

MA2001 Solution to Problem Set 5 Vector Integral Calculus

1. Find the work done in moving a particle from $(0,0)$ to $(1,1)$ in the force field

$\vec{F} = (xy + 2y^2)\vec{i} + (3x^2 + y)\vec{j}$ along the paths, (a) $y = x^2$; (b) $y = x$; (c) the y -axis and then $y = 1$.

What work is done if the particle moves from $(0,0)$ to $(1,1)$ along path (b) and returns to the origin along path (a)?

Solution:

(a)

$C_1 : y = x^2$ from $(0,0)$ to $(1,1)$.

$$\begin{aligned}\int_{C_1} \vec{F} \cdot d\vec{r} &= \int_{C_1} [(xy + 2y^2)\vec{i} + (3x^2 + y)\vec{j}] \cdot (dx\vec{i} + dy\vec{j}) = \int_{C_1} (xy + 2y^2)dx + (3x^2 + y)dy \\ &= \int_0^1 (x^3 + 2x^4)dx + \int_0^1 4ydy = \frac{53}{20}\end{aligned}$$

(b)

$C_2 : y = x$ from $(0,0)$ to $(1,1)$.

$$\begin{aligned}\int_{C_2} \vec{F} \cdot d\vec{r} &= \int_{C_2} [(xy + 2y^2)\vec{i} + (3x^2 + y)\vec{j}] \cdot (dx\vec{i} + dy\vec{j}) = \int_{C_2} (xy + 2y^2)dx + (3x^2 + y)dy \\ &= \int_0^1 (x^2 + 2x^2)dx + \int_0^1 (3y^2 + y)dy = \frac{5}{2}\end{aligned}$$

(c)

$C_3 : \text{the } y\text{-axis and then } y = 1$.

$$\begin{aligned}\int_{C_3} \vec{F} \cdot d\vec{r} &= \int_{C_3} [(xy + 2y^2)\vec{i} + (3x^2 + y)\vec{j}] \cdot (dx\vec{i} + dy\vec{j}) = \int_{C_3} (xy + 2y^2)dx + (3x^2 + y)dy \\ &= \int_0^1 ydy + \int_0^1 (x + 2)dx = 3\end{aligned}$$

The work done when the particle moves from $(0,0)$ to $(1,1)$ along path (b) and returns to the origin along path (a):

$$\oint_{C_2 - C_1} \vec{F} \cdot d\vec{r} = \oint_{C_2} \vec{F} \cdot d\vec{r} - \oint_{C_1} \vec{F} \cdot d\vec{r} = \frac{5}{2} - \frac{53}{20} = -\frac{3}{20}$$

2. Prove that the vector field $\vec{F} = (3x^2 - y)\vec{i} + (2yz^2 - x)\vec{j} + 2y^2z\vec{k}$ is conservative, but not solenoidal.

Hence find a scalar function $f(x, y, z)$ such that $F = \nabla f$ and evaluate $\int_C \vec{F} \cdot d\vec{r}$ along any curve C

joining the point $(0,0,0)$ to the point $(1,2,3)$.

Solution:

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - y & 2yz^2 - x & 2y^2z \end{vmatrix} = (4yz - 4yz)\vec{i} + [-1 - (-1)]\vec{k} = \vec{0}$$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(3x^2 - y) + \frac{\partial}{\partial y}(2yz^2 - x) + \frac{\partial}{\partial z}(2y^2z) = 6x + 2z^2 + 2y^2.$$

$\vec{F}$ is not solenoidal.

$$\nabla f = \vec{F} \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 3x^2 - y \\ \frac{\partial f}{\partial y} = 2yz^2 - x \\ \frac{\partial f}{\partial z} = 2y^2z \end{cases} \Rightarrow f(x, y, z) = x^3 - yx + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = -x + \frac{\partial g}{\partial y}$$

