

EE3109 Applied Electromagnetics
2022/23 Semester B
Assignment 3 Solution

1. A low loss transmission line has the following properties: $R' \ll j\omega L'$; $G' \ll j\omega C'$. Find expressions for α, β, v_p and Z_0 for the low loss transmission line; show that these expressions approach the lossless counterparts when R' and $G' \rightarrow 0$.

Solution:

Characteristic Impedance:

$$Z_0 = \sqrt{\frac{(R' + j\omega L')}{(G' + j\omega C')}} = \sqrt{\frac{j\omega L' \left[\left(\frac{R'}{j\omega L'} \right) + 1 \right]}{j\omega C' \left[\left(\frac{G'}{j\omega C'} \right) + 1 \right]}} \cong \sqrt{\frac{j\omega L'}{j\omega C'}} = \sqrt{\frac{L'}{C'}}.$$

The highlighted terms are small because of the low loss property, and can hence be neglected.

Propagation Constant:

$$\begin{aligned} \gamma &= \sqrt{(R' + j\omega L')(G' + j\omega C')} = \sqrt{R'G' + (j\omega L')(G') + (R')(j\omega C') + (j\omega L')(j\omega C')} \\ &= j\omega\sqrt{L'C'} \sqrt{\left(\frac{R'}{j\omega L'} \right) \left(\frac{G'}{j\omega C'} \right) + \left(\frac{R'}{j\omega L'} \right) + \left(\frac{G'}{j\omega C'} \right) + 1} \end{aligned}$$

All the bracketed items are small because of the low loss property. The first term (highlighted) is second-order small, which will be neglected in a first order approximation. Using the Taylor series $\sqrt{1+x} = 1 + \frac{x}{2}$ for $x \ll 1$, we have:

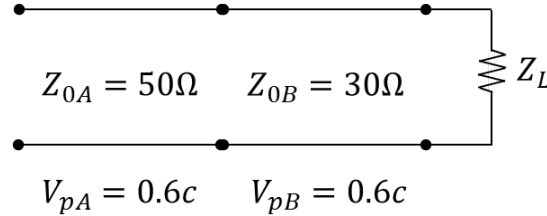
$$\begin{aligned} \gamma &\cong j\omega\sqrt{L'C'} \sqrt{\left(\frac{R'}{j\omega L'} \right) + \left(\frac{G'}{j\omega C'} \right) + 1} \cong j\omega\sqrt{L'C'} \left(1 + \frac{1}{2} \left(\frac{R'}{j\omega L'} + \frac{G'}{j\omega C'} \right) \right) \\ &= j\omega\sqrt{L'C'} + \frac{\sqrt{L'C'}}{2} \left(\frac{R'}{L'} + \frac{G'}{C'} \right) = j\omega\sqrt{L'C'} + \frac{1}{2} \left(\frac{R'}{Z_0} + G'Z_0 \right). \end{aligned}$$

(The last step uses Z_0 derived above.)

$$\begin{aligned} \gamma = \alpha + j\beta &\Rightarrow \alpha = \frac{1}{2} \left(\frac{R'}{Z_0} + G'Z_0 \right); \beta = \omega\sqrt{L'C'}; \\ v_p &= \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}. \end{aligned}$$

From the derivations, we see that Z_0, β and v_p are the same as for a lossless transmission line. α will also vanish as R' and $G' \rightarrow 0$, as expected for a lossless transmission line.

2. (Adapted from a previous test question) A microwave network analyzer with a port impedance of 50Ω measures the input reflection coefficient of a microwave circuit as $\Gamma = 0.1\angle -60^\circ$ at a frequency of 2 GHz. The circuit being measured is pictured below. Both transmission lines are lossless. A time domain measurement with a 5 GHz voltage wave input shows the arrival times of the first two reflection signals at 7.25 ns and 13.5 ns respectively.
- What are the electrical lengths of transmission line sections A and B?
 - What is the reflection coefficient at junction between transmission lines A and B?
 - What is the load impedance?
 - What percentage of the incident power is delivered to the load?
 - Write the current as a function of position along transmission line B.
 - Where does the first voltage minimum appear relative to the load?



