

MA1200 CALCULUS AND BASIC LINEAR ALGEBRA I

LECTURE: CG1

Chapter 7 **Techniques of Differentiation**

Brief table of derivatives of some elementary functions with respect to x

Function $y = f(x)$	Derivative of y with respect to x $\frac{dy}{dx} = \frac{d}{dx}(f(x))$ or $\frac{dy}{dx} = f'(x)$
$y = c$, where c is a constant	$\frac{dy}{dx} = 0$
$y = x^a$, where a is a constant	$\frac{dy}{dx} = ax^{a-1}$
$y = a^x$	$\frac{dy}{dx} = a^x \ln a$
$y = e^x$	$\frac{dy}{dx} = e^x$
$y = \ln x$	$\frac{dy}{dx} = \frac{1}{x}$
$y = \log_a x$, where $a > 0$	$\frac{dy}{dx} = \frac{1}{x} \log_a e$
$y = \sin x$	$\frac{dy}{dx} = \cos x$
$y = \cos x$	$\frac{dy}{dx} = -\sin x$

etc.

Rules of Differentiation

Let $u = u(x)$ and $v = v(x)$ be differentiable functions of x .

Then $\frac{du}{dx}$ (also denoted as $u'(\textcircled{x})$) and $\frac{dv}{dx}$ (also denoted as $v'(\textcircled{x})$) are the derivatives of u and v , respectively, ^{w.r.t.} with respect to $\textcircled{x}$. $\leftarrow \frac{d?}{dx}$

Rule #1: The derivative of any **constant** c is 0.

$$\boxed{\frac{d}{dx}(c) = 0}.$$

Rule #2: For any constant c , the derivative of a **scalar multiple** of $u(x)$ is given by

$$\boxed{\frac{d}{dx}(c \cdot u) = c \cdot \frac{du}{dx}}.$$

It is also denoted as

$$\frac{d}{dx}[c \cdot u(x)] = c \cdot u'(x).$$

Rule #3: The derivative of a **sum** or a **difference** of two functions of x is given by

$$\boxed{\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}}$$

It follows that $\boxed{\frac{d}{dx}(c_1 \cdot u \pm c_2 \cdot v) = c_1 \cdot \frac{du}{dx} \pm c_2 \cdot \frac{dv}{dx}}$ where c_1 and c_2 are constants.

It is also denoted as $\frac{d}{dx}[c_1 \cdot u(x) \pm c_2 \cdot v(x)] = c_1 \cdot u'(x) \pm c_2 \cdot v'(x)$.

Example 1

If $y = 2e^x + 3 \cos x - 5x^3 + 4x - \sqrt{x} + \frac{2}{\sqrt[3]{x}} + 8$, then

$$\begin{aligned} \frac{dy}{dx} &= 2 \frac{d}{dx}(e^x) + 3 \frac{d}{dx}(\cos x) - 5 \frac{d}{dx}(x^3) + 4 \frac{d}{dx}(x) - \frac{d}{dx}(x^{\frac{1}{2}}) + 2 \frac{d}{dx}(x^{-\frac{1}{3}}) + \frac{d}{dx}(8) \\ &= 2e^x + 3(-\sin x) - 5 \cdot 3x^2 + 4 \cdot 1 - \frac{1}{2}x^{-\frac{1}{2}} + 2\left(-\frac{1}{3}\right)x^{-\frac{1}{3}-1} + 0 \\ &= 2e^x - 3 \sin x - 15x^2 + 4 - \frac{1}{2\sqrt{x}} - \frac{2}{3}x^{-\frac{4}{3}} \end{aligned}$$

Note:

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$$

$$\sqrt[3]{x} = x^{\frac{1}{3}}$$

$$\frac{1}{x^a} = x^{-a}$$

etc.

Rule #4: (**Product rule**) The derivative of a **product** of two functions of x is given by

$$\star \quad \boxed{\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}}.$$

Note:
 $\frac{d}{dx}(uv) \neq \frac{du}{dx} \cdot \frac{dv}{dx}$

It is also denoted as

$$\frac{d}{dx}[u(x) \cdot v(x)] = u(x) \cdot v'(x) + v(x) \cdot u'(x).$$

Example 2

If $y = x^5 \cos x$, then

$$\begin{aligned} \frac{dy}{dx} &= x^5 \cdot \frac{d}{dx}(\cos x) + \cos x \cdot \frac{d}{dx}(x^5) \quad \text{by using **Product rule**} \\ &= x^5 \cdot (-\sin x) + \cos x \cdot 5x^4 \\ &= -x^5 \sin x + 5x^4 \cos x. \end{aligned}$$

Rule #5: (Quotient rule) The derivative of a **quotient** of two functions of x is given by

$$\star \quad \boxed{\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}}.$$

Note:

$$\frac{d}{dx} \left(\frac{u}{v} \right) \neq \frac{\left(\frac{du}{dx} \right)}{\left(\frac{dv}{dx} \right)}$$