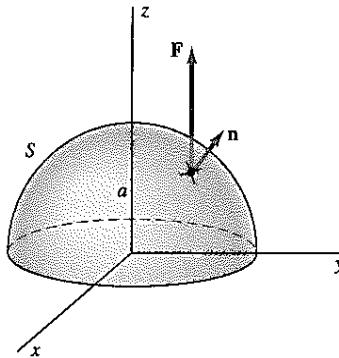
$$2yz^2 - x = -x + \frac{\partial g}{\partial y} \Rightarrow \frac{\partial g}{\partial y} = 2yz^2 \Rightarrow g(y, z) = y^2z^2 + h(z)$$

$$\Rightarrow f(x, y, z) = x^3 - yx + y^2z^2 + h(z)$$

$$\Rightarrow \frac{\partial f}{\partial z} = 2y^2z + h'(z) \Rightarrow 2y^2z = 2y^2z + h'(z) \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$$

$$\text{So } f(x, y, z) = x^3 - yx + y^2z^2 + c. \text{ And } \int_c \vec{F} \cdot d\vec{r} = f(1, 2, 3) - f(0, 0, 0) = 35.$$

3. Calculate the flux $\iint_S \vec{f} \cdot \vec{n} dS$ where $\vec{f} = v_0 \vec{k}$ and S is the hemispherical surface of radius a with equation $z = \sqrt{a^2 - x^2 - y^2}$ and with outer unit normal vector $\vec{n}$.



Solution:

$$z = \sqrt{a^2 - x^2 - y^2} \Rightarrow \varphi(x, y, z) = x^2 + y^2 + z^2 = a^2 \Rightarrow \nabla \varphi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}, \varphi_z = 2z.$$

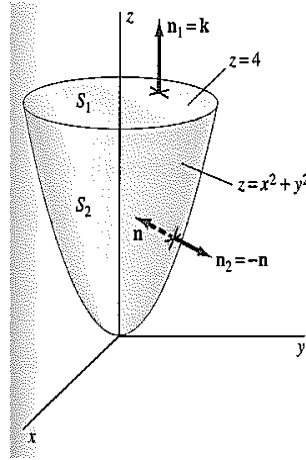
$$\iint_S \vec{f} \cdot \vec{n} dS = \iint_{\sigma_{xy}} \vec{f} \cdot \frac{\nabla \varphi}{|\varphi_z|} dxdy = \iint_{\sigma_{xy}} v_0 \vec{k} \cdot \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{2z} dxdy = \iint_{\sigma_{xy}} v_0 dxdy.$$

σ_{xy} is a circular disk of radius a .

Introducing polar coordinates $dxdy = adrd\theta$ we have

$$\iint_S \vec{f} \cdot \vec{n} dS = \iint_{\sigma_{xy}} \vec{f} \cdot \frac{\nabla \varphi}{|\varphi_z|} dxdy = \iint_{\sigma_{xy}} v_0 dxdy = \int_0^a \left(\int_0^{2\pi} v_0 r d\theta \right) dr = \int_0^a 2\pi v_0 r dr = \pi v_0 r^2 \Big|_0^a = \pi a^2 v_0.$$

4. Find the flux of the vector field $\vec{f} = x\vec{i} + y\vec{j} + 3\vec{k}$ out of S , where S is the closed surface of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. (i.e. to find $\iint_S \vec{F} \cdot d\vec{S}$.)



Solution:

Let S_1 denote the circular top, which has outer unit normal vector $\vec{n}_1 = \vec{k}$.

Let S_2 be the parabolic part of this surface, with outer unit normal vector $\vec{n}_2$.

The flux across S_1 is $\iint_{S_1} (x\vec{i} + y\vec{j} + 3\vec{k}) \cdot \vec{k} dS = \iint_{S_1} 3 dS = 3 \times 2^2 \pi = 12\pi$, because S_1 is a circular disk of radius 2.

To compute $\iint_{S_2} \vec{f} \cdot \vec{n}_2 dS$.

