} \\&= \frac{2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)}{12 \left[\cos \left(-\frac{\pi}{3} \right) + i \sin \left(-\frac{\pi}{3} \right) \right]} \\&= \frac{1}{6} \left[\cos \left(\frac{\pi}{3} - \left(-\frac{\pi}{3} \right) \right) + i \sin \left(\frac{\pi}{3} - \left(-\frac{\pi}{3} \right) \right) \right] \\&= \frac{1}{6} \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \\&= \frac{1}{6} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = -\frac{1}{12} + \frac{\sqrt{3}}{12}i.\end{aligned}$$

Exponential Form (or Euler Form)

Euler (1707-1783) established an important formula which provides a deep relationship between the trigonometric functions and the complex exponential function. The formula states that

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

Given the polar form of a complex number $z = r(\cos \theta + i \sin \theta)$, one can use Euler formula to express the complex number into following form

$$z = re^{i\theta}.$$

This is called **Euler form** (or Exponential form) of a complex number.

Rough Proof of Euler relation ($e^{i\theta} = \cos \theta + i \sin \theta$)

Recall that the Taylor's series of e^x about $x = 0$ is given by

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Substituting $x = i\theta$, we have

$$e^{i\theta} = 1 + i\theta + i^2 \frac{\theta^2}{2!} + i^3 \frac{\theta^3}{3!} + i^4 \frac{\theta^4}{4!} + i^5 \frac{\theta^5}{5!} + i^6 \frac{\theta^6}{6!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i \frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \right)$$

$$= \cos \theta + i \sin \theta.$$

Using the theorem in p.26, one can obtain the following theorem

Theorem: (Multiplication and Division of complex numbers in Euler form)

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}, \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

Take Example 10 as an example, we have

$$1 + \sqrt{3}i = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) = 2e^{i\frac{\pi}{3}},$$

$$3 - \sqrt{3}i = 2\sqrt{3} \left(\cos \left(-\frac{\pi}{6} \right) + i \sin \left(-\frac{\pi}{6} \right) \right) = 2\sqrt{3}e^{-i\frac{\pi}{6}}.$$

$$\begin{aligned} \Rightarrow \frac{1 + \sqrt{3}i}{(3 - \sqrt{3}i)^2} &= \frac{2e^{i\frac{\pi}{3}}}{\left(2\sqrt{3}e^{-i\frac{\pi}{6}}\right)^2} = \frac{1}{6} \frac{e^{i\frac{\pi}{3}}}{e^{-i\frac{\pi}{3}}} = \frac{1}{6} e^{i\frac{\pi}{3} - (-i\frac{\pi}{3})} = \frac{1}{6} e^{i\frac{2\pi}{3}} \\ &= \frac{1}{6} \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right). \end{aligned}$$

Example of exponential form of some complex numbers

1. $-1 = -1 + 0i = \cos \pi + i \sin \pi = e^{i\pi}.$

2. $i = 0 + i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = e^{i\frac{\pi}{2}}$

3. $-i = 0 - i = \cos \left(-\frac{\pi}{2}\right) + i \sin \left(-\frac{\pi}{2}\right) = e^{-i\frac{\pi}{2}}.$

4. (From Example 4)

$$1 + \sqrt{3}i = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) = 2e^{i\frac{\pi}{3}}.$$

Example of non-exponential form

1. $z = ie^{i\theta}$ is not in exponential form! (There is a complex number i appeared in front of $e^{i\theta}$).

However, one can transform this number into exponential form as follows:

$$ie^{i\theta} = e^{i\left(\frac{\pi}{2}\right)} e^{i\theta} = e^{i\left(\theta + \frac{\pi}{2}\right)}.$$

2. $z = -e^{i\theta}$ is not in exponential form! (The leading coefficient is a negative number -1). Using similar technique as in 1, we can transform the number into exponential form:

$$-e^{i\theta} = (-1)e^{i\theta} \underset{-1=e^{i\pi}}{=} e^{i\pi} e^{i\theta} = e^{i(\theta+\pi)}.$$

Example 11

It is clear that the complex number $z = e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{2}}$ is not in Euler's form since there are two exponential terms. Express z in Euler's form.

IDEA: To express the number in exponential form, one has to combine those two exponential terms.

☺Solution:

$$\begin{aligned}
 z &= e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{2}} = e^{i\left(\frac{\pi}{3} + \left(-\frac{\pi}{2}\right)\right)} \left[e^{i\frac{\pi}{3} - i\left(\frac{\pi}{3} + \left(-\frac{\pi}{2}\right)\right)} + e^{-i\frac{\pi}{2} - i\left(\frac{\pi}{3} + \left(-\frac{\pi}{2}\right)\right)} \right] \\
 &= e^{-i\frac{\pi}{12}} \left[e^{i\frac{5\pi}{12}} + e^{-i\frac{5\pi}{12}} \right] = e^{-i\frac{\pi}{12}} \left[2 \cos \frac{5\pi}{12} \right] = \underbrace{2 \cos \frac{5\pi}{12}}_{r>0} \underbrace{e^{i\left(-\frac{\pi}{12}\right)}}_{e^{i\theta}}.
 \end{aligned}$$

The second last equality follows from $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

Exponential Form v.s. Polar Form

- Exponential form is useful in handling the multiplication and division of complex numbers which involves the trigonometric functions, e.g.

$$z_1 = \sin \theta \pm i \cos \theta, \quad z_2 = 1 + \sin \theta + i \cos \theta.$$

- It is hard to change the above complex numbers into polar form, since their arguments $\theta = \arg z_i$ is hard to obtain.
- It is often easier to transform these numbers into exponential form instead of polar form.

How to change these numbers into Euler form?

- Let's recall the Euler relation:

$$e^{i\theta} = \cos \theta + i \sin \theta \dots \dots (1)$$

- On the other hand, we replace θ in (1) by $-\theta$, we obtain another equation:

$$e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) = \cos \theta - i \sin \theta \dots \dots (2)$$

- (1) + (2) implies

$$e^{i\theta} + e^{-i\theta} = 2 \cos \theta \Rightarrow \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

(1) – (2) implies

$$e^{i\theta} - e^{-i\theta} = 2i \sin \theta \Rightarrow \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Example 12

Transform the following numbers into exponential form (where $0 < \theta < \pi$). Find the modulus and the principle value of the argument.