It is also denoted as

$$\frac{d}{dx} \left[\frac{u(x)}{v(x)} \right] = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}.$$

Example 3

If $y = \frac{\sin x}{x^2+1}$, then

$$\frac{dy}{dx} = \frac{(x^2+1) \cdot \frac{d}{dx}(\sin x) - \sin x \cdot \frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

by using **Quotient rule**

$$= \frac{(x^2+1) \cdot \cos x - \sin x \cdot (2x)}{(x^2+1)^2}$$

$$= \frac{(x^2+1) \cos x - 2x \sin x}{(x^2+1)^2}$$

Rule #6: (**Chain rule**) Let f be a function of u , and u be a function of x . Then the derivative of the **composite** function $(f \circ u)(x) = f(u(x))$ is given by

$$\star \quad \frac{d}{dx}[f(u)] = \frac{d[f(u)]}{du} \cdot \frac{du}{dx}$$

↑ inner function
outer function

Diff. outer function w.r.t. inner function Diff. inner function w.r.t. x

It is also denoted as

$$\frac{d}{dx}[f(u(x))] = f'(u(x)) \cdot u'(x).$$

Example 4

{ outer function: $(\cdot)^{\frac{3}{2}}$
inner function: $3x^2 + 5x - 1$

↓

If $y = (3x^2 + 5x - 1)^{\frac{3}{2}}$, then

$$\begin{aligned} \frac{dy}{dx} &= \frac{d[(3x^2 + 5x - 1)^{\frac{3}{2}}]}{d(3x^2 + 5x - 1)} \cdot \frac{d(3x^2 + 5x - 1)}{dx} \\ &= \frac{3}{2} (3x^2 + 5x - 1)^{\frac{3}{2} - 1} \cdot [3(2x) + 5] \\ &= \frac{3}{2} (3x^2 + 5x - 1)^{\frac{1}{2}} \cdot (6x + 5) \end{aligned}$$

$$\frac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2} x^{\frac{1}{2}}$$

$$\therefore \frac{d}{du}(u^{\frac{3}{2}}) = \frac{3}{2} u^{\frac{1}{2}}$$

$$\therefore \frac{d}{d(\dots)}(\dots)^{\frac{3}{2}} = \frac{3}{2} (\dots)^{\frac{1}{2}}$$

Alternative method for Ex. 3 :

$$y = \frac{\sin x}{x^2+1} = (\sin x) \cdot (x^2+1)^{-1}$$

Then using product rule & chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \sin x \cdot \frac{d}{dx}[(x^2+1)^{-1}] + (x^2+1)^{-1} \cdot \frac{d}{dx}(\sin x) \\&= \sin x \cdot (-1)(x^2+1)^{-2} \cdot \frac{d}{dx}(x^2+1) + (x^2+1)^{-1} \cdot \cos x \\&= \sin x \cdot (-1)(x^2+1)^{-2} \cdot 2x + (x^2+1)^{-1} \cdot \cos x \\&= \frac{(x^2+1)\cos x - 2x \sin x}{(x^2+1)^2}\end{aligned}$$

Example 5

Find the derivatives of the following functions:

(a) $\tan x$

(b) $\cot x$

Solution

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\ &= \frac{\cos x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot \frac{d}{dx}(\cos x)}{(\cos x)^2}, && \text{by the **Quotient rule**} \\ &= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{d}{dx}(\cot x) &= \frac{d}{dx}[(\tan x)^{-1}] \quad \leftarrow \begin{cases} \text{outer : } (\cdot)^{-1} \\ \text{inner : } \tan x \end{cases} \\ &= \frac{d[(\tan x)^{-1}]}{d(\tan x)} \cdot \frac{d(\tan x)}{dx}, \quad \text{by the Chain rule} \\ &= (-1)(\tan x)^{-2} \cdot \sec^2 x, \quad \text{by (a)} \\ &= \frac{-1}{\tan^2 x} \cdot \frac{1}{\cos^2 x} \\ &= \frac{-1}{\frac{\sin^2 x}{\cos^2 x}} \cdot \frac{1}{\cos^2 x} \\ &= \frac{-1}{\sin^2 x} \\ &= -\csc^2 x \end{aligned}$$

Note:

$$\begin{aligned} \cot x &= \frac{1}{\tan x} \\ &= (\tan x)^{-1} \\ &\neq \tan^{-1} x \end{aligned}$$

Note:

Similarly, we can show that $\frac{d}{dx}(\sec x) = \sec x \tan x$ and $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

Example 6

Note: $\sinh x \neq \sin(x)$

Find the derivatives of the following hyperbolic functions:

(a) $\sinh x$

(b) $\cosh x$

(c) $\tanh x$

Solution

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx}(\sinh x) &= \frac{d}{dx} \left[\frac{1}{2} (e^x - e^{-x}) \right] \\ &= \frac{1}{2} \left[\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x}) \right] \quad \left\{ \begin{array}{l} \text{outer: } e^{(\cdot)} \\ \text{inner: } -x \end{array} \right. \\ &= \frac{1}{2} \left[\frac{d}{dx}(e^x) - \frac{d(e^{-x})}{d(-x)} \cdot \frac{d(-x)}{dx} \right] \quad \text{by Chain rule} \\ &= \frac{1}{2} [e^x - e^{-x} \cdot (-1)] = \frac{1}{2} (e^x + e^{-x}) = \cosh x \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{d}{dx}(\cosh x) &= \frac{d}{dx} \left[\frac{1}{2} (e^x + e^{-x}) \right] = \frac{1}{2} \left[\frac{d}{dx}(e^x) + \frac{d}{dx}(e^{-x}) \right] \\ &= \frac{1}{2} \left[e^x + \frac{d(e^{-x})}{d(-x)} \cdot \frac{d(-x)}{dx} \right] \quad \text{by Chain rule} \\ &= \frac{1}{2} [e^x + e^{-x} \cdot (-1)] = \frac{1}{2} (e^x - e^{-x}) = \sinh x \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \frac{d}{dx}(\tanh x) &= \frac{d}{dx} \left(\frac{\sinh x}{\cosh x} \right) \\ &= \frac{\cosh x \cdot \frac{d}{dx}(\sinh x) - \sinh x \cdot \frac{d}{dx}(\cosh x)}{(\cosh x)^2} \quad \text{by the Quotient rule} \\ &= \frac{\cosh x \cdot \cosh x - \sinh x \cdot (\sinh x)}{\cosh^2 x} \quad \text{by (a) and (b)} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\ &= \frac{1}{\cosh^2 x} \quad (\text{since } \cosh^2 x - \sinh^2 x = 1, \text{ from Chapter 5}) \\ &= \operatorname{sech}^2 x \end{aligned}$$

Homework:

Given that $\coth x = \frac{1}{\tanh x}$, $\operatorname{sech} x = \frac{1}{\cosh x}$ and $\operatorname{csch} x = \frac{1}{\sinh x}$.

Show that $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$, $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

and $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$.

Table of derivatives of $y = f(u)$ with respect to x , where u is a function of x

Function $y = f(u)$	Derivative of y with respect to x
$y = c$, where c is a constant	$\frac{dy}{dx} = 0$
$y = cu$, where c is a constant	$\frac{dy}{dx} = c \frac{du}{dx}$
$y = u^p$, where p is a constant	$\frac{dy}{dx} = p u^{p-1} \frac{du}{dx}$
$y = u + v$	$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
$y = uv$	$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$ product rule
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$ quotient rule
$y = f(u)$, where u is a function of x	$\frac{df(u)}{du} \cdot \frac{du}{dx}$, the chain rule
$y = \log_a u$, $a > 0$	$\frac{dy}{dx} = \frac{1}{u} \log_a e \frac{du}{dx}$
$y = a^u$, $a > 0$	$\frac{dy}{dx} = a^u \log_e a \frac{du}{dx}$
$y = e^u$	$\frac{dy}{dx} = e^u \frac{du}{dx}$
$y = u^v$	$\frac{dy}{dx} = v u^{v-1} \frac{du}{dx} + u^v \log_e u \frac{dv}{dx}$

Function $y = f(u)$	Derivative of y with respect to x
$y = \sin u$	$\frac{dy}{dx} = \cos u \frac{du}{dx}$
$y = \cos u$	$\frac{dy}{dx} = -\sin u \frac{du}{dx}$
$y = \tan u$	$\frac{dy}{dx} = \sec^2 u \frac{du}{dx}$
$y = \cot u$	$\frac{dy}{dx} = -\operatorname{cosec}^2 u \frac{du}{dx}$
$y = \sec u$	$\frac{dy}{dx} = \sec u \tan u \frac{du}{dx}$
$y = \operatorname{cosec} u$	$\frac{dy}{dx} = -\operatorname{cosec} u \cot u \frac{du}{dx}$
$y = \sin^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$
$y = \cos^{-1} u$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$
$y = \tan^{-1} u$	$\frac{dy}{dx} = \frac{1}{1+u^2} \frac{du}{dx}$
$y = \cot^{-1} u$	$\frac{dy}{dx} = -\frac{1}{1+u^2} \frac{du}{dx}$
$y = \sec^{-1} u$	$\frac{dy}{dx} = \frac{1}{ u \sqrt{u^2-1}} \frac{du}{dx}$
$y = \operatorname{cosec}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{ u \sqrt{u^2-1}} \frac{du}{dx}$