Let $\varphi(x, y, z) = z - x^2 - y^2$, then $\pm \nabla \varphi(x, y, z) = \pm (-2x\vec{i} - 2y\vec{j} + \vec{k})$

We select $-\nabla \varphi(x, y, z) = -(-2x\vec{i} - 2y\vec{j} + \vec{k}) = 2x\vec{i} + 2y\vec{j} - \vec{k}$ and $|\varphi_z| = 1$.

$$\iint_{S_2} \vec{f} \cdot \vec{n}_2 dS = \iint_{\sigma_{xy}} \vec{f} \cdot \frac{-\nabla \varphi}{|\varphi_z|} dx dy = \iint_{\sigma_{xy}} (x\vec{i} + y\vec{j} + 3\vec{k}) \cdot (2x\vec{i} + 2y\vec{j} - \vec{k}) dx dy = \iint_{\sigma_{xy}} (2x^2 + 2y^2 - 3) dx dy$$

σ_{xy} is a circular disk of radius 2.

Introducing polar coordinates $dx dy = r dr d\theta$ we have

$$\iint_{\sigma_{xy}} (2x^2 + 2y^2 - 3) dx dy = \int_0^{2\pi} \left(\int_0^2 (2r^2 - 3) r dr \right) d\theta = 2\pi \left(\frac{r^4}{2} - \frac{3r^2}{2} \right) \Big|_0^2 = 4\pi.$$

Hence the total flux of $\vec{f}$ out of T is 16π .

Remark: You can also evaluate with the use of the Divergence Theorem for this question.

5. Consider the magnetic field $\vec{B} = (x+2)\vec{i} + (1-3y)\vec{j} + 2z\vec{k}$ and evaluate the total magnetic flux through each of the faces of the cube bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

Check that your result is consistent with the divergence theorem.

Solution:

$$\begin{aligned} \iint_S \vec{B} \cdot d\vec{S} &= \iint_{ABCD} \vec{B} \cdot d\vec{S} + \iint_{OEFG} \vec{B} \cdot d\vec{S} + \iint_{OADG} \vec{B} \cdot d\vec{S} + \iint_{EBCF} \vec{B} \cdot d\vec{S} + \iint_{OABE} \vec{B} \cdot d\vec{S} + \iint_{GDCF} \vec{B} \cdot d\vec{S} \\ &= 3 - 2 - 1 - 2 + 0 + 2 = 0 \end{aligned}$$

$$\nabla \cdot \vec{B} = \frac{\partial}{\partial x}(x+2) + \frac{\partial}{\partial y}(1-3y) + \frac{\partial}{\partial z}(2z) = 0. \quad \iint_S \vec{B} \cdot d\vec{S} = \iiint_V \nabla \cdot \vec{B} dx dy dz = \iiint_V 0 dx dy dz = 0.$$

6. Use the divergence theorem to show that $\iint_S (x^2 + y + z) dS = \frac{4}{3}\pi$ where W is the solid ball

$$x^2 + y^2 + z^2 \leq 1 \text{ and } S \text{ is its boundary.}$$

Solution:

$$\text{Let } \phi(x, y, z) = x^2 + y^2 + z^2.$$

$$\text{Then } \vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}} = x\vec{i} + y\vec{j} + z\vec{k}, \text{ note that } x^2 + y^2 + z^2 = 1.$$

$$\vec{F} \cdot \vec{n} = (F_1\vec{i} + F_2\vec{j} + F_3\vec{k}) \cdot (x\vec{i} + y\vec{j} + z\vec{k}) = xF_1 + yF_2 + zF_3 = x^2 + y + z$$

$$\text{Let } xF_1 = x^2, yF_2 = y, zF_3 = z, \text{ then } F_1 = x, F_2 = 1, F_3 = 1. \text{ So } \vec{F} = x\vec{i} + \vec{j} + \vec{k} \text{ and } \nabla \cdot \vec{F} = 1.$$

