

- As a closed-book test, you are not allowed to use any written materials online/offline.
- Write your answers on the question papers directly.
- Non-graphical calculators are allowed. Laptops/phones must be switched off.
- Unless otherwise specified, SI units are assumed in the questions and 3 significant figures should be used in numerical answers.
- The total score is 100. There are a total of **7 pages**.

The following formulas may or may not be useful.

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \mathbf{E}(x, y, z) &= \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \hat{\mathbf{a}}_R & d\mathbf{B} &= \frac{\mu_0}{4\pi} \frac{Id\mathbf{L} \times \hat{\mathbf{a}}_R}{R^2} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} & \oint_S \mathbf{D} \cdot d\mathbf{S} &= Q & \oint_L \mathbf{H} \cdot d\mathbf{L} &= I \\ \nabla \cdot \mathbf{D} &= \rho_v & \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \epsilon_r \mathbf{E} & \mathbf{B} &= \mu_0 (\mathbf{H} + \mathbf{M}) = \mu_0 \mu_r \mathbf{H} \\ \nabla \cdot \mathbf{B} &= 0 & \epsilon_0 &= 8.854 \times 10^{-12} \text{ (F/m)} & \mu_0 &= 4\pi \times 10^{-7} \text{ (H/m)}\end{aligned}$$

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho} \hat{\mathbf{a}}_\rho \quad \mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3} (2\cos\theta \hat{\mathbf{a}}_r + \sin\theta \hat{\mathbf{a}}_\theta) \quad \mathbf{B} = \frac{\mu_0 I}{2\pi\rho} \hat{\mathbf{a}}_\phi \quad \mathbf{B} = \mu_0 n I \hat{\mathbf{a}}_z$$

$$\begin{aligned}V_{AB} &= -\int_B^A \mathbf{E} \cdot d\mathbf{L} & \nabla^2 V &= -\frac{\rho_v}{\epsilon} & V &= -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} \\ \nabla^2 &= \nabla \cdot \nabla\end{aligned}$$

$$\begin{aligned}\nabla &= \frac{\partial}{\partial x} \hat{\mathbf{a}}_x + \frac{\partial}{\partial y} \hat{\mathbf{a}}_y + \frac{\partial}{\partial z} \hat{\mathbf{a}}_z & \nabla \times \mathbf{F} &= \frac{1}{\rho} \begin{vmatrix} \hat{\mathbf{a}}_\rho & \rho \hat{\mathbf{a}}_\phi & \hat{\mathbf{a}}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} \\ \nabla f &= \frac{\partial f}{\partial \rho} \hat{\mathbf{a}}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\mathbf{a}}_\phi + \frac{\partial f}{\partial z} \hat{\mathbf{a}}_z & \nabla \times \mathbf{F} &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{a}}_r & r \hat{\mathbf{a}}_\theta & r \sin\theta \hat{\mathbf{a}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & r F_\theta & r \sin\theta F_\phi \end{vmatrix} \\ \nabla f &= \frac{\partial f}{\partial r} \hat{\mathbf{a}}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\mathbf{a}}_\theta + \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi} \hat{\mathbf{a}}_\phi & \nabla \times \mathbf{F} &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{a}}_r & r \hat{\mathbf{a}}_\theta & r \sin\theta \hat{\mathbf{a}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & r F_\theta & r \sin\theta F_\phi \end{vmatrix}\end{aligned}$$

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho F_\rho) + \frac{\partial}{\partial \phi} F_\phi + \frac{\partial}{\partial z} (\rho F_z) \right] \\ \nabla \cdot \mathbf{F} &= \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r} (r^2 \sin\theta F_r) + \frac{\partial}{\partial \theta} (r \sin\theta F_\theta) + \frac{\partial}{\partial \phi} (r F_\phi) \right]\end{aligned}$$

Question 1 (15 points)

Consider a metallic wire of radius b and length L . The resistance is $40\ \Omega$. A current of 3 A passes through the wire.