Function $y = f(u)$	Derivative of y with respect to x
$y = \sinh u$	$\frac{dy}{dx} = \cosh u \frac{du}{dx}$
$y = \cosh u$	$\frac{dy}{dx} = \sinh u \frac{du}{dx}$
$y = \sinh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$
$y = \cosh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}$
$y = \tanh^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2} \frac{du}{dx}$
$y = \coth^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2} \frac{du}{dx}$

Example 7

Find the derivatives of the following functions using the Table of Derivatives on p.12 – 14:

(a) $\sin^3 x$

(b) $\sin(x^3)$

(c) $\cos^3(x^2)$

(d) $\ln(\tan 2x)$

(e) $\sec\left(\frac{x^2-1}{x^3+3x}\right)$

(d) $e^{\sinh(x^4 \cos 3x)}$

Solution

$$\begin{aligned}
 \text{(a)} \quad \frac{d}{dx}(\sin^3 x) &= \frac{d}{dx}[(\sin x)^3] = 3(\sin x)^2 \cdot \frac{d}{dx}(\sin x) \leftarrow \text{using } \frac{d}{dx}[u^p] = p u^{p-1} \frac{du}{dx} \\
 &\quad \left\{ \begin{array}{l} \text{outer: } (\cdot)^3 \\ \text{inner: } \sin x \end{array} \right. \quad = 3(\sin x)^2 \cdot \cos x \cdot \underbrace{\frac{dx}{dx}}_{=1} \leftarrow \text{using } \frac{d}{dx}[\sin u] = \cos u \frac{du}{dx} \\
 &= 3 \sin^2 x \cos x
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \frac{d}{dx}[\sin(x^3)] &= \cos(x^3) \cdot \frac{d}{dx}(x^3) \leftarrow \text{using } \frac{d}{dx}[\sin u] = \cos u \frac{du}{dx} \\
 &\quad \left\{ \begin{array}{l} \text{outer: } \sin(\cdot) \\ \text{inner: } x^3 \end{array} \right. \quad = \cos(x^3) \cdot 3x^2 \cdot \underbrace{\frac{dx}{dx}}_{=1} \leftarrow \text{using } \frac{d}{dx}[u^p] = p u^{p-1} \frac{du}{dx} \\
 &= 3x^2 \cos(x^3)
 \end{aligned}$$

$$(c) \frac{d}{dx} [\cos^3(x^2)] = \frac{d}{dx} [(\cos(x^2))^3]$$

outer: $(\cdot)^3$
inner: $\cos(x^2)$

$$= 3(\cos(x^2))^2 \cdot \frac{d}{dx} [\cos(x^2)]$$

outer: $\cos(\cdot)$
inner: x^2

$$\leftarrow \text{using } \frac{d}{dx} [u^p] = p u^{p-1} \frac{du}{dx}$$

$$= 3(\cos(x^2))^2 \cdot \left[-\sin(x^2) \cdot \frac{d}{dx} (x^2) \right] \leftarrow \text{using } \frac{d}{dx} [\cos u] = -\sin u \frac{du}{dx}$$

$$= -3 \cos^2(x^2) \cdot \sin(x^2) \cdot 2x \leftarrow \text{using } \frac{d}{dx} [u^p] = p u^{p-1} \frac{du}{dx}$$

$$= -6x \cos^2(x^2) \sin(x^2)$$

$$(d) \frac{d}{dx} [\ln(\tan 2x)] = \frac{1}{\tan 2x} \cdot \frac{d}{dx} (\tan 2x)$$

outer: $\ln(\cdot)$

inner: $\tan 2x$

outer: $\tan(\cdot)$
inner: $2x$

$$\leftarrow \text{using } \frac{d}{dx} [\log_a u] = \frac{1}{u} \log_a e \frac{du}{dx}$$

$$\therefore \frac{d}{dx} [\ln u] = \frac{d}{dx} [\log_e u] = \frac{1}{u} \underbrace{\log_e e}_{=\ln e=1} \frac{du}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$$

$$= \frac{1}{\tan 2x} \cdot \sec^2(2x) \cdot \frac{d}{dx} (2x) \leftarrow \text{using } \frac{d}{dx} [\tan u] = \sec^2 u \frac{du}{dx}$$

$$= \frac{1}{\tan 2x} \cdot \sec^2(2x) \cdot 2 \leftarrow \text{using } \frac{d}{dx} [cu] = c \frac{du}{dx}$$

$$= 2 \cdot \frac{\cos 2x}{\sin 2x} \cdot \frac{1}{\cos^2 2x}$$

$$= \frac{2}{\sin 2x \cos 2x}$$

(e) $\frac{d}{dx} \left[\sec \left(\frac{x^2-1}{x^3+3x} \right) \right]$ { outer: $\sec(\cdot)$
inner: $\frac{x^2-1}{x^3+3x}$

$$= \sec \left(\frac{x^2-1}{x^3+3x} \right) \tan \left(\frac{x^2-1}{x^3+3x} \right) \cdot \frac{d}{dx} \left(\frac{x^2-1}{x^3+3x} \right) \quad \leftarrow \text{using } \frac{d}{dx} [\sec u] = \sec u \tan u \frac{du}{dx}$$

$$= \sec \left(\frac{x^2-1}{x^3+3x} \right) \tan \left(\frac{x^2-1}{x^3+3x} \right) \cdot \frac{(x^3+3x) \cdot \frac{d}{dx}(x^2-1) - (x^2-1) \cdot \frac{d}{dx}(x^3+3x)}{(x^3+3x)^2} \quad \leftarrow \text{using Quotient rule}$$

