

EF2452: Mathematics for Economics and Finance

Lecture 8

Part II of II

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Main Topics

- Linear equations in matrix form
- Solution of linear equations by row operations
- Echelon form
- Gaussian elimination
- Infinitely many solutions; no solutions
- Consistent and inconsistent systems

Linear Equations in Matrix Form

- We will illustrate the method of finding solutions to a system of linear equations through row operations using a system of three linear equations involving three variables x_1, x_2, x_3 .
- For example, consider the following system

$$x_1 + x_2 + x_3 = 2$$

$$2x_1 - x_2 + 3x_3 = -7$$

$$x_1 + x_2 - x_3 = 6$$

- We can write the above equations more compactly in matrix form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & -1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ 6 \end{pmatrix}$$

Augmented Matrix

- The augmented matrix of the system is

$$(A|\mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -1 & 3 & -7 \\ 1 & 1 & -1 & 6 \end{array} \right)$$

Elementary Row Operations

It is a simple observation that the set of solutions of a system of linear equations is unaltered by the following three operations:

- Multiply both sides of an equation by a non-zero constant
- Add a multiple of one equation to another
- Interchange two equations

Method 1

- The augmented matrix is

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array} \right)$$

where * indicates values (or numbers).

- The goal is to use elementary row operations to reduce the augmented matrix to the following echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array} \right)$$

Method 1

- A general step-by-step method utilizes the following:

$$\begin{aligned}
 & \left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array} \right) \\
 & \rightarrow \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} * & * & 0 & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} * & * & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \end{array} \right) \\
 & \rightarrow \left(\begin{array}{ccc|c} * & 0 & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array} \right)
 \end{aligned}$$

Method 1: Example

- Consider the system of equations

$$x_1 + x_2 + x_3 = 2$$

$$2x_1 - x_2 + 3x_3 = -7$$

$$x_1 + x_2 - x_3 = 6$$

- The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -1 & 3 & -7 \\ 1 & 1 & -1 & 6 \end{array} \right)$$

- We will now use elementary row operations to reduce the augmented matrix to echelon form (see next page).

Method 1: Example

$$\begin{aligned}
 &\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -1 & 3 & -7 \\ 1 & 1 & -1 & 6 \end{array} \right) \xrightarrow{R_1 - R_3 \rightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -1 & 3 & -7 \\ 0 & 0 & 2 & -4 \end{array} \right) \\
 &\xrightarrow{-2R_1 + R_2 \rightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & 1 & -11 \\ 0 & 0 & 2 & -4 \end{array} \right) \\
 &\xrightarrow{R_1 - \frac{1}{2}R_3 \rightarrow R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & -3 & 1 & -11 \\ 0 & 0 & 2 & -4 \end{array} \right) \\
 &\xrightarrow{R_3 - 2R_2 \rightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 6 & 0 & 18 \\ 0 & 0 & 2 & -4 \end{array} \right)
 \end{aligned}$$

Method 1: Example

$$\begin{aligned}
 &\left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 6 & 0 & 18 \\ 0 & 0 & 2 & -4 \end{array} \right) \xrightarrow{R_2 - 6R_1 \rightarrow R_3} \left(\begin{array}{ccc|c} -6 & 0 & 0 & -6 \\ 0 & 6 & 0 & 18 \\ 0 & 0 & 2 & -4 \end{array} \right) \\
 &\xrightarrow{\frac{-1}{6}R_1 \rightarrow R_1 \quad \& \quad \frac{1}{6}R_2 \rightarrow R_2 \quad \& \quad \frac{1}{2}R_3 \rightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array} \right)
 \end{aligned}$$

Therefore,

$$x_1 = 1$$

$$x_2 = 3$$

$$x_3 = -2$$

Method 2

- A more popular method is known as Gaussian elimination with back substitution
- The augmented matrix is

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array} \right)$$

where * indicates values (or numbers).

- The goal is to use elementary row operations to reduce the augmented matrix to the following echelon form

$$\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array} \right)$$

Method 2: Gaussian Elimination with Back Substitution

- Gaussian elimination with back substitution involves the following:

$$\begin{pmatrix} * & * & * & | & * \\ * & * & * & | & * \\ * & * & * & | & * \end{pmatrix} \rightarrow \begin{pmatrix} * & * & * & | & * \\ * & * & * & | & * \\ 0 & * & * & | & * \end{pmatrix} \rightarrow \begin{pmatrix} * & * & * & | & * \\ 0 & * & * & | & * \\ 0 & * & * & | & * \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} * & * & * & | & * \\ 0 & * & * & | & * \\ 0 & 0 & * & | & * \end{pmatrix} \rightarrow \begin{pmatrix} 1 & * & * & | & * \\ 0 & 1 & * & | & * \\ 0 & 0 & 1 & | & * \end{pmatrix}$$

- Then, repeated back substitution will give the values of the variables. Let's go through an example.