(a) $-\cos \theta + i \sin \theta$

(b) $1 - \cos \theta + i \sin \theta$

☺Solution

$$\begin{aligned}
 \text{(a)} \quad -\cos \theta + i \sin \theta &= -\frac{e^{i\theta} + e^{-i\theta}}{2} + i \frac{e^{i\theta} - e^{-i\theta}}{2i} \\
 &= -\frac{e^{i\theta} + e^{-i\theta}}{2} + \frac{e^{i\theta} - e^{-i\theta}}{2} \\
 &= -\frac{2e^{-i\theta}}{2} = -e^{-i\theta} \\
 &= e^{i\pi} e^{-i\theta} = e^{i(\pi - \theta)}.
 \end{aligned}$$

Modulus = 1 and Argument = $\pi - \theta$.

$$\begin{aligned}
\text{(b)} \quad 1 - \cos \theta + i \sin \theta &= 1 - \frac{e^{i\theta} + e^{-i\theta}}{2} + i \frac{e^{i\theta} - e^{-i\theta}}{2i} \\
&= 1 - \frac{e^{i\theta} + e^{-i\theta}}{2} + \frac{e^{i\theta} - e^{-i\theta}}{2} \\
&= 1 - e^{-i\theta} \\
&= e^{i0} - e^{-i\theta} \\
&= e^{-i\frac{\theta}{2}} \left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}} \right) \\
&= e^{-i\frac{\theta}{2}} \left(2i \sin \frac{\theta}{2} \right) \\
&= 2 \sin \frac{\theta}{2} i e^{-i\frac{\theta}{2}} \\
&= 2 \sin \frac{\theta}{2} e^{i\frac{\pi}{2}} e^{-i\frac{\theta}{2}} = 2 \sin \frac{\theta}{2} e^{i\left(\frac{\pi}{2} - \frac{\theta}{2}\right)}. \\
\text{Modulus} &= 2 \sin \frac{\theta}{2}; \quad \text{Argument} = \frac{\pi}{2} - \frac{\theta}{2}.
\end{aligned}$$

Example 13

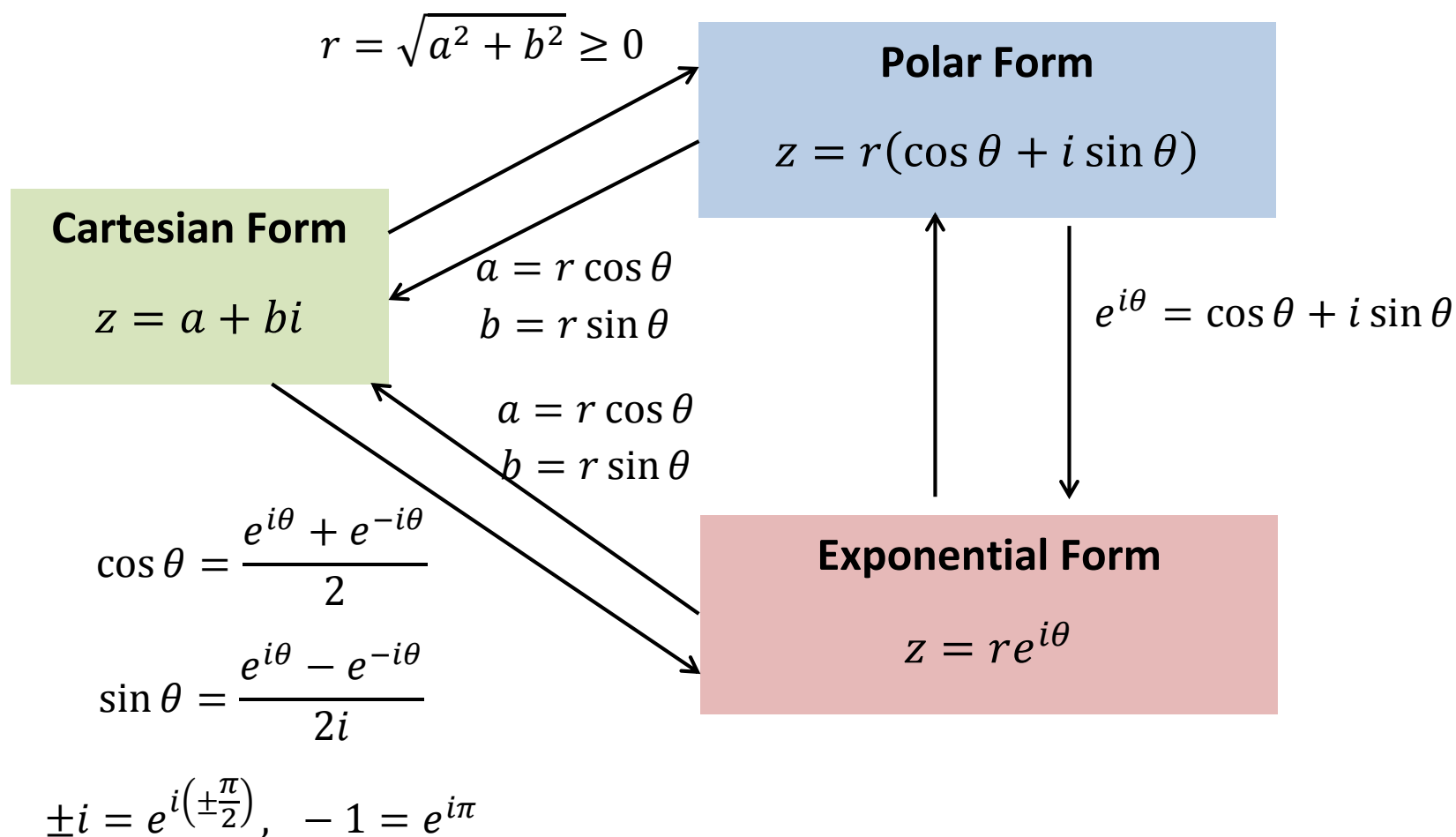
Compute $\frac{1 - \cos \theta + i \sin \theta}{-\cos \theta + i \sin \theta}$.

☺Solution:

Using the result in Example 12, we have

$$\begin{aligned}\frac{1 - \cos \theta + i \sin \theta}{-\cos \theta + i \sin \theta} &= \frac{2 \sin \frac{\theta}{2} e^{i\left(\frac{\pi}{2} - \frac{\theta}{2}\right)}}{e^{i(\pi - \theta)}} \\ &= 2 \sin \frac{\theta}{2} e^{i\left[\left(\frac{\pi}{2} - \frac{\theta}{2}\right) - (\pi - \theta)\right]} \\ &= 2 \sin \frac{\theta}{2} e^{i\left(\frac{\theta}{2} - \frac{\pi}{2}\right)}.\end{aligned}$$

Summary: (Relations between Cartesian form, Polar form and Exponential form (Euler form))



Power and n^{th} root of complex numbers

Question: How to compute $(a + bi)^n$ and $\sqrt[n]{a + bi}$?

