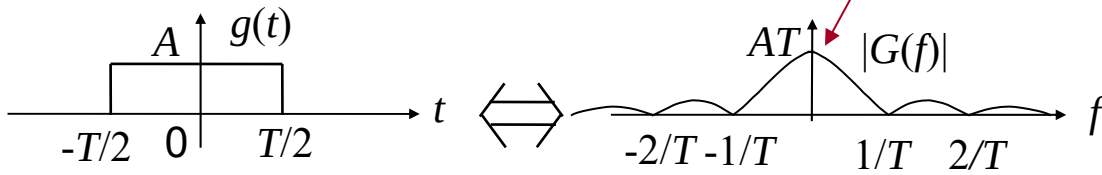


EE 3008. Fourier Transform Tutorial Solutions

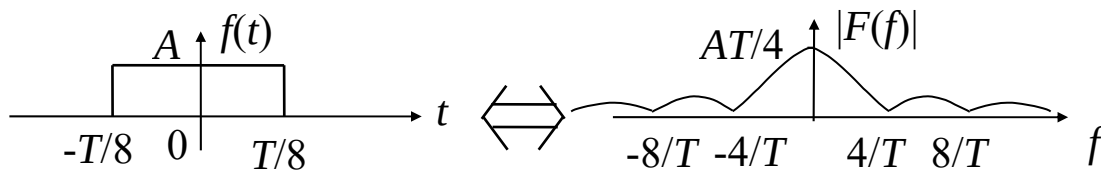
Q1) This is already given in the notes. We repeat the result below.

$$\begin{aligned}
 G(f) &= \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt = \int_{-T/2}^{T/2} A e^{-j2\pi ft} dt \\
 &= A \frac{e^{-j2\pi ft}}{-j2\pi f} \bigg|_{t=-T/2}^{t=T/2} = A \frac{e^{-j\pi fT} - e^{+j\pi fT}}{-j2\pi f} \\
 &= A \frac{e^{+j\pi fT} - e^{-j\pi fT}}{j2} \cdot \frac{1}{\pi f} \\
 &= A \sin(\pi fT) / \pi f \quad \leftarrow \text{Note:} \\
 &= AT \text{sinc}(\pi fT) \quad \leftarrow \begin{array}{l} \text{(1) This is in the } \sin(x)/x \text{ form.} \\ \text{(2) } \sin(x)/x = 1 \text{ when } x \rightarrow 0. \end{array}
 \end{aligned}$$



Q2) (a) Note that $f(t)$ in Q2(a) can be obtained by replacing T in $g(t)$ in Q1 with $T/4$. Then $H(f)$ can also be obtained from $G(f)$ by replacing T with $T/4$.

$$F(f) = G(f) \big|_{T \leftarrow T/4} = A \frac{T}{4} \text{sinc}\left(\pi f \frac{T}{4}\right)$$



You can verify the scaling property as

$$F(f) = \frac{1}{4} \cdot G\left(\frac{f}{4}\right).$$

(b) The time shift property is: $s(t-\tau) \Leftrightarrow S(f) \exp(-j2\pi f\tau)$.

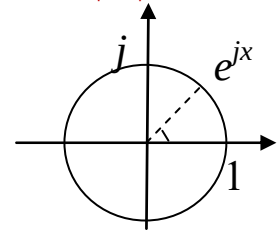
Now $h(t) = f(t-b)$. Thus

$$H(f) = F(f) \exp(-j2\pi fb).$$

We can also directly derive $H(f)$ as follows.

$$\begin{aligned}
 H(f) &= \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt = \int_{b-T/8}^{b+T/8} A e^{-j2\pi ft} dt \\
 &= A \frac{e^{-j2\pi ft}}{-j2\pi f} \bigg|_{t=b-T/8}^{t=b+T/8} = A \frac{e^{-j2\pi f(b+T/8)} - e^{-j2\pi f(b-T/8)}}{-j2\pi f} \\
 &= \left(A \frac{e^{-j2\pi fT/8} - e^{+j2\pi fT/8}}{-j2\pi f} \right) \cdot e^{-j2\pi fb} \\
 &= A \frac{T}{4} \text{sinc}\left(\pi f \frac{T}{4}\right) \cdot e^{-j2\pi fb}
 \end{aligned}$$