- (a) Find the voltage drop across the wire (in Volt).
- (b) Find the power dissipated by the wire (in Watts).
- (c) If the radius becomes $2b$, what would be the value of the resistance?
- (d) If the length becomes $2L$, what would be the value of the resistance?
- (e) If the current increases to 30 A , how does the drift velocity of the charges change?

Solution: (for Version a)

- (a) The voltage dropped across the wire is

$$V = IR = 3 \times 40 = 120\text{ (V)}$$

- (b) The power dissipated by the wire is

$$P = IV = 3 \times 120 = 360\text{ (W)}$$

- (c) The resistance is inversely proportional to area.

If the radius becomes $2b$, the resistance will be reduced by 4 times to $10\ \Omega$.

- (d) The resistance is proportional to length.

If the length becomes $2L$, the resistance will be increased by 2 times to $80\ \Omega$.

- (e) The value of current is proportional to the drift velocity.

If the current increases by 10 times to 30 A , then the drift velocity will increase by 10 times as well.

Question 2 (15 points)

Consider a parallel-plate capacitor with a capacitance of $1\text{ }\mu\text{F}$. Its positive plate stores $3\text{ }\mu\text{C}$. Its negative plate stores $-3\text{ }\mu\text{C}$. Remember to include the units in your answers.

- (a) What is the potential difference between the two plates?
- (b) How much electrical energy is stored by the capacitor?
- (c) How much energy is needed to move $+0.1\text{ pC}$ from the negative plate to the positive plate?
- (d) Suppose the plates are separated by $100\text{ }\mu\text{m}$. What is the magnitude of the electric field?
- (e) If the areas of the plates are doubled, what would be the new capacitance?

Solution: (for Version a)

- (a) The potential difference between the two plates is

$$V = \frac{Q}{C} = \frac{3}{1} = 3\text{ (V)}$$

- (b) The electrical energy stored by the capacitor is

$$W_e = \frac{1}{2}CV^2 = \frac{1}{2} \times 1 \times 10^{-6} \times 3^2 = 4.5 \times 10^{-6}\text{ (J)}$$

- (c) The energy needed to move a charge $dq = +0.1\text{ pC}$ from the negative plate to the positive plate is

$$dW_e = dq \cdot V = 0.1 \times 10^{-12} \times 3 = 3 \times 10^{-13}\text{ (J)}$$

- (d) The potential drops by $V = 3\text{ (V)}$ across a distance of $d = 100\text{ }\mu\text{m}$. Since the electric field is the negative potential gradient, its magnitude is

$$E = \frac{V}{d} = \frac{3}{1 \times 10^{-4}} = 3 \times 10^4\text{ (V/m)}$$

- (e) The value of capacitance is proportional to the area of the plate.

If the areas of the plates are doubled, the new capacitance will be doubled to $2\text{ }\mu\text{F}$.

Question 3 (20 points)

Consider a sphere of radius L (m) centered at the origin with a uniform charge density ρ_v (C/m³). The spherical coordinate system (r, θ, ϕ) is adopted.

- Find the total charge Q_0 in the sphere.
- What is the direction of $\mathbf{D}(r, \theta, \phi)$? Express the answers using a unit vector.
- Does the magnitude $D(r, \theta, \phi)$ depend on r ? Yes/No.
Does it depend on θ ? Yes/No.
Does it depend on ϕ ? Yes/No.
Please briefly state the reason(s) here.
- Suppose $L = 1$ m and $\rho_v = 0.2$ C/m³. At point $(r, \theta, \phi) = (L/2, \pi/2, 0)$, find the electric field $\mathbf{E}$ using Gauss's law. Express the final answer using $\hat{x}$, $\hat{y}$, and/or $\hat{z}$.

Solution:

- (a) Since the density is uniform (i.e. constant), the total charge Q_0 in the sphere is

$$Q_0 = \text{Volume} \times (\text{Charge density}) = \frac{4}{3}\pi L^3 \rho_v$$

- (b) By symmetry, $\mathbf{D}$ points along $\hat{\mathbf{r}}$ (which is the radial direction from the origin.)