- It is not efficient to do the computation if the complex number is expressed in the form of $a + bi$.
- The computation will be easier if we express the complex number into polar form: $r(\cos \theta + i \sin \theta)$.

Power of complex number

Let $z = r(\cos \theta + i \sin \theta)$, then for any positive integer n ($n = 1, 2, 3 \dots$), we have

$$\begin{aligned} [r(\cos \theta + i \sin \theta)]^n &= r^n(\cos \theta + i \sin \theta)^n \\ &= r^n[\cos(\theta + \theta + \dots + \theta) + i \sin(\theta + \theta + \dots + \theta)] \\ &= r^n(\cos n\theta + i \sin n\theta) \end{aligned}$$

This theorem is known as **DeMoivre's theorem**.

DeMoivre's theorem (Non-negative integer index power)

If n is nonnegative integer, then

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

or in more general,

$$[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta).$$

Remark:

This theorem only works when the complex number is expressed in POLAR FORM!

We **cannot** adopt the DeMoivre's theorem directly to the following:

$$(\sin \theta + i \cos \theta)^n \neq (\sin n\theta + i \cos n\theta),$$

$$(-\cos \theta + i \sin \theta)^n \neq -\cos n\theta + i \sin n\theta.$$

Example 14

Compute $\left(\frac{1 - \sqrt{3}i}{1 + i}\right)^{12}$

☺Solution:

Step 1: Transform the complex numbers into polar form.

One can find that

$$1 - \sqrt{3}i = 2 \left[\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right], \quad 1 + i = \sqrt{2} \left(\cos\left(\frac{\pi}{4}\right) + i \sin\frac{\pi}{4} \right).$$

$$\Rightarrow \frac{1 - \sqrt{3}i}{1 + i} = \frac{2 \left[\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right]}{\sqrt{2} \left(\cos\left(\frac{\pi}{4}\right) + i \sin\frac{\pi}{4} \right)}$$

$$= \sqrt{2} \left[\cos\left(-\frac{\pi}{3} - \frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{3} - \frac{\pi}{4}\right) \right] = \sqrt{2} \left[\cos\left(-\frac{7\pi}{12}\right) + i \sin\left(-\frac{7\pi}{12}\right) \right].$$

Step 2: Use DeMoivre's theorem

$$\begin{aligned}\left(\frac{1 - \sqrt{3}i}{1 + i}\right)^{12} &= \left(\sqrt{2} \left[\cos\left(-\frac{7\pi}{12}\right) + i \sin\left(-\frac{7\pi}{12}\right)\right]\right)^{12} \\&= (\sqrt{2})^{12} \left[\cos\left(12\left(-\frac{7\pi}{12}\right)\right) + i \sin\left(12\left(-\frac{7\pi}{12}\right)\right)\right] \\&= 64[\cos(-7\pi) + i \sin(-7\pi)] \\&= 64(-1 + 0i) \\&= -64.\end{aligned}$$

DeMoivre's Theorem for negative integer index m

We write $m = -n$ (where n is positive integer), then

$$\begin{aligned}
 (\cos \theta + i \sin \theta)^m &= (\cos \theta + i \sin \theta)^{-n} = \left(\frac{1}{\cos \theta + i \sin \theta} \right)^n \\
 &= \left(\frac{\cos 0 + i \sin 0}{\cos \theta + i \sin \theta} \right)^n \\
 &= (\cos(-\theta) + i \sin(-\theta))^n \\
 &= \cos(-n\theta) + i \sin(-n\theta) \\
 &= \cos m\theta + i \sin m\theta.
 \end{aligned}$$

So we have the similar result

$$(\cos \theta + i \sin \theta)^m = \cos m\theta + i \sin m\theta.$$

where m is a negative integer. Hence, the DeMoivre's theorem can be applied for any integer power n !

Example 15

If n is an integer and $z = \cos \theta + i \sin \theta$, show that

$$z^n + \frac{1}{z^n} = 2 \cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i \sin n\theta,$$

☺Solution:

Note that

$$z^n = (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta,$$

$$\frac{1}{z^n} = z^{-n} = (\cos \theta + i \sin \theta)^{-n} = \cos(-n\theta) + i \sin(-n\theta) = \cos n\theta - i \sin n\theta.$$

So by simple computation, we get

$$z^n + \frac{1}{z^n} = 2 \cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i \sin n\theta.$$

n^{th} root of a complex number $\sqrt[n]{z} = z^{\frac{1}{n}}$

With the help of polar form, one can find a simple way to compute the n^{th} root of the complex number $z^{\frac{1}{n}}$.

As an example, we would like to find the value of $\sqrt[3]{2 - 2i}$. According to the definition of cubic root, it is equivalent to find a complex number z satisfying

$$z^3 = 2 - 2i.$$

To solve for z , first express the complex numbers z and $2 - 2i$ into polar form.

Let $z = \underbrace{\rho}_{\text{modulus}} \left(\cos \underbrace{\phi}_{\text{argument}} + i \sin \phi \right)$ and $2 - 2i = \sqrt{8} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right)$. Using DeMoivre's theorem, we have

$$[\rho(\cos \phi + i \sin \phi)]^3 = \sqrt{8} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right)$$

$$\Rightarrow \underbrace{\rho^3}_{\text{modulus of } z^3} \left(\cos \underbrace{3\phi}_{\text{argument of } z^3} + i \sin 3\phi \right) = \underbrace{\sqrt{8}}_{\text{modulus of } 2-2i} \left[\cos \underbrace{\left(-\frac{\pi}{4}\right)}_{\text{argument of } 2-2i} + i \sin \left(-\frac{\pi}{4}\right) \right]$$

Note that if the two complex numbers z^3 and $2 - 2i$ are equal, then we have

$$\underbrace{\rho^3 = \sqrt{8}}_{\text{modulus}}, \quad \underbrace{3\phi = 2k\pi + \left(-\frac{\pi}{4}\right)}_{\text{argument}}, \quad k = 0, 1, 2, \dots$$

$$\Rightarrow \rho = (\sqrt{8})^{\frac{1}{3}} = \sqrt{2}, \quad \phi = \frac{2k\pi}{3} - \frac{\pi}{12}, \quad k = 0, 1, 2, \dots$$

Thus $z = \sqrt[3]{2 - 2i}$ is then given by

$$z = \sqrt{2} \left[\cos \left(\frac{2k\pi}{3} - \frac{\pi}{12} \right) + i \sin \left(\frac{2k\pi}{3} - \frac{\pi}{12} \right) \right], \quad k = 0, 1, 2, \dots$$

