

Suggested Solution

MA1201 Calculus and Basic Linear Algebra II Midterm Test

Sections
CE, CF, CG, CH

Name:

Student ID:

Score:

Time allowed: 100 minutes

1. Calculate the following indefinite and definite integrals

(i) $\int 6x\sqrt{2x^2+5} dx$

(8 points)

Solution

Let $y = 2x^2 + 5$, then $\frac{dy}{dx} = 4x \Rightarrow dx = \frac{1}{4x} dy$

The integral is transformed into

$$\int 6x\sqrt{2x^2+5} dx = \int 6x\sqrt{2x^2+5} \left(\frac{1}{4x} dy\right)$$

$$\begin{aligned} \sqrt{y} = y^{1/2} &= \frac{3}{2} \int \sqrt{y} dy \\ &= \frac{3}{2} \frac{y^{1/2+1}}{1/2+1} + C = (2x+5)^{3/2} + C. \end{aligned}$$

(ii) $\int \frac{x}{3x+2} dx$

(8 points)

Solution

Let $y = 3x+2 \Rightarrow \frac{dy}{dx} = 3 \Rightarrow dx = \frac{1}{3} dy$

Then the integral becomes

$$\int \frac{x}{3x+2} dx = \frac{1}{3} \int \frac{\frac{y-2}{3}}{y} dy = \frac{1}{9} \int \left(1 - \frac{2}{y}\right) dy$$

$$= \frac{1}{9} y - \frac{2}{9} \ln y + C$$

$$= \frac{1}{9} (3x+2) - \frac{2}{9} \ln(3x+2) + C$$

(iii) $\int x^2 \cos(2x) dx$

(8 points)

Using integration by parts with $u=x^2$, $dv=\cos 2x dx$

$$\int x^2 \cos 2x dx = \frac{1}{2} x^2 \sin 2x - \int \frac{1}{2} \sin 2x d(x^2) \quad \Rightarrow v = \int \cos 2x dx = \frac{1}{2} \sin 2x$$

$$= \frac{1}{2} x^2 \sin 2x - \int x \sin 2x dx \quad \leftarrow u=x$$

$$= \frac{1}{2} x^2 \sin 2x + \frac{1}{2} x \cos 2x$$

$$dv = \sin 2x dx$$

$$\Rightarrow v = \int \sin 2x dx$$

$$- \int \frac{1}{2} \cos 2x dx$$

$$= -\frac{1}{2} \cos 2x$$

$$= \frac{1}{2} x^2 \sin 2x + \frac{1}{2} x \cos 2x - \frac{1}{4} \sin 2x + C.$$

(iv) $\int \sin^2(2x) dx$

product-to-sum

(8 points)

$$\int \sin^2(2x) dx \stackrel{\downarrow}{=} -\frac{1}{2} \int [\cos(2x+2x) - \cos(2x-2x)] dx$$

$$= -\frac{1}{2} \int (\cos 4x - 1) dx$$

$$= -\frac{1}{8} \sin 4x + \frac{1}{2} x + C.$$

(v) $\int \frac{1}{x^3+4x} dx$

(8 points)

Note that Partial fraction

$$\frac{1}{x^3+4x} = \frac{1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\Rightarrow 1 = A(x^2+4) + x(Bx+C)$$

• Put $x=0$, $1=4A \Rightarrow A=\frac{1}{4}$

• Put $x=1$, $1=5A+B+C \Rightarrow B+C=-\frac{1}{4}$

• Put $x=-1$, $1=5A-(-B+C) \Rightarrow B-C=-\frac{1}{4}$

$$\Rightarrow B=-\frac{1}{4} \text{ and } C=0$$

$$\Rightarrow \int \frac{1}{x^3+4x} dx$$

$$= \frac{1}{4} \int \frac{1}{x} dx - \frac{1}{4} \int \frac{x}{x^2+4} dx$$

$$= \frac{1}{4} \ln|x| - \frac{1}{8} \int \frac{1}{y} dy$$

$$= \frac{1}{4} \ln|x| - \frac{1}{8} \ln|y| + C$$

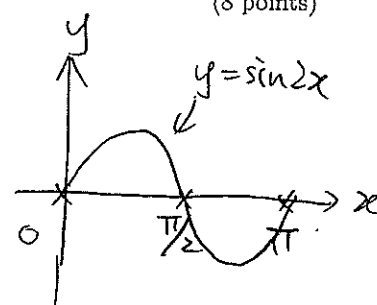
$$= \frac{1}{4} \ln|x| - \frac{1}{8} \ln|x^2+4| + C.$$

(vi) $\int_0^\pi |\sin(2x)| dx$

(8 points)