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

$$= \sec \left(\frac{x^2-1}{x^3+3x} \right) \tan \left(\frac{x^2-1}{x^3+3x} \right) \cdot \frac{(x^3+3x)(2x) - (x^2-1)(3x^2+3)}{(x^3+3x)^2} \quad \leftarrow \text{using } \frac{d}{dx} [u^p] = p u^{p-1} \frac{du}{dx}$$

$$= \sec \left(\frac{x^2-1}{x^3+3x} \right) \tan \left(\frac{x^2-1}{x^3+3x} \right) \cdot \frac{(-x^4+6x^2+3)}{(x^3+3x)^2}$$

$$(f) \quad \frac{d}{dx} \left[e^{\sinh(x^4 \cos 3x)} \right]$$

$\left\{ \begin{array}{l} \text{outer: } e^{(\cdot)} \\ \text{inner: } \sinh(x^4 \cos 3x) \end{array} \right\}$
 $\left\{ \begin{array}{l} \text{outer: } \sinh(\cdot) \\ \text{inner: } x^4 \cos 3x \end{array} \right\}$

$$= e^{\sinh(x^4 \cos 3x)} \cdot \frac{d}{dx} [\sinh(x^4 \cos 3x)] \quad \leftarrow \text{using } \frac{d}{dx} [e^u] = e^u \frac{du}{dx}$$

$$= e^{\sinh(x^4 \cos 3x)} \cdot \cosh(x^4 \cos 3x) \cdot \frac{d}{dx} (x^4 \cos 3x) \quad \leftarrow \text{using } \frac{d}{dx} [\sinh u] = \cosh u \frac{du}{dx}$$

$$= e^{\sinh(x^4 \cos 3x)} \cdot \cosh(x^4 \cos 3x) \cdot \left[x^4 \cdot \frac{d}{dx} (\cos 3x) + \cos 3x \cdot \frac{d}{dx} (x^4) \right]$$

$$\left\{ \begin{array}{l} \text{outer: } \cos(\cdot) \\ \text{inner: } 3x \end{array} \right\} \quad \leftarrow \text{using Product rule } \frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$= e^{\sinh(x^4 \cos 3x)} \cdot \cosh(x^4 \cos 3x) \cdot \left[x^4 \cdot (-\sin 3x) \cdot \underbrace{\frac{d}{dx} (3x)}_{=3} + \cos 3x \cdot (4x^3) \right]$$

$$\leftarrow \text{using } \frac{d}{dx} [\cos u] = -\sin u \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx} [u^p] = p u^{p-1} \frac{du}{dx}$$

$$= e^{\sinh(x^4 \cos 3x)} \cdot \cosh(x^4 \cos 3x) \cdot [-3x^4 \sin 3x + 4x^3 \cos 3x]$$

Example

Find $\frac{dy}{dx}$ for the function $y = \sqrt[3]{x^2 + \sqrt{x^4 + 1}}$.

Solution:

$$y = \sqrt[3]{x^2 + \sqrt{x^4 + 1}} = \left[x^2 + (x^4 + 1)^{\frac{1}{2}} \right]^{\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{1}{3} \left[x^2 + (x^4 + 1)^{\frac{1}{2}} \right]^{-\frac{2}{3}} \cdot \frac{d}{dx} \left[x^2 + (x^4 + 1)^{\frac{1}{2}} \right]$$

$$= \frac{1}{3} \left[x^2 + (x^4 + 1)^{\frac{1}{2}} \right]^{-\frac{2}{3}} \cdot \left[2x + \frac{1}{2}(x^4 + 1)^{-\frac{1}{2}} \cdot \frac{d}{dx}(x^4 + 1) \right]$$

$$= \frac{1}{3} \left[x^2 + (x^4 + 1)^{\frac{1}{2}} \right]^{-\frac{2}{3}} \cdot \left[2x + \frac{1}{2}(x^4 + 1)^{-\frac{1}{2}} \cdot 4x^3 \right]$$

$$= \frac{1}{3} \left[x^2 + (x^4 + 1)^{\frac{1}{2}} \right]^{-\frac{2}{3}} \cdot \left[2x + 2x^3(x^4 + 1)^{-\frac{1}{2}} \right]$$

Differentiation of functions in the form of u^p , a^u , e^u and u^v

(where u and v are functions of x , and $a (> 0)$ and p are constants.)