Putting $k = 0, 1, 2, \dots$, we have

$$z_0 = \sqrt{2} \left[\cos \left(-\frac{\pi}{12} \right) + i \sin \left(-\frac{\pi}{12} \right) \right] = \dots$$

$$z_1 = \sqrt{2} \left[\cos \left(\frac{2\pi}{3} - \frac{\pi}{12} \right) + i \sin \left(\frac{2\pi}{3} - \frac{\pi}{12} \right) \right] = \dots$$

$$z_2 = \sqrt{2} \left[\cos \left(\frac{4\pi}{3} - \frac{\pi}{12} \right) + i \sin \left(\frac{4\pi}{3} - \frac{\pi}{12} \right) \right] = \dots$$

$$z_3 = \sqrt{2} \left[\cos \left(\frac{6\pi}{3} - \frac{\pi}{12} \right) + i \sin \left(\frac{6\pi}{3} - \frac{\pi}{12} \right) \right] = \sqrt{2} \left[\cos \left(-\frac{\pi}{12} \right) + i \sin \left(-\frac{\pi}{12} \right) \right]$$

$$z_4 = \sqrt{2} \left[\cos \left(\frac{8\pi}{3} - \frac{\pi}{12} \right) + i \sin \left(\frac{8\pi}{3} - \frac{\pi}{12} \right) \right] = \sqrt{2} \left[\cos \left(\frac{2\pi}{3} - \frac{\pi}{12} \right) + i \sin \left(\frac{2\pi}{3} - \frac{\pi}{12} \right) \right]$$

... (repeated)

Thus, we conclude that

$$z = \sqrt{2} \left[\cos \left(\frac{2k\pi}{3} - \frac{\pi}{12} \right) + i \sin \left(\frac{2k\pi}{3} - \frac{\pi}{12} \right) \right], \quad k = 0, 1, 2.$$

General Case

Suppose we would like to find the value of $\sqrt[n]{a + bi}$ (where n is a positive integer), one can let $z = \sqrt[n]{a + bi}$ and find the values of x and y by solving

$$z^n = a + bi \dots \dots (*)$$

Following the idea of the above numerical example, we first express the complex numbers z and $a + bi$ into polar forms: Let $z = \rho(\cos \phi + i \sin \phi)$ and $a + bi = r(\cos \theta + i \sin \theta)$.

From equation (*), we then have

$$[\rho(\cos \phi + i \sin \phi)]^n = r(\cos \theta + i \sin \theta),$$

$$\Rightarrow \underbrace{\rho^n}_{\text{modulus of } z^n} \left(\cos \underbrace{n\phi}_{\text{argument of } z^n} + i \sin n\phi \right) = \underbrace{r}_{\text{modulus of } a+bi} \left(\cos \underbrace{\theta}_{\text{argument of } a+bi} + i \sin \theta \right)$$

Since two complex numbers are equal, then we must have

$$\underbrace{\rho^n = r}_{\text{modulus}} \quad \text{and} \quad \underbrace{n\phi = 2k\pi + \theta}_{\text{argument}}, \quad k = 0, 1, 2, 3, \dots$$

$$\Rightarrow \rho = r^{\frac{1}{n}}, \quad \phi = \frac{2k\pi + \theta}{n}, k = 0, 1, 2, \dots$$

Since the value of ϕ will be repeated after $k = n, n + 1, \dots$, thus we just need to concentrate on the values of ϕ for $k = 0, 1, 2, \dots, n - 1$. Thus we have the following result:

Theorem: (n^{th} root of a complex number)

If $z = r(\cos \theta + i \sin \theta)$ is the complex number expressed in polar form, then

$$\sqrt[n]{r(\cos \theta + i \sin \theta)} = \underbrace{r^{\frac{1}{n}}}_{\rho} \left(\cos \underbrace{\frac{2k\pi + \theta}{n}}_{\phi} + i \sin \frac{2k\pi + \theta}{n} \right),$$

We see that the n^{th} root of a complex number has n possible values.

Example 16

Compute $\sqrt[4]{1-i}$. Express your answer in polar form.

☺Solution:

Step 1: Write $1 - i$ in polar

One can find that

$$1 - i = \sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right).$$

Step 2: Do computation

$$\begin{aligned} \sqrt[4]{1-i} &= (1-i)^{\frac{1}{4}} = \left[\sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right) \right]^{\frac{1}{4}} \\ &= 2^{\frac{1}{8}} \left[\cos \frac{2k\pi - \frac{\pi}{4}}{4} + i \sin \frac{2k\pi - \frac{\pi}{4}}{4} \right], \quad k = 0, 1, 2, 3. \end{aligned}$$

Take $k = 0$, then

$$z_0 = 2^{\frac{1}{8}} \left[\cos \left(-\frac{\pi}{16} \right) + i \sin \left(-\frac{\pi}{16} \right) \right] = 1.0696 - 0.2128i.$$

Take $k = 1$, then

$$z_1 = 2^{\frac{1}{8}} \left[\cos \left(\frac{7\pi}{16} \right) + i \sin \left(\frac{7\pi}{16} \right) \right] = 0.2128 + 1.0696i.$$

Take $k = 2$, then

$$z_2 = 2^{\frac{1}{8}} \left[\cos \left(\frac{15\pi}{16} \right) + i \sin \left(\frac{15\pi}{16} \right) \right] = -1.0696 + 0.2128i.$$

Take $k = 3$, then

$$\begin{aligned} z_3 &= 2^{\frac{1}{8}} \left[\cos \left(\frac{23\pi}{16} \right) + i \sin \left(\frac{23\pi}{16} \right) \right] = 2^{\frac{1}{8}} \left[\cos \left(\frac{-9\pi}{16} \right) + i \sin \left(\frac{-9\pi}{16} \right) \right] \\ &= -0.2128 - 1.0696i. \end{aligned}$$

Example 17

In real number world, we know that $\sqrt[5]{1} = (1)^{\frac{1}{5}} = 1$. What are the values of $\sqrt[5]{1}$ in complex number world?

😊Solution

Step 1: Write 1 in polar form

One can find that

$$1 = \cos 0 + i \sin 0.$$

Step 2: Do computation

$$\begin{aligned}\sqrt[5]{1} &= (\cos 0 + i \sin 0)^{\frac{1}{5}} \\ &= \left[\cos \frac{2k\pi + 0}{5} + i \sin \frac{2k\pi + 0}{5} \right], \quad k = 0, 1, 2, 3, 4.\end{aligned}$$

The values of $\sqrt[5]{1}$ are given by

$$z_0 = \cos 0 + i \sin 0 = 1,$$

$$z_1 = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} = 0.3090 + 0.9511i$$

$$z_2 = \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5} = -0.8090 + 0.5878i$$

$$z_3 = \cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5} = \cos \frac{-4\pi}{5} + i \sin \frac{-4\pi}{5} = -0.8090 - 0.5878i$$

$$z_4 = \cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5} = \cos \frac{-2\pi}{5} + i \sin \frac{-2\pi}{5} = 0.3090 - 0.9511i$$

Remark:

The number $z = \sqrt[n]{1}$ (or $z^n = 1$) is also called the n^{th} root of unity and is given by

$$\sqrt[n]{1} = \left[\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \right], \quad k = 0, 1, 2, \dots, n-1.$$

Example 18 (Fractional power of a complex number)

Find all possible values of $(-\sqrt{3} + i)^{\frac{2}{5}}$.