Thus, by Divergence Theorem,

$$\begin{aligned} \iint_S (x^2 + y + z) dS &= \iint_S (x\vec{i} + \vec{j} + \vec{k}) \cdot (x\vec{i} + y\vec{j} + z\vec{k}) dS \\ &= \iiint_W \nabla \cdot (x\vec{i} + y\vec{j} + z\vec{k}) dx dy dz = \iiint_W 1 dx dy dz = \text{Volume}(W) = \frac{4}{3}\pi (\text{radius of the ball})^3 \\ &= \frac{4}{3}\pi \times 1^3 = \frac{4}{3}\pi \end{aligned}$$

7. Verify the divergence theorem for the vector field $\vec{F} = (8+z)\vec{j} + z^2\vec{k}$ and the region bounded by the planes $z=0$, $z=6$, $x=2$, $y=0$ and the surface $y^2=8x$ in the first octant.

Solutions:

$$\text{For the plane in } x=2, \text{ the outward normal to it is } \vec{i}. \text{ So } \vec{F} \cdot \vec{i} = 0 \text{ and } \iint_{x=2} \vec{F} \cdot \vec{i} dS = 0.$$

$$\text{For the plane in } z=0, \text{ the outward normal to it is } -\vec{k}. \text{ So } \vec{F} \cdot -\vec{k} = -z^2, \text{ however, } z=0 \text{ implies the surface integral on it } = 0.$$

$$\text{For the plane in } y=0, \text{ the outward normal to it is } -\vec{j}. \text{ So } \vec{F} \cdot -\vec{j} = 8+z \text{ and}$$

$$\iint_{y=0} -(8+z) dx dz = \int_0^2 \left[\int_0^6 -(8+z) dz \right] dx = -132.$$

$$\text{For the plane in } z=6, \text{ the outward normal to it is } \vec{k}. \vec{F} \cdot \vec{k} = z^2 \text{ and project the plane in } z=6 \text{ onto the } x\text{-}y \text{ plane. So } \int_0^2 \left(\int_0^{\sqrt{8x}} 36 dy \right) dx = 192$$

$$\text{For the curved surface in } y^2=8x, \text{ let } \phi(x, y, z) = y^2 - 8x \text{ then } \nabla \phi = -8\vec{i} + 2y\vec{j}. \text{ Project the curved surface in } y^2=8x \text{ onto the } x\text{-}z \text{ plane, suppose we have, say, } \sigma_{xz}.$$

$$\iint_{\sigma_{xz}} \vec{F} \cdot \frac{\nabla \phi}{|\phi_y|} dx dz = \int_0^2 \left[\int_0^6 \frac{2y(8+z)}{2y} dz \right] dx = 132$$

The total surface integral = 192.

We have $\nabla \cdot \vec{F} = 2z$ and project the region bounded by the planes $z = 0, z = 6, x = 2, y = 0$ and the surface $y^2 = 8x$ onto the x - y plane, suppose we have, say, σ_{xz} . Then

$$\iiint_V 2z dx dy dz = \iint_{\sigma_{xy}} \left(\int_0^6 z^2 dz \right) dx dy = \int_0^2 \left(\int_0^{\sqrt{8x}} 36 dy \right) dx = 192$$

8. Verify Stokes's theorem for the vector field $\vec{F} = (x - y)\vec{i} + 2z\vec{j} + x^2\vec{k}$ where S is the cone $z = \sqrt{x^2 + y^2}$ for $x^2 + y^2 \leq 4$.

Solution:

$$z = \sqrt{x^2 + y^2} \Rightarrow \phi(x, y, z) = z - \sqrt{x^2 + y^2} \Rightarrow \nabla \phi = \pm \left(-\frac{x}{z}\vec{i} - \frac{y}{z}\vec{j} + \vec{k} \right)$$

Now, we choose $\nabla \phi = \left(-\frac{x}{z}\vec{i} - \frac{y}{z}\vec{j} + \vec{k} \right)$, the upper normal which makes an acute angle with the positive direction of z -axis.