For any real x
 $|e^{jx}| = 1$



Therefore

$$|H(f)| = |F(f)|.$$

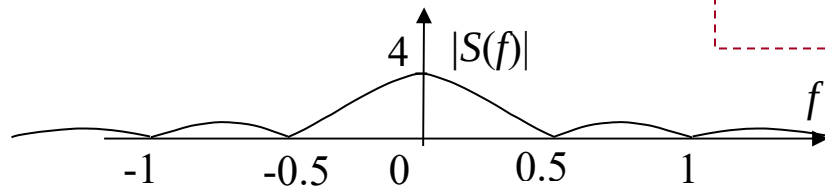
This means that delay will not change the magnitude spectrum.

Q3) (a) Refer to the previous result. Substituting $A=2$ and $\tau=2$ into the result of Q1, we have

$$S(f) = A\tau \text{sinc}(\pi f \tau) = 4 \text{sinc}(2\pi f).$$

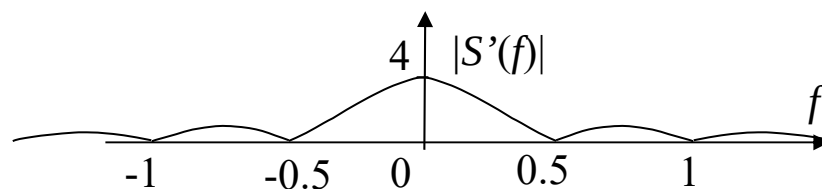
The first zero is at $1/2$.

It is useful to memorize
 $S(f) = A\tau \text{sinc}(\pi f \tau)$
 for a rectangular pulse.



(b) Refer to the time shift property. The spectrum is the same above.

The first zero is still at $1/2$.

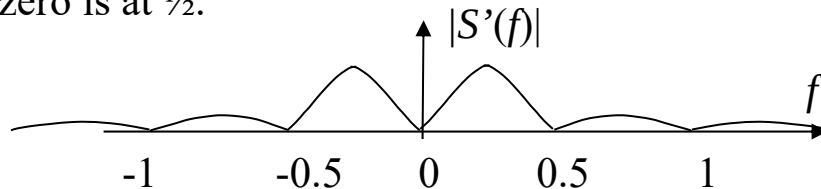


Q4) Note that $s'(t) = s(t) - s(t-2)$. The Fourier transform of $s(t)$ has been derived earlier. Using the linearity and time shift properties, we have

$$\begin{aligned}
 S'(f) &= S(f) - S(f)e^{-j2\cdot 2\pi f} = S(f)(1 - e^{-j2\cdot 2\pi f}) \\
 &= e^{-j2\pi f} S(f)(e^{j2\pi f} - e^{-j2\pi f}) \\
 &= 2je^{-j2\pi f} S(f) \sin(2\pi f)
 \end{aligned}$$

Time domain shifting will not cause any difference in the magnitude spectrum. This is a direct consequence of the time shift property.

Therefore $|S'(f)| = |2S(f)\sin(2\pi f)|$. The first zero is at 0. The second zero is at $\frac{1}{2}$.