- (c) By symmetry, D is dependent on r , independent of θ , and independent of ϕ .
The symmetry can be elaborated by imagining the rotation as either angle is varied.

- (d) We can construct a Gaussian Surface S as shown in the figure below. It is centered at the origin with a radius of $L/2$. Therefore, the point P is on the surface S .

From Gauss's law, we have

$$\oint_S \mathbf{D}(r, \theta, \phi) \cdot d\mathbf{S} = Q_{enc},$$

where Q_{enc} is the enclosed charge within the surface.

Using $\mathbf{D}(L/2, \theta, \phi) = D(L/2)\hat{\mathbf{r}}$

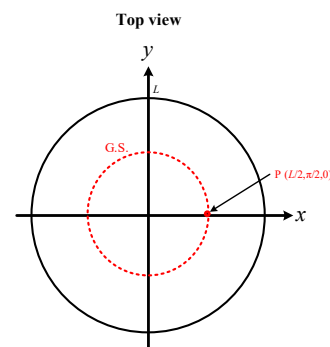
$$\oint_S (D(L/2)\hat{\mathbf{r}}) \cdot (d\mathbf{S}\hat{\mathbf{r}}) = Q_{enc}$$

So

$$D(L/2) \oint_S (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) dS = Q_{enc}$$

$$D(L/2) \oint_S dS = Q_{enc}$$

$$D(L/2) = \frac{Q_{enc}}{4\pi \left(\frac{L}{2}\right)^2} = \frac{Q_{enc}}{\pi L^2}$$



Here the charges are enclosed only partially by the sphere of radius $L/2$. Thus,

$$Q_{enc} = \frac{4\pi}{3} \left(\frac{L}{2}\right)^3 \rho_v$$

Hence,

$$D(L/2) = \frac{\frac{4\pi}{3} \left(\frac{L}{2}\right)^3 \rho_v}{\pi L^2} = \frac{L \rho_v}{6}$$

Since $\mathbf{E} = \mathbf{D}/\epsilon_0$, we have $\mathbf{E} = 3.76 \times (10)^9 \hat{\mathbf{x}}$ (V/m).

(For other versions, the direction would be different because P is located differently.)

Question 4 (25 points)

In the spherical coordinate system, there is an electric field:

$$\mathbf{E}(r, \theta, \varphi) = \frac{22k}{r} \hat{\mathbf{r}} + 11h \hat{\boldsymbol{\theta}}$$

for some constants k and h .

- What is the unit of k ?
- Given that the electric field is *conservative*, deduce that $h = 0$. The reasoning should be expressed clearly.
- Find the divergence of $\mathbf{E}$. (Hint: Based on part (b), it is now known that $h = 0$).
- Suppose the space is uniformly filled with a dielectric of susceptibility $\chi = 2.1$. Find the charge density (in C/m^3) at position (r, θ, φ) .

Solution: (for Version a, while other versions have different susceptibilities)

(a) The unit of k is V.

(b) Since the electric field is conservative, $\nabla \times \mathbf{E}$ should be zero.

Then, we can get $\nabla \times \mathbf{E} = \frac{1}{r^2 \sin \theta} \left(\frac{\partial}{\partial r} (r 11h) \right) r \sin \theta \hat{\boldsymbol{\phi}} = 0$.

So, $h = 0$.