Solution:

- (a) We will define l_A, λ_A and l_B, λ_B to be the lengths and phase constants of TL A and TL B respectively.

The first reflection signal arrives when the signal reaches the junction of TLs A and B, then reflects back to the generator. The second reflection signal arrives when the signal reaches the end of TL B then reflects back to the generator. From the round-trip times, we can calculate:

$$2l_A = (7.25 \text{ ns})(V_{pA}) = (7.25 \times 10^{-9} \text{ s})(1.8 \times 10^8 \text{ m/s}) = 1.305 \text{ m} \Rightarrow l_A = 0.653 \text{ m}$$

Similarly:

$$2l_B = ((13.5 - 7.25) \text{ ns})(V_{pB}) = (6.25 \times 10^{-9} \text{ s})(1.8 \times 10^8 \text{ m/s}) = 1.125 \text{ m} \Rightarrow l_B = 0.563 \text{ m}$$

The wavelengths for the lines are:

$$\lambda_A = \frac{V_{pA}}{f} = \frac{0.6 \times 3 \times 10^8}{2 \times 10^9} = 0.09000 \text{ m}; \quad V_{pB} = V_{pA} \Rightarrow \lambda_B = \lambda_A = 0.09000 \text{ m}.$$

So the electrical lengths are:

$$l_A = \frac{0.6525}{0.09000} = 7.250\lambda_A; \quad l_B = \frac{0.5625}{0.09000} = 6.250\lambda_B.$$

- (b) Let Γ_{jxn} be the reflection coefficient at the junction. Then the measured reflection coefficient is the input reflection coefficient when Γ_{jxn} is connected to the generator with TL A:

$$0.1\angle -60^\circ = 0.1e^{-j\frac{\pi}{3}} = \Gamma(z = l_A) = \Gamma_{jxn}e^{-j2\beta_A l_A}.$$

Now $\beta_A l_A = \left(\frac{2\pi}{\lambda_A}\right)(7.250\lambda_A) = 14.5\pi \text{ rad} = 0.5\pi \text{ rad}$. Substitution into the above, and rearranging, gives:

$$\Gamma_{jxn} = \Gamma(z = l_A)e^{+j2\beta_A l_A} = \left(0.1e^{-j\frac{\pi}{3}}\right)(e^{j\pi}) = 0.1e^{j\frac{2\pi}{3}} = 0.1e^{j2.093}.$$

- (c) The impedance at the junction of TL A and TL B is

$$Z_{jxn} = Z_{0A} \frac{1 + \Gamma_{jxn}}{1 - \Gamma_{jxn}} = (50) \frac{1 + 0.1e^{j2.093}}{1 - 0.1e^{j2.093}} = 44.59 + j7.80 \Omega.$$

The load impedance can be considered as the “input impedance” of a transmission line same as TL B, but with a length of $-\lambda_B$, and with Z_{jxn} at the load. That is:

$$Z_L = Z_{0B} \frac{Z_{jxn} + jZ_{0B} \tan(-\beta_B l_B)}{Z_{0B} + jZ_{jxn} \tan(-\beta_B l_B)} = Z_{0B} \frac{Z_{jxn} - jZ_{0B} \tan(\beta_B l_B)}{Z_{0B} - jZ_{jxn} \tan(\beta_B l_B)}.$$

Now $\beta_B l_B = \left(\frac{2\pi}{\lambda_B}\right)(6.25\lambda_B) = 12.5\pi = 0.5\pi \text{ rad} \Rightarrow \tan(\beta_B l_B) = \tan(0.5\pi) = \infty$. Substitute into the above to get

$$Z_L = (30) \frac{(44.59 + j7.80) - j(30)(\infty)}{(30) - j(44.59 + j7.80)(\infty)} = (30) \frac{-j(\infty)(30)}{-j(\infty)(44.59 + j7.80)} = 19.58 - j3.43 [\Omega] .$$