Note that $|\sin 2x| = \begin{cases} \sin 2x & \text{if } \sin 2x \geq 0 \\ -\sin 2x & \text{if } \sin 2x < 0 \end{cases}$

$$= \begin{cases} \sin 2x & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ -\sin 2x & \text{if } \frac{\pi}{2} \leq x \leq \pi. \end{cases}$$



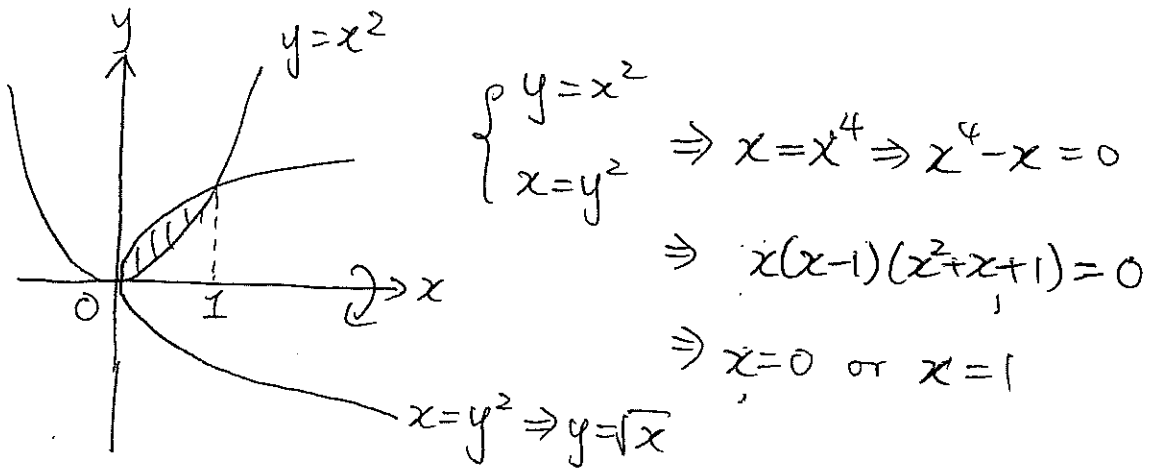
Thus

$$\int_0^\pi |\sin 2x| dx = \int_0^{\pi/2} \sin 2x dx - \int_{\pi/2}^\pi \sin 2x dx$$

$$= -\frac{1}{2} \cos 2x \Big|_0^{\pi/2} + \frac{1}{2} \cos 2x \Big|_{\pi/2}^\pi$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2.$$

- 2.(i) Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^2$ and $x = y^2$ about the x -axis. (12 points)



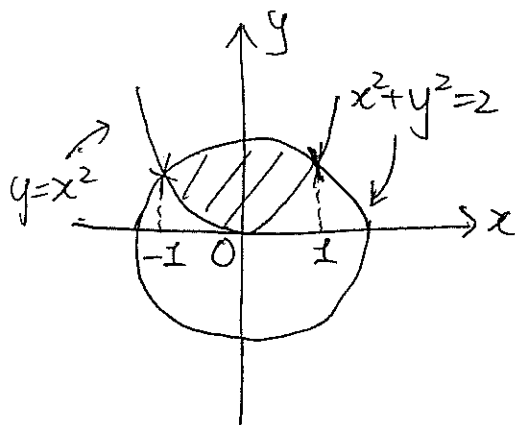
The volume is

$$= \int_0^1 \pi (\sqrt{x})^2 dx - \int_0^1 \pi (x^2)^2 dx$$

$$= \int_0^1 \pi (x - x^4) dx = \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \right) \Big|_0^1 = \frac{3\pi}{10}.$$

2.(ii) Calculate the area of the region below the curve $x^2 + y^2 = 2$ and above the curve $y = x^2$.

(18 points)



$$\begin{cases} x^2 + y^2 = 2 \\ y = x^2 \end{cases} \Rightarrow x^2 + x^4 = 2 \Rightarrow x^4 + x^2 - 2 = 0 \\ \Rightarrow (x^2 + 2)(x^2 - 1) = 0 \\ \Rightarrow x = \pm 1.$$