(c) From the formula for the spherical coordinates:

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2 \sin \theta} \left(\frac{\partial}{\partial r} \left(r^2 \sin \theta \frac{22k}{r} \right) \right) = \frac{22k}{r^2}$$

(d) From Gauss's law, the charge density:

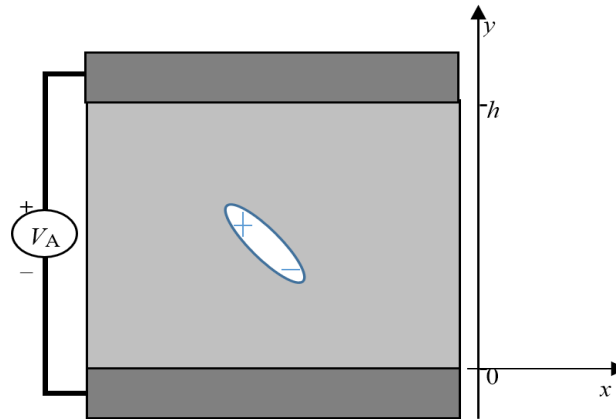
$$\rho_v = \nabla \cdot \mathbf{D} = \epsilon_0 \epsilon_r \nabla \cdot \mathbf{E} = \epsilon_0 (1 + \chi) \frac{22k}{r^2} = \frac{68.2 \epsilon_0 k}{r^2} = \frac{6.04 \times (10)^{-10} k}{r^2}$$

Question 5 (25 points)

The diagram below shows a parallel plate capacitor. A piece of dielectric at $y = 0$ to h is sandwiched between two parallel metallic plates.

- The lower plate is grounded.
- A positive voltage of $+V_A$ is applied to the upper plate.
- The cross-section area of each plate is M . It is considered to be very large.
- The permittivity of the dielectric is ϵ .

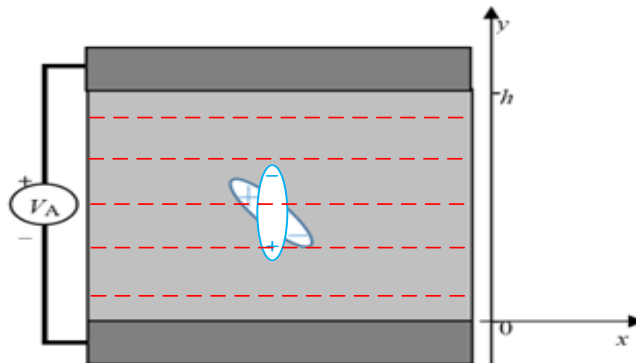
As an example, a dipole of the dielectric is shown below before the voltage is applied.



- When the voltage V_A is applied, describe the motion of the dipole. Draw the final position and orientation of the dipole in the above diagram.
- In the above diagram, sketch a few equipotential lines. State the reasoning.
- Using the Laplace's equation, deduce the potential $V(x, y)$ in the dielectric. Express the answer in terms of h and V_A .
- What is the electric field inside the upper plate?
- What is the electric field at $y = h/2$ in the middle of the dielectric?

Solution:

- The dipole will start rotating counter-clockwise and finally reaches the orientation below.
- The equipotential lines (in red) are all parallel to the plates. This is because the potential $V(x, y) = V(y)$ only varies along y direction.



(c) The Laplace's equation can be applied:

$$\nabla^2 V = 0$$

Here the voltage depends only on y :

$$\frac{\partial^2}{\partial y^2} V(y) = 0$$

$$\frac{\partial}{\partial y} V(y) = A$$

$$V(y) = Ay + B$$

where A and B are integration constants.

From the boundary conditions:

$$\begin{cases} V(h) = V_A \\ V(0) = 0 \end{cases} \Rightarrow \begin{cases} Ah = V_A \\ B = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{V_A}{h} \\ B = 0 \end{cases}$$

Therefore,

$$V(y) = \frac{V_A}{h} y$$

(d) The plate is a metal, so the field is zero inside.

(e) Since electric field is the negative gradient of potential,

$$\mathbf{E} = -\nabla V$$

$$\mathbf{E} = -\frac{\partial}{\partial y} \left(\frac{V_A}{h} y \right) \hat{\mathbf{y}} = -\frac{V_A}{h} \hat{\mathbf{y}}$$

Hence, the electric field at $y = h/2$ in the middle of the dielectric is $-\frac{V_A}{h} \hat{\mathbf{y}}$

~END~