

Part A: Basic Concept**Problem 1**

Simplify the following expression and express your answer in the form of $a + bi$.

(a) $i^5 - i^7 + i^{10}$

(b) $\frac{1-i}{3+2i}$

(c) $(2+i)^2(3-i)$

(d) $(1-3i)^{-1}$

Problem 2

Let $z = a + bi$ be a complex numbers satisfying $\frac{1}{z} = 1 + 3i$. Find the value of z .

Problem 3

Express the following complex numbers in the Polar and Euler form. Write down the modulus and principal argument of each of the complex numbers.

(a) $z_1 = 3 - 3i$

(b) $z_2 = \sqrt{6} + \sqrt{2}i$

(c) $z_3 = -2i$

(d) $z_4 = -4 - \sqrt{48}i$

(e) $z_5 = -1 + 5i$

(f) $z_6 = 5 - 8i$

(g) $z_7 = ie^{\frac{i\pi}{4}}$

(h) $z_8 = -2e^{\frac{i\pi}{3}}$

(i) $z_9 = e^{\frac{i\pi}{5}} + e^{-i\pi}$

(j) $z_{10} = 1 - e^{\frac{i\pi}{4}}$

Problem 4

Express the following complex number in the Euler form and polar form. Here, $0 < \theta < \frac{\pi}{2}$.

(a) $-\cos \theta - i \sin \theta$

(b) $-\sin \theta + i \cos \theta$

(c) $1 - \sin \theta + i \cos \theta$

(d) $1 + \cos \theta - i \sin \theta$

Problem 5

Simplify the following expression using suitable methods. Express your answer in the Polar form or Euler form.

(a) $z_1 = \frac{1 + \sqrt{3}i}{2 - 2i}$

(b) $z_2 = (1 + i)^{-5}$

(c) $z_3 = \left(\frac{(1-i)(\sqrt{3}+i)}{2i} \right)^{12}$

(d) $z_4 = \sqrt[6]{-\sqrt{48} + 4i}$

(e) $z_5 = \sqrt{\frac{4+4i}{(-2+\sqrt{12}i)^3}}$

(f) $z_6 = (-2 - 2i)^{\frac{3}{4}}$

(g) $z_7 = (\sin \theta - i \cos \theta)^5$

(h) $z_8 = \sqrt[4]{1 + e^{\frac{i\pi}{4}}}$

(Hint: For (g) and (h), you need to express the complex number inside the bracket in the polar form first. The techniques used in Problem 3(g)-(j) and Problem 4 will be useful.)

Problem 6

Show that for any $0 < \theta < \frac{\pi}{2}$, we have

$$\left(\frac{1 + i \tan \theta}{1 - i \tan \theta}\right)^5 = \frac{1 + i \tan 5\theta}{1 - i \tan 5\theta}.$$

(Hint: Note that $\tan \theta = \frac{\sin \theta}{\cos \theta}$).

Problem 7 (A bit harder)

Let z be a complex number with $|z| = 1$ and $z \neq \pm 1$.

- (a) Show that the complex number $z_0 = \frac{1+z}{1-\bar{z}}$ is *purely imaginary*. (i.e. $z_0 = bi$ for some real number b .)

(Hint: You may consider the polar form of z : $z = r(\cos \theta + i \sin \theta)$. What can you say about the value of r ?)

- (b) Using the similar technique, show that if $\theta \neq k\pi$, (k is integer) then $z_1 = \frac{1+\bar{z}}{1-z}$ is also purely imaginary.

Problem 8

Let $z = 1 + 3i$ be a complex number, compute the following

- (a) $\frac{3z}{1+\bar{z}}$ (b) $(z + 3\bar{z})^2$
 (c) $\sqrt[4]{z - \bar{z}}$

Problem 9

Using the fact that $z + \bar{z} = 2\operatorname{Re}(z)$ and $z - \bar{z} = 2\operatorname{Im}(z)i$, where $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ represent the real part and imaginary part of z respectively, evaluate

$$\operatorname{Re}\left(\frac{z-1}{z+1}\right) \text{ and } \operatorname{Im}\left(\frac{z-1}{z+1}\right),$$

where z is the complex number with $|z| = 1$. Express your answer in terms of $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ if necessary.

Problem 10

Let z_1 and z_2 be two complex numbers, show that

$$|z_1 + z_2|^2 - |z_1 - z_2|^2 = 4\operatorname{Re}(\bar{z}_1 z_2).$$

(☺Hint: $|z|^2 = z\bar{z}$.)

Application of Complex Number

Problem 11

Solve the following equations

- (a) $z^6 = -3 + \sqrt{3}i$
 (b) $(1-z)^7 + (1+z)^7 = 0$
 (c) $z^{10} - 5z^5 - 6 = 0$
 (d) $z^8 - 2\sqrt{3}z^4 + 4 = 0$
 (e) $\frac{z^5}{1+z^5} = \sqrt{3}i.$

Problem 12

Solve the equation $z^4 - 8z^3 + 27z^2 - 50z + 50 = 0$ given that $3 + i$ is one of the root.

Problem 13

- (a) By considering the expression $(\cos \theta + i \sin \theta)^5$, show that
$$\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta.$$
- (b) Using similar techniques as in (a), show that
$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta.$$

Problem 14

- (a) We let $z = \cos \theta + i \sin \theta$ be a complex number.
By considering the expression $\left(z - \frac{1}{z}\right)^5$ and using the fact that $z^n - \frac{1}{z^n} = 2i \sin n\theta$,
show that
$$\sin^5 \theta = \frac{1}{16} (\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta).$$
- (b) By considering the expression $\left(z - \frac{1}{z}\right)^3 \left(z + \frac{1}{z}\right)^4$ and using the fact that $z^n - \frac{1}{z^n} = 2i \sin n\theta$ and $z^n + \frac{1}{z^n} = 2 \cos n\theta$, show that
$$\sin^3 \theta \cos^4 \theta = -\frac{1}{64} (\sin 7\theta + \sin 5\theta - 3 \sin 3\theta - 3 \sin \theta).$$
- (c) Hence, compute the integral $\int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^4 \theta d\theta$.