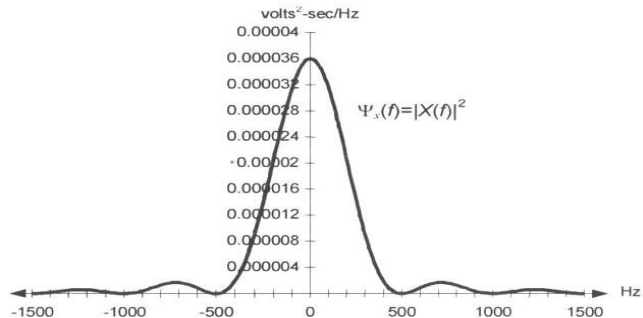
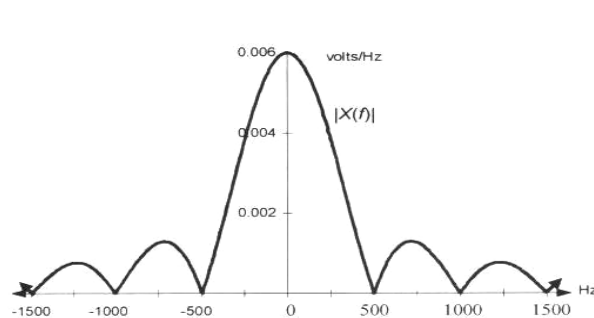
This example shows how to obtain the spectrum of two pulses.

Q5) (a) The derivation is similar to that in Q1:

$$X(f) = AT \text{sinc}(\pi fT) \Big|_{A=3 \text{ and } T=0.002} = 0.006 \text{sinc}(0.002\pi f)$$

Magnitude Spectrum:

Energy Spectrum:

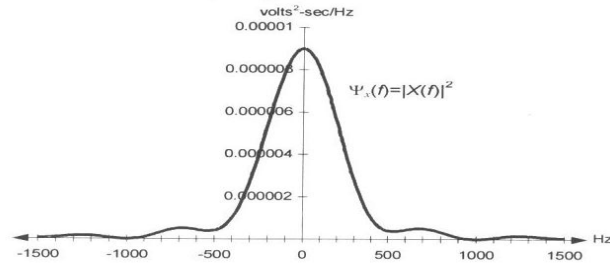
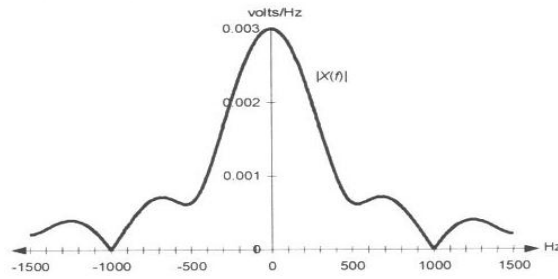


(b) The function is the sum of two rectangular pulses (one from 0 to 1 and the other from 1 to 2). Thus the Fourier transform is the sum of the two Fourier transforms.

$$\begin{aligned}
 X(f) &= 0.002 \text{sinc}(0.001\pi f) e^{-j0.001\pi f} + 0.001 \text{sinc}(0.001\pi f) e^{-j0.003\pi f} \\
 &= 0.001 \text{sinc}(0.001\pi f) (2e^{-j0.001\pi f} + e^{-j0.003\pi f})
 \end{aligned}$$

It is difficult to draw this function by hand. The figures are done by a computer.

Magnitude Spectrum:



Q6) We will use Fourier transform properties to solve these questions.

(1) $\mathcal{F}\{2\} = 2\delta(f)$.

(2) $\mathcal{F}\{\sin(2\pi f_0 t)\} = (\delta(f - f_0) - \delta(f + f_0)) / (2j)$;

$\mathcal{F}\{\cos(2\pi f_0 t)\} = (\delta(f - f_0) + \delta(f + f_0)) / 2$.

(3) $x(t) \exp(j2\pi f_0 t)$ (using the frequency shift property).

(4) Note that $\sin(2\pi f_0 t) = (e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}) / (2j)$.

Thus $\mathcal{F}\{x(t) \sin(2\pi f_0 t)\} = (\mathcal{F}\{x(t) e^{j2\pi f_0 t}\} - \mathcal{F}\{x(t) e^{-j2\pi f_0 t}\}) / (2j)$
 $= (X(f - f_0) - X(f + f_0)) / (2j)$.