The area

$$= \int_{-1}^1 (\sqrt{2-x^2} - x^2) dx$$

$$= \int_{-1}^1 \sqrt{2-x^2} dx - \int_{-1}^1 x^2 dx$$

$$\begin{aligned} x &= \sqrt{2} \sin \theta \\ \frac{dx}{d\theta} &= \sqrt{2} \cos \theta \end{aligned} \Rightarrow \int_{-\pi/4}^{\pi/4} \sqrt{2-2\sin^2 \theta} \sqrt{2} \cos \theta d\theta - \frac{x^3}{3} \Big|_{-1}^1$$

$$= \int_{-\pi/4}^{\pi/4} 2 \cos^2 \theta d\theta - \frac{2}{3}$$

Product to sum $\Rightarrow \int_{-\pi/4}^{\pi/4} (1 + \cos 2\theta) d\theta - \frac{2}{3}$

$$= \theta + \frac{\sin 2\theta}{2} \Big|_{-\pi/4}^{\pi/4} - \frac{2}{3}$$

$$= \frac{\pi}{2} - \frac{1}{3}.$$

3. Let $A(1, 0, 6)$, $B(2, 3, -8)$, $C(8, -5, 6)$ and $O(0, 0, 0)$ be four points in $\mathbb{R}^3$,

(i) Calculate the $\vec{OA} \cdot \vec{OC}$ and the angle between the vectors $\vec{OA}$ and $\vec{OC}$.

(8 points)

$$\vec{OA} = \vec{i} + 6\vec{k}$$

$$\vec{OC} = 8\vec{i} - 5\vec{j} + 6\vec{k}$$

$$\Rightarrow \vec{OA} \cdot \vec{OC} = 1 \times 8 + 0 \times (-5) + 6 \times 6 \\ = 44.$$

Let θ be the angle between $\vec{OA}$ and $\vec{OC}$.

$$\text{Then } \cos \theta = \frac{\vec{OA} \cdot \vec{OC}}{|\vec{OA}| |\vec{OC}|}$$

$$|\vec{OA}| = \sqrt{1^2 + 6^2} = \sqrt{37}$$

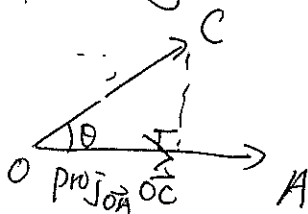
$$|\vec{OC}| = \sqrt{8^2 + (-5)^2 + 6^2} = \sqrt{125}$$

$$\Rightarrow \frac{44}{\sqrt{37} \sqrt{125}} \Rightarrow \theta = 49.7^\circ$$

(ii) Find the projection vector of $\vec{OC}$ on $\vec{OA}$, i.e. $\text{proj}_{\vec{OA}} \vec{OC}$.

(7 points)

Since $\theta < 90^\circ$, the vectors $\vec{OA}$, $\vec{OC}$ should be of the following form:



$$\text{Length of } \text{proj}_{\vec{OA}} \vec{OC} = |\vec{OC}| \cos \theta = \frac{44}{\sqrt{37}}.$$

$$\text{Direction of } \text{proj}_{\vec{OA}} \vec{OC} = \hat{OA} = \frac{\vec{OA}}{|\vec{OA}|} = \frac{1}{\sqrt{37}} (\vec{i} + 6\vec{k})$$

↑
same as $\vec{OA}$

$$\text{Therefore } \text{proj}_{\vec{OA}} \vec{OC} = \underbrace{\frac{44}{\sqrt{37}}}_{\text{magnitude}} \times \underbrace{\frac{1}{\sqrt{37}} (\vec{i} + 6\vec{k})}_{\text{direction}} = \frac{44}{37} \vec{i} + \frac{264}{37} \vec{k}.$$

3.(iii) Find the volume of the parallelepiped determined by $\vec{OA}$, $\vec{OB}$ and $\vec{OC}$.

(7 points)

The volume of parallelepiped

$$\begin{aligned}
 &= \left| \vec{OA} \cdot (\vec{OB} \times \vec{OC}) \right| \quad \vec{OB} \times \vec{OC} \\
 &= \left| (1\vec{i} + 6\vec{k}) \cdot (-22\vec{i} - 52\vec{j} - 34\vec{k}) \right| = (2\vec{i} + 3\vec{j} - 8\vec{k}) \times (8\vec{i} - 5\vec{j} + 6\vec{k}) \\
 &= \left| 1 \times (-22) + 0 \times (-52) + 6 \times (-34) \right| = 16(\vec{i} \times \vec{i}) - 10(\vec{i} \times \vec{j}) + 12(\vec{i} \times \vec{k}) \\
 &= \left| -22 + 0 - 204 \right| = 226. \quad + 24(\vec{j} \times \vec{i}) - 15(\vec{j} \times \vec{j}) + 18(\vec{j} \times \vec{k}) \\
 &\quad - 64(\vec{k} \times \vec{i}) + 40(\vec{k} \times \vec{j}) - 48(\vec{k} \times \vec{k}) \\
 &= -22\vec{i} - 52\vec{j} - 34\vec{k}
 \end{aligned}$$

-End-

