

Write your answers on **paper** using the **following format**:

Answer to Question X

Scan your answers **or** take **photos** of them.

Submit **scanned** answers **or** their **photos** in a **single file** to Canvas **before 4:10 pm**.

Name your file in following format: **Quiz1_SID_LastName_FirstName**

Part 1: Short questions:

40 marks

Write down all the steps in your calculations.

Q1) Find the floating-point representation of the following number with 5 decimal points.

0.00126678

Answer:

Mantissa: 0.12668 (2 mark)

Exponent: -2 (2 mark)

Q2) Consider the function $f(x) = x^3 + 3x^2 + 2x + 4$. Compute the first derivative of $f(x)$ around $x_i = 0.5$ using the forward, backward, and centered methods. Use the step-size $h = 0.01$.

Answer:

Forward: $\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = 5.7951$ (4 marks)

Backward: 5.7051 (4 marks)

Forward: $\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = 5.7501$ (4 marks)

Q3) Suppose that $\tilde{x} = 1.5$ is an approximation of x with an error of $\Delta\tilde{x} = 0.02$. We would like to approximate the value of $f(x) = x^2 + 2x + 2$ using $f(\tilde{x})$. Compute the approximation error.

Answer: $\Delta f(\tilde{x}) = (2\tilde{x} + 2)\Delta\tilde{x} = 5 \times 0.02 = 0.1$ (4 marks)

Q4) Consider the following vectors v_1 and v_2 . Can v_1 and v_2 be eigenvectors of a symmetric matrix?

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

Answer1: Yes, because they are orthogonal. (4 marks)

Q5) Find the determinant of the following matrix.

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 1 \\ 2 & 4 & 3 \end{bmatrix}$$

Answer: $\text{Det}(A) = -20$. (4 marks)

Q6) Does the following matrix have any eigenvalues equal to zero? Why?

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -3 & 3 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Answer: Yes. $\text{Det}(A) = 0$. So, it has at least one zero eigenvalue. (4 marks)

Q7) Consider matrix A and vector v below. Is v , an eigenvector of A ? Why?

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 1 \\ 2 & 4 & 3 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Answer: No. We cannot find any λ such that $Av = \lambda v$

$$Av = \begin{bmatrix} 1 \\ 0 \\ 13 \end{bmatrix} \text{ (4 marks)}$$

Q8) Can the polynomial $\lambda^2 + 2\lambda + 1$ be the characteristic polynomial of a symmetric 2-by-2 matrix? Why?

Answer: Yes. Because the roots of $\lambda^2 + 2\lambda + 1$ are real numbers. The eigenvalues of symmetric matrices are real. (4 marks)

Part 2: Long questions:

Q1) Consider the function $f(x) = 1 + x^2 + x^3 + x^4 + 3x^5$

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20 marks

- a) Find the first-order Taylor approximation of $f(x)$ around $x_i=1$.

$$\text{Answer: } f^1(x) = 15x^4 + 4x^3 + 3x^2 + 2x$$

$$f^1(1) = 24$$

$$\text{Taylor's approximation } 24x - 17$$

(3 marks)

- b) Find the percentage of the relative error when the first-order Taylor approximation of $f(x)$ is used to evaluate $f(2)$.

$$\text{Percentage of relative error} = \frac{376}{5} = 75.2 \text{ (2 marks)}$$

- c) Find the second-order Taylor approximation of $f(x)$ around $x_i=1$.

$$\text{Answer: } f^2(x) = 60x^3 + 12x^2 + 6x + 2$$

$$f^2(1) = 80$$

$$\text{Taylor's approxim } 24x + 40(x - 1)^2 - 17$$

(3 marks)

- d) Find the percentage of the relative error when the second-order Taylor approximation of $f(x)$ is used to evaluate $f(2)$.

$$\text{Percentage of relative error} = 216/5 = 43.2 \text{ (2 marks)}$$

- e) Find the third-order Taylor approximation of $f(x)$ around $x_i=1$.