- (d) Since we deal with lossless transmission lines, and since the load is the only resistive component in the microwave circuit, what doesn't get reflected back to the source is transmitted to the load. Hence the percentage of power that is transmitted to the load is:

$$|\tau_L|^2 = 1 - |\Gamma|^2 = 1 - (0.1)^2 = 0.99 = 99\% .$$

- (e) We write the current as a function of position z along TL B, with the load located at $z = 0$:

$$I(z) = \frac{V_0^+}{Z_{0B}} e^{-j\beta_B z} (1 - \Gamma_L e^{j2\beta_B z})$$

Here

$$\beta_B = \frac{2\pi f}{V_{pB}} = \frac{(2\pi \times 2 \times 10^9)}{(0.6 \times 3 \times 10^8)} = 69.81 \text{ rad/m},$$

$$\Gamma_L = \frac{Z_L - Z_{0B}}{Z_L + Z_{0B}} = \frac{19.58 - j3.43 - 30}{19.58 - j3.43 + 30} = -0.2044 - j0.0833 = 0.2207 e^{-j2.753} .$$

Substitution into $I(z)$ above gives

$$I(z) = \frac{V_0^+}{30} e^{-j69.81z} (1 - 0.2207 e^{-j2.753} e^{j139.6z}) [\text{A}] .$$

- (f) The first voltage minimum appear when the phase of the reflected wave component reaches $-\pi$, i.e.

$$-\pi = 2.753 - 139.6 d_{min} \Rightarrow d_{min} = \frac{-\pi + 2.753}{-139.6} = 2.772 [\text{mm}] .$$

Therefore the first voltage minimum appears 2.772 mm away from the load.

3. A $30\ \Omega$ lowloss transmission line is terminated with a microwave device, whose load can be modelled by a series connection of a resistor $R = 15\ \Omega$ and a capacitor $C = 50\ \text{pF}$. The line length is 25cm. The phase velocity for a 500 MHz wave along this line is $1.5 \times 10^8\ \text{[m/s]}$. A voltage wave taking a round trip along the transmission line experiences a loss of 0.1 dB.
- Find the load impedance.
 - Find the phase constant, attenuation constant and wavelength of a voltage wave along this TL.
 - Design an L-section matching circuit that matches this load impedance to the transmission line.
 - Design a single stub tuner that matches this load impedance to the transmission line. Choose the solution that leads to the shortest stub length for an open-circuited shunt stub.
 - Use a shunt reactive load and a quarter-wave transformer to match the load impedance to the transmission line. What is the length and impedance of the quarter-wave transformer?

Solution

- (a) The load impedance is

$$Z_L = R_L - j \left(\frac{1}{\omega C} \right) = 15 - \frac{j}{(2\pi \times 500 \times 10^6)(50 \times 10^{-12})} = 15.00 - j6.37\ [\Omega]$$

- (b) The phase constant is

$$\beta = \frac{\omega}{V_p} = \frac{(2\pi \times 500 \times 10^6)}{(1.5 \times 10^8)} = \frac{20\pi}{3} = 6.667\ \text{rad/m},$$

The attenuation constant is

$$\alpha_{dB} = \frac{0.1\ \text{dB}}{2 \times 0.25\ \text{m}} = 0.2\ \text{dB/m} \Rightarrow \alpha = (0.2)(0.1151) = 0.02302\ [\text{Np/m}].$$

The wavelength is

$$\lambda = \frac{V_p}{f} = \frac{(1.5 \times 10^8)}{(500 \times 10^6)} = 0.3\ [\text{m}].$$