(i) If $\boxed{y = u^p}$, then $\boxed{\frac{dy}{dx} = p u^{p-1} \frac{du}{dx}}$.

u^p : (function of x)^(constant)

Example 8

$$\begin{aligned}\frac{d}{dx} \left[\frac{1}{\sqrt{x+\sqrt{x}}} \right] &= \frac{d}{dx} \left[\left(x + x^{\frac{1}{2}} \right)^{-\frac{1}{2}} \right] \\ &= -\frac{1}{2} \left(x + x^{\frac{1}{2}} \right)^{-\frac{1}{2}-1} \cdot \frac{d}{dx} \left(x + x^{\frac{1}{2}} \right) \\ &= -\frac{1}{2} \left(x + x^{\frac{1}{2}} \right)^{-\frac{3}{2}} \cdot \left(1 + \frac{1}{2} x^{\frac{1}{2}-1} \right) \\ &= -\frac{1}{2} \left(x + x^{\frac{1}{2}} \right)^{-\frac{3}{2}} \cdot \left(1 + \frac{1}{2} x^{-\frac{1}{2}} \right)\end{aligned}$$

(ii) If $y = a^u$, then $\frac{dy}{dx} = a^u \log_e a \frac{du}{dx}$, where $\log_e a = \ln a$.

a^u : (constant)^(function of x)

Example 9

$$\begin{aligned}
 \frac{d}{dx} [3^{\sin^{-1}(5x)}] &= 3^{\sin^{-1}(5x)} \log_e 3 \cdot \frac{d}{dx} [\sin^{-1}(5x)] \\
 &= 3^{\sin^{-1}(5x)} (\ln 3) \cdot \frac{1}{\sqrt{1-(5x)^2}} \cdot \underbrace{\frac{d}{dx}(5x)}_{=5} \\
 &= 3^{\sin^{-1}(5x)} (\ln 3) \cdot \frac{5}{\sqrt{1-(5x)^2}}
 \end{aligned}$$

Handwritten notes for Example 9:

- For $3^{\sin^{-1}(5x)}$: { outer: $3^{(\cdot)}$, inner: $\sin^{-1}(5x)$ }
- For $\sin^{-1}(5x)$: { outer: $\sin^{-1}(\cdot)$, inner: $5x$ }

An alternative method called logarithmic differentiation will be discussed later.

Remark:

$$\sin^{-1}x \neq (\sin x)^{-1}.$$

($\sin^{-1}x$ is the inverse function of $\sin x$, and $(\sin x)^{-1}$ is the reciprocal of $\sin x$.)

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \quad \text{but} \quad \frac{d}{dx}[(\sin x)^{-1}] = \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

(iii) If $y = e^u$, then $\frac{dy}{dx} = e^u \frac{du}{dx}$.

Example 10

$$\begin{aligned}\frac{d}{dx} [e^{\tan^2(3x)}] &= e^{\tan^2(3x)} \cdot \frac{d}{dx} [\tan^2(3x)] \\ &= e^{\tan^2(3x)} \cdot 2[\tan(3x)]^{2-1} \cdot \frac{d}{dx} (\tan 3x) \\ &= e^{\tan^2(3x)} \cdot 2 \tan(3x) \cdot \sec^2(3x) \cdot 3 \\ &= 6 e^{\tan^2(3x)} \cdot \tan(3x) \cdot \sec^2(3x)\end{aligned}$$

Handwritten notes in red:

- For $e^{\tan^2(3x)}$:
 - outer: $e(\cdot)$
 - inner: $\tan^2(3x)$
- For $\tan^2(3x)$ (in the first step):
 - outer: $(\cdot)^2$
 - inner: $\tan 3x$
- For $\tan 3x$ (in the second step):
 - outer: $\tan(\cdot)$
 - inner: $3x$

(iv) If $y = u^v$, then $\frac{dy}{dx} = v u^{v-1} \frac{du}{dx} + u^v \log_e u \frac{dv}{dx}$, where $\log_e u = \ln u$.

Example 11

u^v : (function of x)^(function of x)

$$\begin{aligned}
 \frac{d}{dx} [(\log_e(2x))^{x^3}] &= x^3 [\ln(2x)]^{x^3-1} \cdot \frac{d}{dx} [\ln(2x)] + [\ln(2x)]^{x^3} \cdot \ln(\ln(2x)) \cdot \frac{d}{dx} (x^3) \\
 &= x^3 [\ln(2x)]^{x^3-1} \cdot \frac{1}{2x} \cdot \underbrace{\frac{d}{dx} (2x)}_{=2} + [\ln(2x)]^{x^3} \cdot \ln(\ln(2x)) \cdot 3x^2 \\
 &= x^2 [\ln(2x)]^{x^3} \cdot \left[\frac{1}{\ln(2x)} + 3 \ln(\ln(2x)) \right]
 \end{aligned}$$

$(\ln(2x))^{x^3}$
 $\uparrow$
 u^v form
 where $u = \ln(2x)$
 $v = x^3$

An alternative method called logarithmic differentiation will be discussed later.