Similarly, note that $\cos(2\pi f_0 t) = (e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}) / 2$,

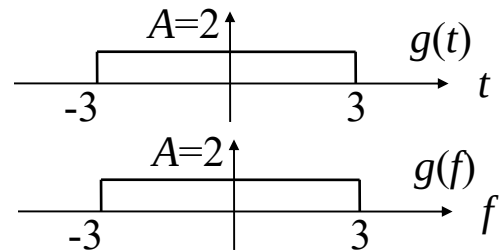
$\mathcal{F}\{x(t) \cos(2\pi f_0 t)\} = (X(f - f_0) + X(f + f_0)) / 2$.

(5) Let $g(t)$ be the rectangular pulse in Q1) with $A=2$ and $T=6$. Then,

$\mathcal{F}(g(t)) = 12 \text{sinc}(6\pi f)$.

From the duality property, we have

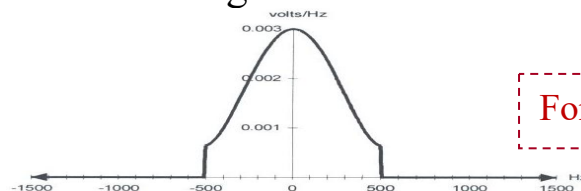
$\mathcal{F}(12 \text{sinc}(6\pi f)) = g(-f) = g(f)$.



(6) An idea low pass filter.

You may learn this in signal analysis.

Q7) The filter removes the signal $> 500\text{Hz}$ and $< -500\text{Hz}$. See below.



For Illustration only

Q8) The time domain function with such a spectrum is a sinc function. It has an infinite span in time. This is impossible to realize in practice.

$$\text{Q9)} \quad X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt = \int_{-\infty}^{+\infty} x(t) \cos(2\pi f) dt - j \int_{-\infty}^{+\infty} x(t) \sin(2\pi f) dt$$

$$X(-f) = \int_{-\infty}^{+\infty} x(t) e^{+j2\pi ft} dt = \int_{-\infty}^{+\infty} x(t) \cos(2\pi f) dt + j \int_{-\infty}^{+\infty} x(t) \sin(2\pi f) dt$$

If $x(t)$ is real, then

$$|X(f)|^2 = \left(\int_{-\infty}^{+\infty} x(t) \cos(2\pi f) dt \right)^2 + \left(\int_{-\infty}^{+\infty} x(t) \sin(2\pi f) dt \right)^2 = |X(-f)|^2.$$

Q10) (1) The magnitude spectrum of $x(t-t_0)$ is the same as of $x(t)$. It is symmetric to the y-axis.

(2) Using the frequency shift property, we have

$$\mathcal{F}\{x(t)e^{j2\pi f_0 t}\} = X(f - f_0).$$

The magnitude spectrum is plotted below, which is no longer symmetric (since $x(t)e^{j2\pi f_0 t}$ is not real).



Q11) As shown in the lecture notes, the Fourier transform of C is $C\delta(f)$.

From the frequency shift property, we have $S(f) = C\delta(f - f_0)$.

$$s(t) = Ce^{j2\pi f_0 t} \iff \begin{array}{c} \uparrow \\ f_0 \end{array} \quad \begin{array}{c} S(f) = C\delta(f - f_0) \\ \leftarrow \end{array}$$

frequency domain shifting property

$$\text{Q12)} \quad X(f) = [2A\tau \text{sinc}(2\pi f\tau) + A\tau \text{sinc}(\pi f\tau)] e^{-j3\pi\tau f}$$

time domain shifting property

Q13) (a) Recall from Q1 that $A\tau \text{sinc}(\pi\tau f)$ is the Fourier transform of a rectangular pulse with duration τ and amplitude A . Using the duality property, we have $x(t) = A\tau \text{sinc}(\pi\tau t)$.

$$(b) \quad \int_{-\infty}^{+\infty} x(t) dt = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt \Big|_{f=0} = X(0) = A$$

$X(0)$ equals the total area of $x(t)$.