In addition, $\nabla \times \vec{F} = -2\vec{i} - 2x\vec{j} + \vec{k}$.

$$\begin{aligned} \iint_S \nabla \times \vec{F} \cdot d\vec{S} &= \iint_{x^2+y^2 \leq 4} (-2\vec{i} - 2x\vec{j} + \vec{k}) \cdot \left(-\frac{x}{z}\vec{i} - \frac{y}{z}\vec{j} + \vec{k} \right) dx dy = \iint_{x^2+y^2 \leq 4} \left(\frac{2x}{\sqrt{x^2+y^2}} + \frac{2xy}{\sqrt{x^2+y^2}} + 1 \right) dx dy \\ &= \int_0^{2\pi} \left[\int_0^2 \left(\frac{2r \cos \theta}{r} + \frac{2r^2 \cos \theta \sin \theta}{r} + 1 \right) r dr \right] d\theta = 4\pi \end{aligned}$$

According to right-handed rule, use the representation $x = 2 \cos \theta, y = 2 \sin \theta, z = 2, 0 \leq \theta \leq 2\pi$ (anti-clockwise)

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} ((x - y)\vec{i} + 2z\vec{j} + x^2\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k}) = \int_0^{2\pi} (x - y)dx + 2zdy + x^2dz \\ &= \int_0^{2\pi} (2 \cos \theta - 2 \sin \theta)(-2 \sin \theta)d\theta + 8 \cos \theta d\theta = 4\pi \end{aligned}$$

9. Verify Stokes's theorem by evaluating both sides of $\iint_S \nabla \times \vec{F} \cdot \vec{n} dS = \oint_C \vec{F} \cdot d\vec{r}$ for the vector field

$\vec{F} = (2x - y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ where S is the curved surface of the hemisphere $x^2 + y^2 + z^2 = 16, z \leq 0$ and C is its boundary.

Solution:

$$x^2 + y^2 + z^2 = 16, z \leq 0 \Rightarrow \phi(x, y, z) = 16 - x^2 - y^2 - z^2 \Rightarrow \pm \nabla \phi = \pm (-2x\vec{i} - 2y\vec{j} - 2z\vec{k}).$$

Now, we choose $\nabla \phi = -2x\vec{i} - 2y\vec{j} - 2z\vec{k}$, the upper normal which makes an acute angle with the positive direction of z -axis (notice $z \leq 0$).

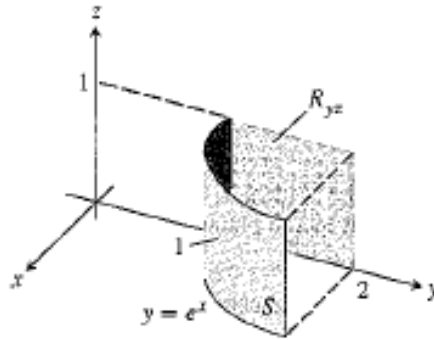
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \vec{k}$$

$$\begin{aligned} \iint_S \nabla \times \vec{F} \cdot d\vec{S} &= \iint_{x^2+y^2 \leq 16} \vec{k} \cdot \left(\frac{-2x\vec{i} - 2y\vec{j} - 2z\vec{k}}{-2z} \right) dx dy = \iint_{x^2+y^2 \leq 16} \vec{k} \cdot \left(\frac{-2x\vec{i} - 2y\vec{j} - 2z\vec{k}}{-2z} \right) dx dy \\ &= \iint_{x^2+y^2 \leq 16} 1 dx dy = 16\pi \end{aligned}$$

According to right-handed rule, use the representation $x = 4 \cos \theta, y = 4 \sin \theta, z = 0$
 $0 \leq \theta \leq 2\pi$ (anti-clockwisely)