☺Solution:

The polar form of $-\sqrt{3} + i$ is given by $-\sqrt{3} + i = 2 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$.

Then

$$\begin{aligned}
 (-\sqrt{3} + i)^{\frac{2}{5}} &= \left\{ \left[2 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) \right]^2 \right\}^{\frac{1}{5}} \\
 &= \left[2^2 \left(\cos \frac{10\pi}{6} + i \sin \frac{10\pi}{6} \right) \right]^{\frac{1}{5}} = 2^{\frac{2}{5}} \left(\cos \frac{2k\pi + \frac{5\pi}{3}}{5} + i \sin \frac{2k\pi + \frac{5\pi}{3}}{5} \right) \\
 &= 2^{\frac{2}{5}} \left(\cos \left(\frac{2k\pi}{5} + \frac{\pi}{3} \right) + i \sin \left(\frac{2k\pi}{5} + \frac{\pi}{3} \right) \right), \quad k = 0, 1, 2, 3, 4.
 \end{aligned}$$

Example 19 (DeMoivre's Theorem for fractional index)

Compute $(\cos \theta + i \sin \theta)^{\frac{m}{n}}$ where m is an integer and n is a positive integer.

☺Solution:

$$(\cos \theta + i \sin \theta)^{\frac{m}{n}} = [(\cos \theta + i \sin \theta)^m]^{\frac{1}{n}}$$

$$= [\cos m\theta + i \sin m\theta]^{\frac{1}{n}}$$

$$= \cos \frac{2k\pi + m\theta}{n} + i \sin \frac{2k\pi + m\theta}{n},$$

where $k = 0, 1, 2, \dots, (n - 1)$.

Solving algebraic equations using DeMoivre's Theorem

Example 20

(a) Solve $(1 + z)^8 = (1 + i)$

(b) Solve $(1 + z)^8 = (2 - z)^8$

Express your answer in form of $a + bi$.

☺Solution of (a)

$$(1 + z)^8 = (1 + i) \Rightarrow (1 + z) = (1 + i)^{\frac{1}{8}} \Rightarrow z = (1 + i)^{\frac{1}{8}} - 1$$

$$\Rightarrow z = \left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^{\frac{1}{8}} - 1$$

$$\Rightarrow z = 2^{\frac{1}{16}} \left(\cos \frac{2k\pi + \frac{\pi}{4}}{8} + i \sin \frac{2k\pi + \frac{\pi}{4}}{8} \right) - 1, \quad k = 0, 1, 2, \dots, 7.$$

$$\Rightarrow z = \left(2^{\frac{1}{16}} \cos \frac{2k\pi + \frac{\pi}{4}}{8} - 1 \right) + 2^{\frac{1}{16}} \sin \frac{2k\pi + \frac{\pi}{4}}{8} i, \quad k = 0, 1, 2, \dots, 7.$$

☺Solution of (b)

$$(1 + z)^8 = (2 - z)^8 \Rightarrow \left(\frac{1 + z}{2 - z} \right)^8 = 1$$

$$\Rightarrow \frac{1 + z}{2 - z} = 1^{\frac{1}{8}} = (\cos 0 + i \sin 0)^{\frac{1}{8}}$$

$$\Rightarrow \frac{1 + z}{2 - z} = \cos \frac{2k\pi}{8} + i \sin \frac{2k\pi}{8} \quad \text{for } k = 0, 1, 2, \dots, 7.$$

For simplicity, we let $\omega_k = \cos \frac{2k\pi}{8} + i \sin \frac{2k\pi}{8}$, then

$$\frac{1 + z}{2 - z} = \omega_k \Rightarrow \dots \Rightarrow z_k = \frac{2\omega_k - 1}{1 + \omega_k}.$$

$$\begin{aligned}
z_k &= \frac{2 \cos \frac{2k\pi}{8} - 1 + 2i \sin \frac{2k\pi}{8}}{1 + \cos \frac{2k\pi}{8} + i \sin \frac{2k\pi}{8}} \\
&= \frac{2 \cos \frac{2k\pi}{8} - 1 + 2i \sin \frac{2k\pi}{8}}{1 + \cos \frac{2k\pi}{8} + i \sin \frac{2k\pi}{8}} \left(\frac{1 + \cos \frac{2k\pi}{8} - i \sin \frac{2k\pi}{8}}{1 + \cos \frac{2k\pi}{8} - i \sin \frac{2k\pi}{8}} \right) \\
&= \dots \\
&= \frac{1 + \cos \frac{2k\pi}{8} + 3 \sin \frac{2k\pi}{8} i}{\left(1 + \cos \frac{2k\pi}{8}\right)^2 + \left(\sin \frac{2k\pi}{8}\right)^2} = \left(\frac{1 + \cos \frac{2k\pi}{8}}{2 + 2 \cos \frac{2k\pi}{8}} \right) + \left(\frac{3 \sin \frac{2k\pi}{8}}{2 + 2 \cos \frac{2k\pi}{8}} \right) i, \\
&= \frac{1}{2} + \left(\frac{3 \sin \frac{k\pi}{4}}{2 + 2 \cos \frac{k\pi}{4}} \right) i, \quad k = 0, 1, \dots, 7.
\end{aligned}$$

Example 21

Solve $z^6 - z^3 - 2 = 0$.

☺Solution:

To make your life easier, we first let $y = z^3$, then the equation becomes

$$\begin{aligned} y^2 - y - 2 &= 0 \\ \Rightarrow (y - 2)(y + 1) &= 0 \\ \Rightarrow y = 2 \text{ or } y &= -1 \end{aligned}$$

For $y = 2$, then $z^3 = 2$. So

$$z = \sqrt[3]{2} = [2(\cos 0 + i \sin 0)]^{\frac{1}{3}} = 2^{\frac{1}{3}} \left(\cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3} \right), \quad k = 0, 1, 2.$$

For $y = -1$, then $z^3 = -1$. So

$$\begin{aligned} z &= \sqrt[3]{-1} = (\cos \pi + i \sin \pi)^{\frac{1}{3}} = \left(\cos \frac{2k\pi + \pi}{3} + i \sin \frac{2k\pi + \pi}{3} \right) \\ &= \cos \frac{(2k + 1)\pi}{3} + i \sin \frac{(2k + 1)\pi}{3}, \quad k = 0, 1, 2. \end{aligned}$$

Example 22 (A bit harder)

Let z be a complex number such that $z + 3i$ is the 5th root of unity, find and list out all possible values of z . Express your answer in the form of $a + bi$.