$$\text{Answer: } f^3(x) = 180x^2 + 24x + 6$$

$$f^3(1) = 210$$

$$\text{Taylor's approxim } 24x + 40(x - 1)^2 + 35(x - 1)^3 - 17$$

(3 marks)

- f) Find the percentage of the relative error when the third-order Taylor approximation of $f(x)$ is used to evaluate $f(2)$.

$$\text{Percentage of relative error} = \frac{76}{5} = 15.2 \text{ (2 marks)}$$

- g) Find the fourth-order Taylor approximation of $f(x)$ around $x_i=1$.

$$\text{Answer: } f^4(x) = 360x + 24$$

$$f^4(1) = 384$$

$$\text{Taylor's approxim } 24x + 40(x - 1)^2 + 35(x - 1)^3 + 16(x - 1)^4 - 17$$

(3 marks)

- h) Find the percentage of the relative error when the fourth-order Taylor approximation of $f(x)$ is used to evaluate $f(2)$.

$$\text{Percentage of relative error} = \frac{12}{5} = 2.4$$

(2 marks)

Write down all the steps in your calculations.

20 marks

Q2) Consider the following matrix.

$$A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{bmatrix}$$

- a) Find the characteristic polynomial of A .
- b) The vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector of A . Find its corresponding eigenvalue.
- c) Find the other two eigenvalues of A

Write down all the steps in your calculations.

a) $-\lambda^3 + 16\lambda^2 - 81\lambda + 126$ (10 marks)

b) $A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. So, the corresponding eigenvalue is 6. (2 mark)

c) The other eigenvalues are 7 and 3. (8marks)

20 marks

Q3)

Let A be an invertible $n \times n$ matrix. Suppose that v is an eigenvector of A and λ is the eigenvalue corresponding to v . Assume that λ is non-zero.

- a) Show that v is an eigenvector of $(A^{-1})^n$ and $(\frac{1}{\lambda})^n$ is its corresponding eigenvalue.
- b) Show that λ^n is an eigenvalue of the matrix A^n and v is its corresponding eigenvector.

Answer: Give full marks only if the answer is complete and logically consistent.

a) Since v is an eigenvector of A , we have $\lambda v = Av$. Multiplying both sides of this equality from left by A^{-1} , we have

$$\begin{aligned} A^{-1}\lambda v &= A^{-1}Av \\ &= Iv \\ &= v \end{aligned}$$

So, we have $A^{-1}v = \frac{1}{\lambda}v$. **(5 marks)**

$$\begin{aligned}
 (A^{-1})^n v &= \underbrace{A^{-1} \times \dots \times A^{-1}}_n v \\
 &= \underbrace{A^{-1} \times \dots \times A^{-1}}_{n-1} \frac{1}{\lambda} v \\
 &= \underbrace{A^{-1} \times \dots \times A^{-1}}_{n-2} \left(\frac{1}{\lambda}\right)^2 v \\
 &\quad \vdots \\
 &= \left(\frac{1}{\lambda}\right)^n v
 \end{aligned}$$

Which implies we have $(A^{-1})^n v = \left(\frac{1}{\lambda}\right)^n v$. Thus, v is an eigenvector of A^n **(5 marks)** and $\left(\frac{1}{\lambda}\right)^n$ is its corresponding eigenvalue **(5 marks)**.

b) Since λ is an eigenvalue of matrix A , we have $\lambda v = Av$. To show v is an eigenvector of A^n , we compute $A^n v$ as follows

$$\begin{aligned}
 A^n v &= \underbrace{A \times \dots \times A}_n v \\
 &= \underbrace{A \times \dots \times A}_{n-1} \lambda v \\
 &= \underbrace{A \times \dots \times A}_{n-2} \lambda^2 v \\
 &\quad \vdots \\
 &= \lambda^n v
 \end{aligned}$$

Which implies we have $A^n v = \lambda^n v$. Thus, v is an eigenvector of A^n **(2.5 marks)** and λ^n is its corresponding eigenvalue. **(2.5 marks)**