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \left((2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k} \right) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k}) \\ &= \int_0^{2\pi} (2x-y)dx - yz^2dy - y^2zdz \\ &= \int_0^{2\pi} (8 \cos \theta - 4 \sin \theta)(-4 \sin \theta) d\theta = \int_0^{2\pi} 16 \sin^2 \theta d\theta \\ &= \int_0^{2\pi} 16 \left(\frac{1 - \cos 2\theta}{2} \right) d\theta = 16\pi \end{aligned}$$

10. Let S be the portion of the cylinder $y = e^x$ in the first octant that projects parallel to the x -axis onto the rectangle $R_{yz} : 1 \leq y \leq 2, 0 \leq z \leq 1$ in the yz -plane. Let $\vec{n}$ be the unit vector normal to S that points away from the yz -plane. Find the flux of the field $\vec{F}(x, y, z) = -2\vec{i} + 2y\vec{j} + z\vec{k}$ across S in the direction of $\vec{n}$.



Solutions:

Let $\varphi(x, y, z) = y - e^x$. Then $y = e^x \Leftrightarrow \varphi(x, y, z) = 0$. $\varphi(x, y, z) = y - e^x \Rightarrow \pm \nabla \varphi = \pm(-e^x\vec{i} + \vec{j})$.

Observe that the normal to S makes an acute angle with $\vec{i}$, as a result, we choose

$-\nabla \varphi = -(-e^x\vec{i} + \vec{j}) = e^x\vec{i} - \vec{j}$ with $|\varphi_x| = e^x$.

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \iint_S \vec{F} \cdot \vec{n} dS = \iint_S \vec{F} \cdot (-\nabla \varphi) \frac{1}{|\varphi_x|} dy dz = \iint_S (-2\vec{i} + 2y\vec{j} + z\vec{k}) \cdot (e^x\vec{i} - \vec{j}) \frac{1}{e^x} dy dz \\ &= \iint_S \left(\frac{-2e^x - 2y}{e^x} \right) dy dz = \iint_{y=e^x, R_{yz}} \left(\frac{-2y - 2y}{y} \right) dy dz = \iint_{R_{yz}} -4 dy dz = \int_0^1 \left(\int_1^2 -4 dy \right) dz = -4 \end{aligned}$$

11. Let $\vec{F} = (y^2 - z^2)\vec{i} + (z^2 - x^2)\vec{j} + (x^2 - y^2)\vec{k}$. Use Stoke's Theorem to calculate $\oint_C \vec{F} \cdot d\vec{r}$ where C is the path which is the intersection of the plane $x + y + z = 2$ and the faces of a parallelepiped bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 2$. The direction of the path C is anticlockwise when looking from the positive direction of x -axis.

Solution:

According to Stokes's Theorem, $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot d\vec{S}$.

$$\nabla \times \vec{F} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{pmatrix} = (-2y - 2z)\vec{i} - (2x + 2z)\vec{j} + (-2x - 2y)\vec{k}.$$

$$\phi(x, y, z) = x + y + z \Rightarrow \nabla \phi = \vec{i} + \vec{j} + \vec{k} \text{ and } \phi_z = 1.$$

The normal to $x + y + z = 2$ makes an acute angle with the positive direction of z -axis, and the projection of the plane enclosed by C onto xy -plane is $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$

$$\begin{aligned} \iint_S \nabla \times \vec{F} \cdot d\vec{S} &= \iint_{\substack{0 \leq x \leq 1 \\ 0 \leq y \leq 1}} [(-2y - 2z)\vec{i} - (2x + 2z)\vec{j} + (-2x - 2y)\vec{k}] \cdot \frac{\vec{i} + \vec{j} + \vec{k}}{1} dx dy \\ &= \iint_{\substack{0 \leq x \leq 1 \\ 0 \leq y \leq 1}} (-4x - 4y - 4z) dx dy = \int_0^1 \left(\int_0^1 [-4x - 4y - 4(2 - x - y)] dy \right) dx = \int_0^1 \left(\int_0^1 -8 dy \right) dx = -8 \end{aligned}$$