☺Solution:

Given that $z + 3i$ is the 5th root of unity, we have

$$(z + 3i)^5 = 1$$

$$\Rightarrow z + 3i = \sqrt[5]{1} = \sqrt[5]{\cos 0 + i \sin 0}$$

$$\Rightarrow z + 3i = \cos \frac{2k\pi + 0}{5} + i \sin \frac{2k\pi + 0}{5}, \quad k = 0, 1, 2, 3, 4.$$

$$\Rightarrow z = \cos \frac{2k\pi}{5} + i \left(\sin \frac{2k\pi}{5} - 3 \right), \quad k = 0, 1, 2, 3, 4.$$

Substitute $k = 0, 1, 2, 3, 4$, we then get

$$z_0 = \cos 0 + i(\sin 0 - 3) = 1 - 3i;$$

$$z_1 = \cos \frac{2\pi}{5} + i \left(\sin \frac{2\pi}{5} - 3 \right) = 0.3090 - 2.0489i;$$

$$z_2 = \cos \frac{4\pi}{5} + i \left(\sin \frac{4\pi}{5} - 3 \right) = -0.8090 - 2.4122i;$$

$$z_3 = \cos \frac{6\pi}{5} + i \left(\sin \frac{6\pi}{5} - 3 \right) = -0.8090 - 3.5878i;$$

$$z_4 = \cos \frac{8\pi}{5} + i \left(\sin \frac{8\pi}{5} - 3 \right) = 0.3090 - 3.9511i;$$

Application of Complex Numbers

1. Deriving some useful identities of trigonometric functions

Example 23

By considering $(\cos \theta + i \sin \theta)^4$, show that

$$\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1.$$

😊IDEA:

One will compute $(\cos \theta + i \sin \theta)^4$ in two (or more) different ways and compare the result obtained.

😊Solution:

Way 1: Use DeMoivre's theorem, we get

$$(\cos \theta + i \sin \theta)^4 = \cos 4\theta + i \sin 4\theta \dots (1)$$

Way 2: Use Binomial theorem to expand $(\cos \theta + i \sin \theta)^4$ by brute force

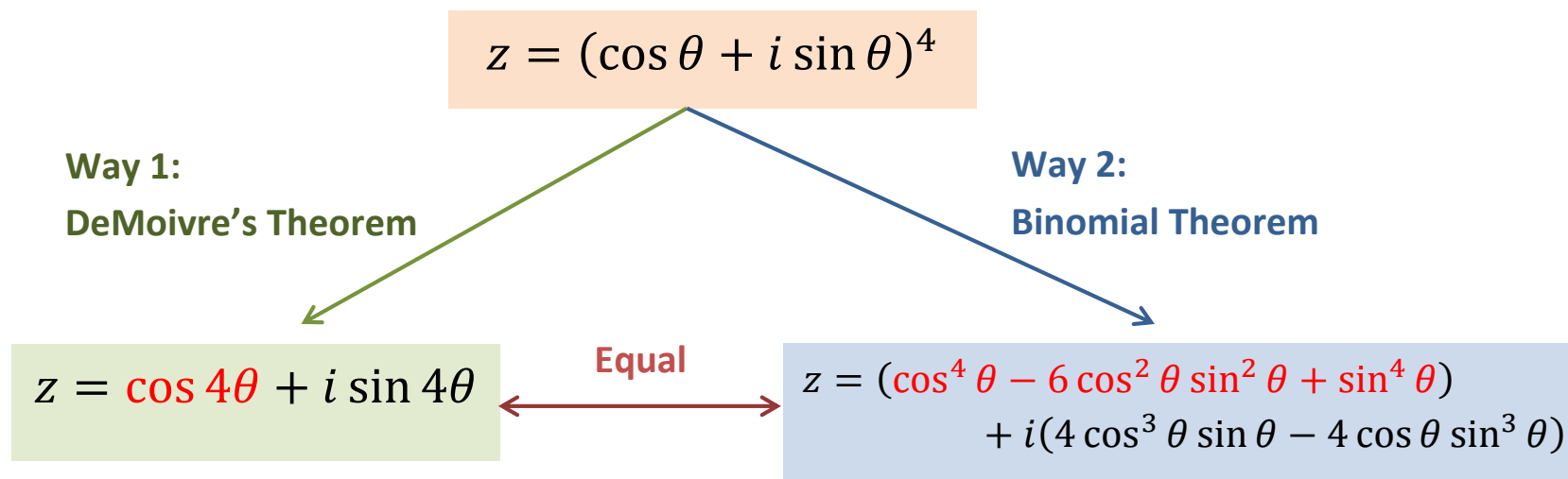
Recall: Binomial Theorem

For any numbers a , b and positive integer n , we have

$$(a + b)^n = \sum_{r=0}^n C_r^n a^{n-r} b^r = a^n + C_1^n a^{n-1} b + C_2^n a^{n-2} b^2 + \dots + b^n$$

where $C_r^n = \frac{n!}{r!(n-r)!}$ and $n! = n \times (n-1) \times (n-2) \times \dots \times 3 \times 2 \times 1$.

$$\begin{aligned} (\cos \theta + i \sin \theta)^4 &= \sum_{r=0}^4 C_r^4 (\cos \theta)^{4-r} (i \sin \theta)^r \\ &= \cos^4 \theta + 4 \cos^3 \theta \sin \theta i + 6 \cos^2 \theta \sin^2 \theta i^2 + 4 \cos \theta \sin^3 \theta i^3 + \sin^4 \theta i^4 \\ &= \cos^4 \theta + 4 \cos^3 \theta \sin \theta i - 6 \cos^2 \theta \sin^2 \theta - 4 \cos \theta \sin^3 \theta i + \sin^4 \theta \\ &= (\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta) + i(4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta) \dots (2) \end{aligned}$$



We compare the real part of the expression (1) and (2), we get

$$\cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta$$

$$= \cos^4 \theta - 6 \cos^2 \theta (1 - \cos^2 \theta) + (1 - \cos^2 \theta)^2$$

$$= \cos^4 \theta - 6 \cos^2 \theta + 6 \cos^4 \theta + 1 - 2 \cos^2 \theta + \cos^4 \theta$$

$$= 8 \cos^4 \theta - 8 \cos^2 \theta + 1.$$

Example 24

By considering $\left(z + \frac{1}{z}\right)^6$ where $z = \cos \theta + i \sin \theta$ and using the fact that $z^n + \frac{1}{z^n} = 2 \cos n\theta$ for all n . Show that

$$\cos^6 \theta = \frac{1}{32} (\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10).$$

☺Solution:

Similar to Example 23, we compute $\left(z + \frac{1}{z}\right)^6$ in two different ways and compare the results again.