Implicit Differentiation

A function written in the form $y = f(x)$ is called an **explicit function**, where x is the independent variable, y is the dependent variable, and $f(x)$ is a function of x .

The first derivative of y with respect to x is $\frac{dy}{dx} = f'(x)$.

When the function is written as the form $F(x, y) = 0$ instead of $y = f(x)$, where $F(x, y)$ is a function of both x and y , then we say that $F(x, y) = 0$ is an **implicit function**.

↑
 y depends on x

Question: How to find $\frac{dy}{dx}$ when the function is expressed in **implicit** form?

Answer: By using **implicit differentiation**:

Step 1: Differentiate (using chain rule) both sides of $F(x, y) = 0$ w.r.t. x .

Step 2: Rearrange the expression to get

$$\frac{dy}{dx} = \dots \text{ (where the R.H.S. may contain both } x \text{ and } y.)$$

Assume that y is a function of x .

Then $\frac{d(x^2)}{dx} = 2x$

$$\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} = 2y \frac{dy}{dx}$$

chain rule

$$\frac{d(\sin y)}{dx} = \cos y \cdot \frac{dy}{dx}$$

$$\frac{d[\cos(y^2)]}{dx} = \frac{d[\cos(y^2)]}{dy} \cdot \frac{dy}{dx}$$

chain rule

$$= -\sin(y^2) \cdot \frac{d(y^2)}{dy} \cdot \frac{dy}{dx}$$

chain rule

$$= -\sin(y^2) \cdot 2y \cdot \frac{dy}{dx}$$

$$\frac{d(x^2 y^3)}{dx} = x^2 \cdot \frac{d}{dx}(y^3) + y^3 \cdot \frac{d(x^2)}{dx}$$

product rule

$$= x^2 \cdot 3y^2 \cdot \frac{dy}{dx} + y^3 \cdot 2x$$

$$\begin{aligned}
 \frac{d[\ln(xe^y+1)]}{dx} &= \frac{d[\ln(xe^y+1)]}{d(xe^y+1)} \cdot \frac{d(xe^y+1)}{dx} \quad \text{chain rule} \\
 &= \frac{1}{xe^y+1} \cdot \left[x \cdot \frac{d(e^y)}{dx} + e^y \cdot \frac{dx}{dx} + 0 \right] \quad \text{product rule} \\
 &= \frac{1}{xe^y+1} \left(x \cdot \boxed{e^y \cdot \frac{dy}{dx}} + e^y \cdot 1 \right)
 \end{aligned}$$

Example 12

Given $y^2 = x^2 + x \sin y$. Find $\frac{dy}{dx}$.

(y depends on x)

Solution

Note that $y^2 = x^2 + x \sin y$ is an **implicit** function.

Differentiate both sides of the equation with respect to (w.r.t.) x , we get

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(x \sin y)$$

chain rule: $\frac{d}{dy}(y^2) \cdot \frac{dy}{dx}$ $\Rightarrow 2y \frac{dy}{dx} = 2x + \underbrace{\left[x \frac{d}{dx}(\sin y) + \sin y \cdot \frac{d}{dx}(x) \right]}_{\text{by product rule}}$ chain rule: $\frac{d}{dx}(\sin y)$

$\Rightarrow 2y \frac{dy}{dx} = 2x + \left[x \cdot \cos y \frac{dy}{dx} + \sin y \cdot 1 \right]$ $= \frac{d}{dy}(\sin y) \cdot \frac{dy}{dx}$

$\Rightarrow (2y - x \cos y) \frac{dy}{dx} = 2x + \sin y$ $= \cos y \cdot \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2x + \sin y}{2y - x \cos y}$

Note that when differentiating an implicit function, the answer may contain both x and y .

Example 13

Find $\frac{dy}{dx}$ of the function $\ln(\sec x) + \tan^{-1} y = e^{xy}$.

Solution

Differentiate both sides w.r.t. x :

$$\begin{aligned}
 & \frac{d}{dx} [\ln(\sec x)] + \frac{d}{dx} (\tan^{-1} y) = \frac{d}{dx} (e^{xy}) \\
 \Rightarrow & \frac{1}{\sec x} \cdot \frac{d(\sec x)}{dx} + \frac{1}{1+y^2} \cdot \frac{dy}{dx} = e^{xy} \cdot \frac{d(xy)}{dx} \quad \frac{d}{dy}(\tan^{-1} y) \cdot \frac{dy}{dx} \\
 \Rightarrow & \frac{1}{\sec x} \cdot \sec x \tan x + \frac{1}{1+y^2} \frac{dy}{dx} = e^{xy} \left(x \frac{dy}{dx} + y \right) \quad \text{product rule} \\
 \Rightarrow & \tan x + \frac{1}{1+y^2} \frac{dy}{dx} = e^{xy} \left(x \frac{dy}{dx} + y \right) \\
 \Rightarrow & \left(\frac{1}{1+y^2} - x e^{xy} \right) \frac{dy}{dx} = y e^{xy} - \tan x \\
 \Rightarrow & \frac{dy}{dx} = \frac{y e^{xy} - \tan x}{\frac{1}{1+y^2} - x e^{xy}}
 \end{aligned}$$

Example 14

Given $xy^2 + 4(x + y)^3 = \frac{y}{x}$. Find $\frac{dy}{dx}$.