- (c-e) The normalized impedance is $z'_L = \frac{Z_L}{Z_0} = 0.5 - j0.21$. See the Smith Charts in the following pages.
- (c) We use the configuration where the load first encounters the series component, as shown on the Smith Chart. We find that a series capacitor of $C = 36.59\ \text{pF}$ and a shunt inductor $L = 9.549\ \text{nH}$ are needed.
- (d) We use a shunt, open-circuited stub to perform matching. We obtain $d = 41.4\ \text{mm}$; $l = 31.2\ \text{mm}$.
- (e) The diagram of the matching circuit is drawn on the Smith Chart. We first use the shunt reactive component to match the imaginary part of the load impedance. We then transform the real part into the TL impedance of $30\ \Omega$ using a quarter-wave transformer. We used a shunt inductor $L = 13.64\ \text{nH}$ and a quarter-wave transformer with characteristic impedance $Z_m = 23.01\ \Omega$.

L-Section Matching : $Z_L' = 0.5 - j0.2$

$Z_0 = 30\Omega$; $\lambda = 0.3m$;

$f = 500MHz$

From S.C.:

$$Z_A' = 0.5 - j0.5$$

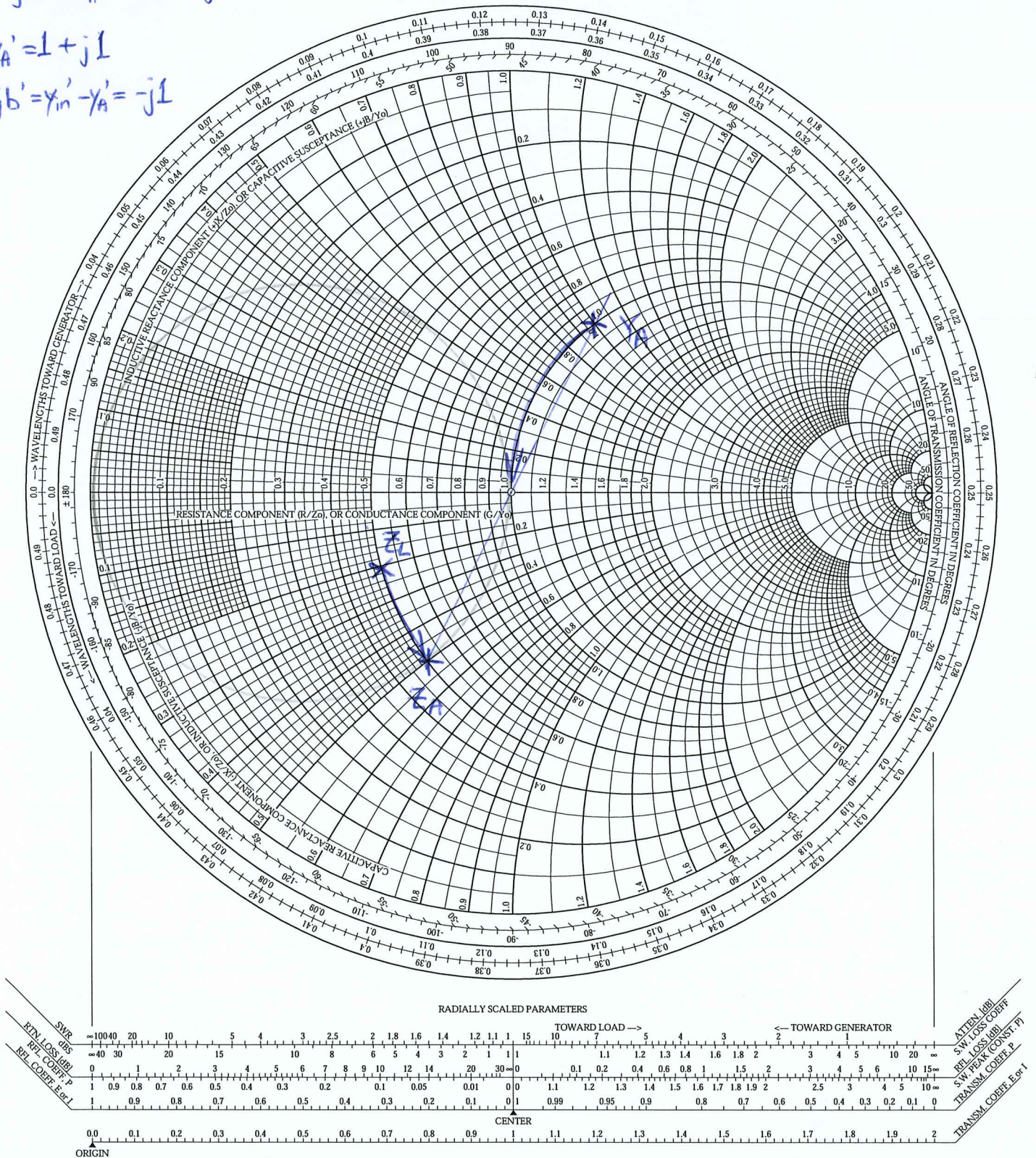
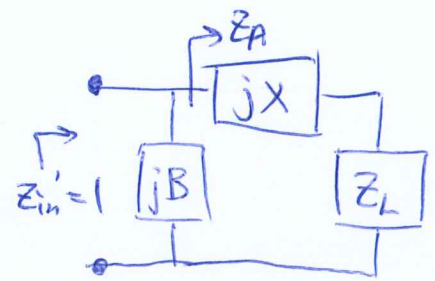
$$\Rightarrow jX' = Z_A' - Z_L' = -j0.29$$