Way 1: Using the given fact (put $n = 1$), we get

$$\left(z + \frac{1}{z}\right)^6 = (2 \cos \theta)^6 = 64 \cos^6 \theta \dots (1)$$

Way 2: Expand the expression by brute force using binomial theorem

$$\begin{aligned}\left(z + \frac{1}{z}\right)^6 &= \sum_{r=0}^6 C_r^6 z^{6-r} \left(\frac{1}{z}\right)^r = \sum_{r=0}^6 C_r^6 z^{6-2r} \\ &= z^6 + 6z^4 + 15z^2 + 20z^0 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6} \\ &= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20\end{aligned}$$

Using the given fact with $n = 2, 4, 6$ we get

$$= 2 \cos 6\theta + 12 \cos 4\theta + 30 \cos 2\theta + 20 \dots (2)$$

We compare the result in (1) and (2), we get

$$\begin{aligned}64 \cos^6 \theta &= 2 \cos 6\theta + 12 \cos 4\theta + 30 \cos 2\theta + 20 \\ \Rightarrow \cos^6 \theta &= \frac{1}{32} (\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10).\end{aligned}$$

2. Computation of integral

Sometimes, complex number theory allows to compute some integrals in a easier way. It can also allow us to compute some integrals which cannot be computed under real number framework (say $\int_0^\infty e^{-x^2} dx$)

Example 25

Using complex number and compute the integral

$$\int e^{3x} \cos 2x \, dx.$$

☺IDEA:

The Euler formula ($e^{i\theta} = \cos \theta + i \sin \theta$) in complex number theory allows us to “transform” the trigonometric functions into exponential function so that they can “combine” with the exponential function in the integral and simplify the calculation.

☺Solution:

Using the fact that $\cos 2x = \frac{1}{2}(e^{i(2x)} + e^{i(-2x)})$, the integral can be computed as

$$\begin{aligned}\int e^{3x} \cos 2x \, dx &= \int e^{3x} \left[\frac{1}{2}(e^{i(2x)} + e^{i(-2x)}) \right] dx \\&= \frac{1}{2} \int e^{(3+2i)x} dx + \frac{1}{2} \int e^{(3-2i)x} dx \\&= \frac{1}{2(3+2i)} e^{(3+2i)x} + \frac{1}{2(3-2i)} e^{(3-2i)x} + C \\&= \frac{1}{2(3+2i)} \left(\frac{3-2i}{3-2i} \right) e^{(3+2i)x} + \frac{1}{2(3-2i)} \left(\frac{3+2i}{3+2i} \right) e^{(3-2i)x} + C \\&= \frac{3-2i}{26} e^{(3+2i)x} + \frac{3+2i}{26} e^{(3-2i)x} + C \\&= \frac{3-2i}{26} e^{3x} e^{2ix} + \frac{3+2i}{26} e^{3x} e^{-2ix} + C\end{aligned}$$

$$\begin{aligned}
&= \frac{3-2i}{26} e^{3x} (\cos 2x + i \sin 2x) + \frac{3+2i}{26} e^{3x} \left(\underbrace{\cos(-2x)}_{\cos 2x} + i \underbrace{\sin(-2x)}_{-\sin 2x} \right) + C \\
&= \frac{e^{3x}}{26} \left[\left(3 \cos 2x + 3 \sin 2x i - 2 \cos 2x i - 2 \sin 2x \underbrace{i^2}_{=-1} \right) \right. \\
&\quad \left. + \left(3 \cos 2x - 3 \sin 2x i + 2 \cos 2x i - 2 \sin 2x \underbrace{i^2}_{=-1} \right) \right] + C \\
&= \frac{e^{3x}}{26} [(3 \cos 2x + 2 \sin 2x + 3 \cos 2x + 2 \sin 2x) \\
&\quad + (3 \sin 2x - 2 \cos 2x - 3 \sin 2x + 2 \cos 2x)i] + C \\
&= \frac{e^{3x}}{26} (6 \cos 2x + 4 \sin 2x) = \frac{3}{13} e^{3x} \cos 2x + \frac{2}{13} e^{3x} \sin 2x + C.
\end{aligned}$$

Complex Conjugate

- In doing the division of complex number $\frac{a+bi}{c+di}$, we try to multiply both numerator and denominator by a common factor $(c - di)$. By doing so, the "i" in the denominator can be eliminated.
- When we solve the quadratic equation such as $x^2 + x + 1 = 0$, we find that the roots are given by $x = \frac{-1 \pm \sqrt{1^2 - 4(1)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. We observe that if $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is a root of the equation, then $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ is also a root.
- Given a complex number $z = a + bi$, the complex number $a - bi$ plays an important role. We call this number to be complex conjugate of z .

Definition (Complex Conjugate)

The *complex conjugate* of a complex number $z = a + bi$, denoted by $\bar{z}$ is defined as