Solution

Differentiate both sides w.r.t. x :

$$\begin{aligned}
 & \frac{d}{dx}(xy^2) + 4 \frac{d}{dx}[(x+y)^3] = \frac{d}{dx}\left(\frac{y}{x}\right) \\
 \Rightarrow & \underbrace{x \cdot \frac{d(y^2)}{dx} + y^2 \cdot \frac{d(x)}{dx}}_{\text{product rule}} + 4 \left\{ \underbrace{3(x+y)^2 \cdot \frac{d(x+y)}{dx}}_{\text{chain rule}} \right\} = \underbrace{\frac{x \cdot \frac{dy}{dx} - y \cdot \frac{d(x)}{dx}}{x^2}}_{\text{Quotient rule}} \\
 \Rightarrow & x \cdot 2y \frac{dy}{dx} + y^2 + 4 \left\{ 3(x+y)^2 \cdot \left(1 + \frac{dy}{dx}\right) \right\} = \frac{x \frac{dy}{dx} - y}{x^2} \\
 \Rightarrow & 2xy \frac{dy}{dx} + y^2 + 12(x+y)^2 + 12(x+y)^2 \frac{dy}{dx} = \frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2} \\
 \Rightarrow & \left[2xy + 12(x+y)^2 - \frac{1}{x} \right] \frac{dy}{dx} = -\frac{y}{x^2} - y^2 - 12(x+y)^2 \\
 \Rightarrow & \frac{dy}{dx} = \frac{-\frac{y}{x^2} - y^2 - 12(x+y)^2}{2xy + 12(x+y)^2 - \frac{1}{x}}
 \end{aligned}$$

Homework: Given that $x^3 + 2xy^2 - y^3 + x - 1 = 0$, find $\frac{dy}{dx}$. [Ans.: $\frac{dy}{dx} = \frac{3x^2 + 2y^2 + 1}{3y^2 - 4xy}$]

Differentiation of inverse functions

Let $f: I \rightarrow \mathbb{R}$ be a differentiable and strictly monotonic (i.e. strictly increasing or strictly decreasing) function of x . Thus, f is a one-to-one function.

The inverse of f , denoted by f^{-1} , is defined as

$$f^{-1}(y) = x, \quad \text{where } y = f(x).$$

If $f'(x) \neq 0$ for all x in I and f^{-1} is differentiable at y , where $y = f(x)$, then

or written as

$$\begin{aligned} x &= f^{-1}(y) \\ \text{Diff. both sides} \\ \text{w.r.t. } y: \\ \frac{dx}{dy} &= (f^{-1})'(y) \\ &= \frac{1}{f'(x)} = \frac{1}{\frac{dy}{dx}} \end{aligned}$$

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

$$\begin{aligned} y &= f(x) \Rightarrow x = f^{-1}(y) \\ \text{Diff. both sides w.r.t. } x: \\ \underbrace{\frac{dx}{dx}}_{=1} &= (f^{-1})'(y) \cdot \underbrace{\frac{dy}{dx}}_{=f'(x)} \\ \Rightarrow (f^{-1})'(y) &= \frac{1}{f'(x)} \end{aligned}$$

This is known as the **inverse function theorem**. Hence,

$$\star \quad \boxed{\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}}$$

Example 15

Consider $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \mathbb{R}$, where $f(x) = \sin x$. Then $f(x)$ is one-to-one and its inverse function is $f^{-1}(x) = \sin^{-1} x$. If $y = \sin^{-1} x$, what is $\frac{dy}{dx}$?

Solution

Method 1: Use the **inverse function theorem**

$$y = \sin^{-1} x \Rightarrow x = \sin y$$

Differentiate both sides with respect to y :

$$\frac{dx}{dy} = \cos y$$

By the **inverse function theorem**,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}},$$

Do not write
 $\frac{1}{\cos(\sin^{-1} x)}$

which is only defined when $-1 < x < 1$.

$$\cos^2 y + \sin^2 y = 1$$

$$\Rightarrow \cos y = \pm \sqrt{1 - \sin^2 y}$$

However, $\cos y > 0$ when $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

$\therefore$ Take positive $\sqrt{}$.

S	A
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