Gaussian Elimination with Back Substitution: Example

- Consider the system of equations

$$x_1 + 2x_2 - 3x_3 = 1$$

$$2x_1 - x_2 + x_3 = 1$$

$$x_1 + 4x_2 - 2x_3 = 9$$

- The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & -1 & 1 & 1 \\ 1 & 4 & -2 & 9 \end{array} \right)$$

- We will now use elementary row operations to reduce the augmented matrix to echelon form (see next page).

Gaussian Elimination with Back Substitution: Example

$$\begin{aligned}
 &\left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & -1 & 1 & 1 \\ 1 & 4 & -2 & 9 \end{array} \right) \xrightarrow{R_3 - R_1 \rightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & -1 & 1 & 1 \\ 0 & 2 & 1 & 8 \end{array} \right) \\
 &\xrightarrow{2R_1 - R_2 \rightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 5 & -7 & 1 \\ 0 & 2 & 1 & 8 \end{array} \right) \\
 &\xrightarrow{2R_2 - 5R_3 \rightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 5 & -7 & 1 \\ 0 & 0 & -19 & -38 \end{array} \right) \\
 &\xrightarrow{\frac{1}{5}R_2 \rightarrow R_2 \quad \& \quad \frac{-1}{19}R_3 \rightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & -7/5 & 1/5 \\ 0 & 0 & 1 & 2 \end{array} \right)
 \end{aligned}$$

Gaussian Elimination with Back Substitution: Example

- The final augmented matrix is in the following Echelon form

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & -7/5 & 1/5 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

- The above represents the system

$$x_1 + 2x_2 - 3x_3 = 1$$

$$x_2 - 7/5x_3 = 1/5$$

$$x_3 = 2$$

- Then, the solution is obtained by back-substitution

$$x_3 = 2, \quad x_2 = 1/5 + \left(\frac{7}{5}\right)(2) = 3, \quad x_1 = 1 - (2)(3) + (3)(2) = 1$$

Consistent and Inconsistent Systems

- We now know how to perform elementary row operations on the augmented matrix representing a linear system and reduce it to the standard echelon form.
- So far, we have systems with exactly one solution. However, a linear system may have no solutions at all or infinitely many solutions.

Inconsistent System: No Solutions

- Consider the system of equations

$$x_1 + 2x_2 + x_3 = 1$$

$$2x_1 + 2x_2 = 2$$

$$3x_1 + 4x_2 + x_3 = 2$$

- The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & 2 \\ 3 & 4 & 1 & 2 \end{array} \right)$$

- Use elementary row operations to reduce the augmented matrix to echelon form (see next page).

Inconsistent System: No Solutions

$$\begin{aligned}
 \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & 2 \\ 3 & 4 & 1 & 2 \end{array} \right) &\xrightarrow{3R_1 - R_3 \rightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & 2 \\ 0 & 2 & 2 & 1 \end{array} \right) \\
 &\xrightarrow{2R_1 - R_2 \rightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 1 \end{array} \right) \\
 &\xrightarrow{R_3 - R_2 \rightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \\
 &\xrightarrow{\frac{1}{2}R_2 \rightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)
 \end{aligned}$$

Inconsistent System: No Solutions

- Thus, the original system of equations is equivalent to the system

$$x_1 + 2x_2 + x_3 = 1$$

$$x_2 + x_3 = 0$$

$$0x_1 + 0x_2 + 0x_3 = 1$$

- This system has no solutions since there are no values of x_1, x_2, x_3 which satisfy the last equation. It reduces to the false statement $0 = 1$, whatever values we give the unknowns.
- We deduce, therefore, that the original system has no solutions.

Inconsistent System

- In the preceding example it turned out that the echelon form has a row of the kind $(0\ 0\ \dots\ 0\ |\ a)$, with $a \neq 0$.
- This means that the original system is equivalent to one in which there is an equation

$$0x_1 + 0x_2 + \dots + 0x_n = a \quad (a \neq 0)$$

- Clearly, the above equation cannot be satisfied by any values of the x_i s, and we say that such a system is ***inconsistent***.
- If there is no row of this kind, the system has at least one solution, and we say that it is ***consistent***.

References

Following chapters of the textbook:

16 Linear Equations I

17 Linear Equations II