$$\bar{z} = a - bi$$

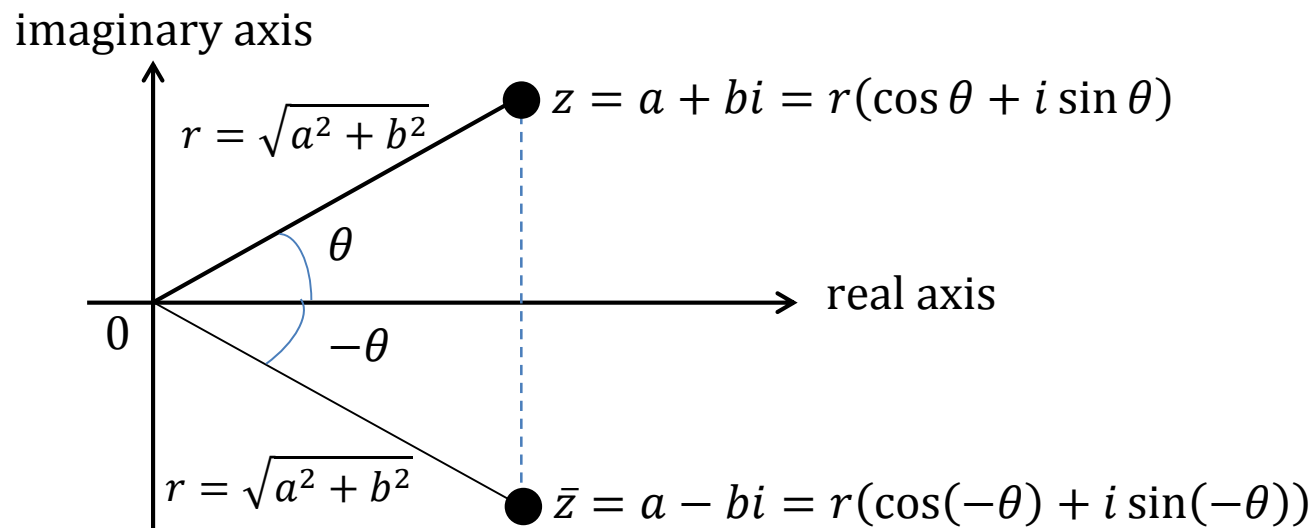
Example 26

- The complex conjugate of $4 + 3i$ is $4 - 3i$.
- The complex conjugate of $4 - 3i$ is $4 - (-3)i = 4 + 3i$.
- The complex conjugate of a real number a is also a .

(Reason: Note that $a = a + 0i$, then the complex conjugate of a is $\bar{a} = a - 0i = a$)

Geometric Representation of Complex Conjugate

- The real parts of z and its complex conjugate $\bar{z}$ are equal and the imaginary parts are equal in magnitude but opposite in sign.
- In an Argand diagram, z and $\bar{z}$ are symmetrical about the real axis (the x -axis).



Properties of Complex Conjugate

Let $z = a + bi$ be a complex number. Then

$$(1) \text{ If } z \text{ is real, then } \bar{z} = z.$$

$$(2) \bar{\bar{z}} = z$$

$$(3) z\bar{z} = a^2 + b^2 = |z|^2$$

$$(4) |\bar{z}| = |z|$$

$$(5) \arg \bar{z} = -\theta = -\arg z$$

$$(6) z + \bar{z} = 2a = 2\operatorname{Re}(z). \text{ (Here, } a = \operatorname{Re}(z) \text{ denote real part of } z)$$

$$(7) z - \bar{z} = 2bi = 2i \operatorname{Im}(z). \text{ (Here, } b = \operatorname{Im}(z) \text{ denote imaginary part of } z)$$

(Operation)

$$(8) \overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$$

$$(9) \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$$(10) \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

The complex conjugate allows us to prove various identities about complex numbers and derive other facts.

Example 27

Prove that, for any complex numbers z_1 and z_2 , we have

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

☺Solution:

Note that

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\ &= z_1\bar{z}_1 + z_2\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_2 \end{aligned}$$

$$\begin{aligned} |z_1 - z_2|^2 &= (z_1 - z_2)\overline{(z_1 - z_2)} = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\ &= z_1\bar{z}_1 - z_2\bar{z}_1 - z_1\bar{z}_2 + z_2\bar{z}_2 \end{aligned}$$

Therefore

$$\begin{aligned} & |z_1 + z_2|^2 + |z_1 - z_2|^2 \\ &= (z_1\bar{z}_1 + z_2\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_2) + (z_1\bar{z}_1 - z_2\bar{z}_1 - z_1\bar{z}_2 + z_2\bar{z}_2) \\ &= 2z_1\bar{z}_1 + 2z_2\bar{z}_2 \\ &= 2|z_1|^2 + 2|z_2|^2 \end{aligned}$$

Example 28

If $|z - 3| = |z + 3|$, show that $z = bi$ for some real number b .

IDEA: Let play around with the condition given $|z - 3| = |z + 3|$ and see how the thing evolve.

☺Solution:

$$|z - 3| = |z + 3| \Rightarrow |z - 3|^2 = |z + 3|^2$$

$$\Rightarrow (z - 3)\overline{(z - 3)} = (z + 3)\overline{(z + 3)}$$

$$\Rightarrow (z - 3)(\bar{z} - \bar{3}) = (z + 3)(\bar{z} + \bar{3})$$

$$\Rightarrow (z - 3)(\bar{z} - 3) = (z + 3)(\bar{z} + 3)$$

$$\Rightarrow z\bar{z} - 3z - 3\bar{z} + 9 = z\bar{z} + 3z + 3\bar{z} + 9$$

$$\Rightarrow 6z + 6\bar{z} = 0$$

$$\Rightarrow z + \bar{z} = 0 \Rightarrow 2\operatorname{Re}(z) = 0$$

$$\Rightarrow \operatorname{Re}(z) = 0.$$

This implies the real part of z must be 0. So $z = bi$ for some real number b .

Roots of Polynomials with real coefficient

Recall that if $a + bi$ is a solution of a quadratic equation $a_2x^2 + a_1x + a_0 = 0$, then $a - bi$ is also a solution of the same equation.

Question: Does the statement hold for the general polynomial equation?

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0.$$

Theorem

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial with real coefficients and degree n ($n \geq 2$).

If $z = a + bi$ is a root of $f(x) = 0$, then $\bar{z} = a - bi$ is also a root of $f(x) = 0$.

Proof of the theorem

It is sufficient to show $f(\bar{z}) = 0$.

Note that

$$\begin{aligned}
 f(\bar{z}) &= a_n \bar{z}^n + a_{n-1} \bar{z}^{n-1} + \cdots + a_2 \bar{z}^2 + a_1 \bar{z} + a_0 \\
 &= a_n \overline{z^n} + a_{n-1} \overline{z^{n-1}} + \cdots + a_2 \overline{z^2} + a_1 \bar{z} + a_0 \\
 &= \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \cdots + \overline{a_2 z^2} + \overline{a_1 z} + \overline{a_0} \\
 &= \overline{(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0)} \\
 &= \overline{f(z)} \\
 &= \bar{0} \\
 &= 0.
 \end{aligned}$$

Example 29

Solve the equation $2x^3 - 7x^2 + 6x + 5 = 0$ given that $2 + i$ is one of the roots.

☺Solution:

Note that $2 - i$ is also the solution of the same equation.

By factor theorem, $(x - (2 + i))$ and $(x - (2 - i))$ are “factor” of $2x^3 - 7x^2 + 6x + 5$. This implies that $(x - (2 + i))(x - (2 - i)) = x^2 - 4x + 5$ is also a factor of the same expression.

Using long division, the expression on L.H.S. can be factorized as:

$$2x^3 - 7x^2 + 6x + 5 = 0$$

$$\Rightarrow (x^2 - 4x + 5)(2x + 1) = 0$$

$$\Rightarrow x^2 - 4x + 5 = 0 \quad \text{or} \quad 2x + 1 = 0$$

$$\Rightarrow x = 2 + i \quad \text{or} \quad x = 2 - i \quad \text{or} \quad x = -\frac{1}{2}.$$