$$Y_A' = 1 + j1$$

$$\Rightarrow jB' = Y_A' - Y_L' = -j1$$

The Complete Smith Chart

Black Magic Design



Use a capacitor for jX :

$$jX = jX' Z_0 = \frac{1}{j\omega C}$$

$$\Rightarrow C = -\frac{1}{\omega Z_0 X'} = 36.59 \text{ pF}$$

Use an inductor for jB :

$$jB = jB' Y_0 = \frac{1}{j\omega L}$$

$$\Rightarrow L = -\frac{Z_0}{\omega B'} = 9.549 \text{ nH}$$

Single Stub Matching : $Z_L' = 0.5 - j0.21$;
 $Z_0 = 30\Omega$; $\lambda = 0.3\text{ m}$

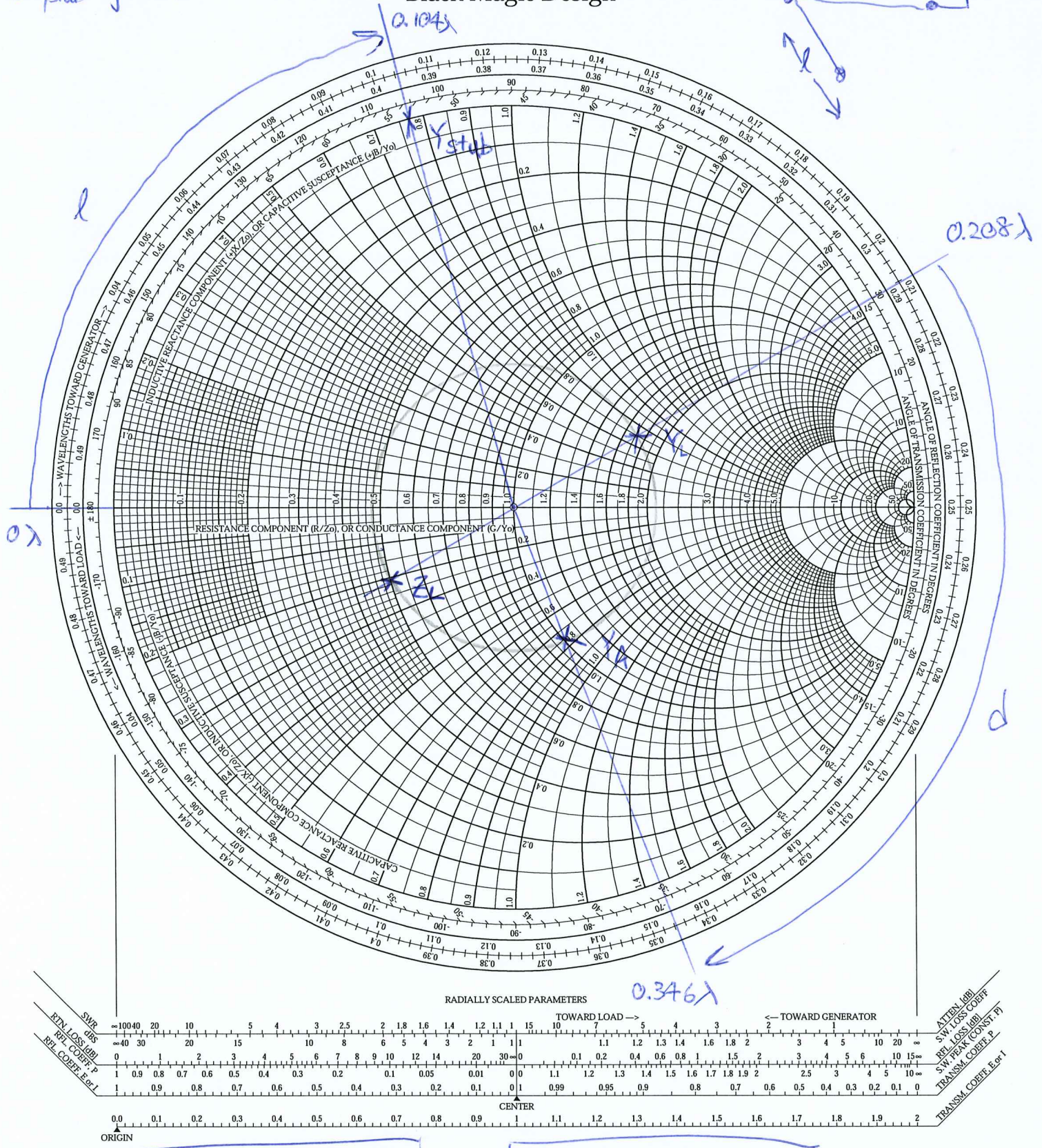
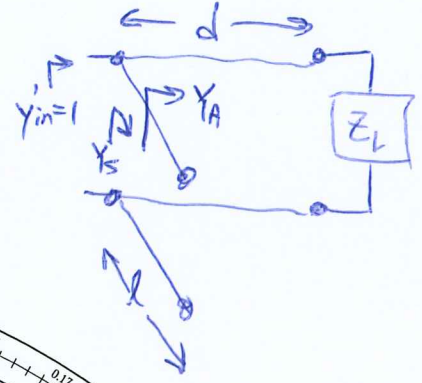
From S.C.:

$$Y_A' = 1 - j0.77$$

$$\Rightarrow Y_{\text{stub}} = j0.77$$

The Complete Smith Chart

Black Magic Design



$$d = (0.346 - 0.208)\lambda$$

$$= 0.138\lambda$$

$$= 41.4\text{ mm}$$

$$l = (0.104 - 0)\lambda$$

$$= 0.104\lambda$$

$$= 31.2\text{ mm}$$

Shunt load + $\frac{\lambda}{4}$ Transformer Matching
 $Z_L' = 0.5 - j0.21$; $Z_0 = 30 \Omega$; $f = 500 \text{ MHz}$

From S.C.:

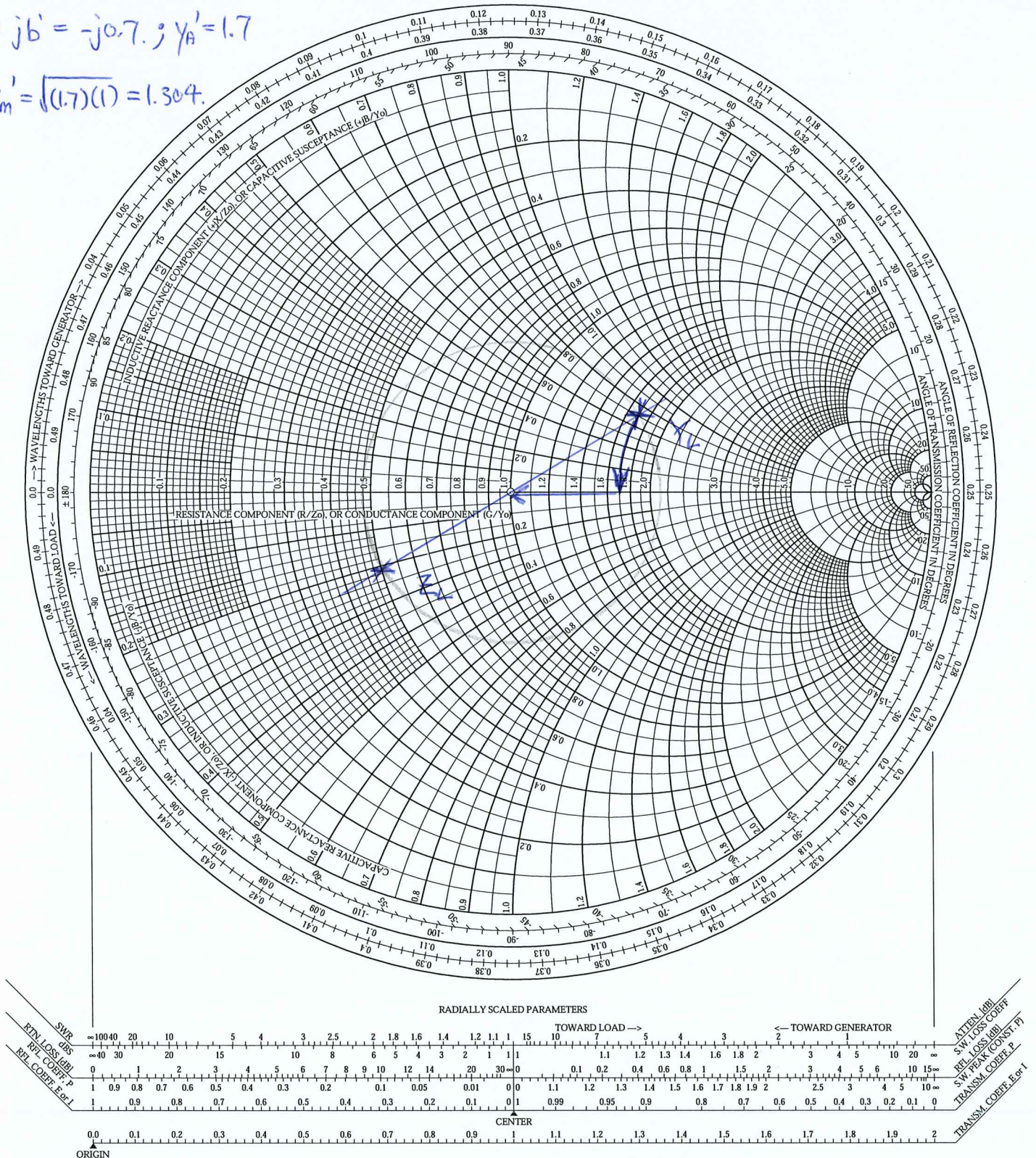
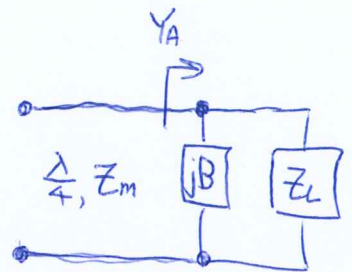
$$Y_L' = 1.7 + j0.7$$

$$\Rightarrow jB' = -j0.7; Y_A' = 1.7$$

$$\Rightarrow Y_m' = \sqrt{(1.7)(1)} = 1.304$$

The Complete Smith Chart

Black Magic Design



Use an inductor for jB :

$$jB = jB' Y_0 = \frac{1}{j\omega L}$$

$$\Rightarrow L = \frac{-Z_0}{\omega B'} = 13.64 \text{ nH}$$

Implement the $\frac{\lambda}{4}$ Transformer:

$$Z_m = \frac{1}{Y_m} = \frac{1}{Y_m' Y_0} = \frac{Z_0}{Y_m} = 23.01 \Omega$$

(The length of the TH section is $\frac{\lambda}{4}$.)

4. The goal of this question is to design a co-axial transmission line, with an inner and outer conductor radii of a and b respectively.
- Derive the per-unit-length capacitance and inductance of the co-axial transmission line. (Most of this is done for you in the course notes.)
 - Find the characteristic impedance and phase velocity of the lossless co-axial transmission line.
 - For a dielectric material with $\epsilon_r = 2.2$, design a co-axial TL with $Z_0 = 75 \Omega$. Choose the outer conductor radius to be the same as that for an RG-59 cable.

Solution:

- (a) See the notes for Lecture 8, Slides 13-14, for derivations on L and C over a unit length l . The per unit length capacitance and inductance are

$$L' = \frac{L}{l} = \frac{\frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right)}{l} = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) \quad \text{and} \quad C' = \frac{C}{l} = \frac{2\pi\epsilon l \left[\ln\left(\frac{b}{a}\right)\right]^{-1}}{l} = 2\pi\epsilon \left[\ln\left(\frac{b}{a}\right)\right]^{-1}$$

- (b) The characteristic impedance and phase velocity are

$$Z_0 = \sqrt{\frac{L'}{C'}} = \sqrt{\frac{\frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right)}{2\pi\epsilon \left[\ln\left(\frac{b}{a}\right)\right]^{-1}}} = \frac{1}{2\pi} \sqrt{\frac{\mu_0}{\epsilon}} \ln\left(\frac{b}{a}\right) = \frac{\eta_0}{2\pi\sqrt{\epsilon_r}} \ln\left(\frac{b}{a}\right) ;$$

$$V_p = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{\left(\frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right)\right) \left(2\pi\epsilon \left[\ln\left(\frac{b}{a}\right)\right]^{-1}\right)}} = \frac{1}{\sqrt{\mu_0\epsilon}}$$

- (c) We find from internet sources (e.g. Wikipedia) that the RG-59 has a diameter of 6.1mm, hence $b = 3.05$ mm.

Rearranging the equation for Z_0 gives

$$\ln\left(\frac{b}{a}\right) = 2\pi\sqrt{\epsilon_r} \frac{Z_0}{\eta_0} = 2\pi\sqrt{2.2} \frac{75}{120\pi} = 1.854 \Rightarrow \frac{b}{a} = e^{1.854} = 6.386 \Rightarrow a = \frac{b}{6.386} = 0.4776 \text{ [mm]} .$$

5. **(10% BONUS)** Prove that one can find the admittance point on the Smith Chart at a location diametrically opposite (same distance from the origin, but in the opposite direction) to the impedance point. Hint: remember the Smith Chart's relationship with the reflection coefficient.

Solution:

We map the point $y' = 1/z'$ onto the Γ -plane as follows:

$$\Gamma_y = \frac{Y - Y_0}{Y + Y_0} = \frac{y' - 1}{y' + 1} = \frac{\frac{1}{z'} - 1}{\frac{1}{z'} + 1} = \frac{1 - z'}{1 + z'} = -\frac{z' - 1}{z' + 1} = -\Gamma_z$$

Hence y' and z' appear on diametrically opposite points on the Smith Chart.