

EE2104 Introduction to Electromagnetics
Assignment 4 Solution

1. Given that an electromagnetic wave propagating in free-space has:

$$\mathbf{E} = E_0 \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \hat{\mathbf{x}}$$
$$\mathbf{H} = \frac{E_0}{c\mu_0} \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \hat{\mathbf{y}}$$

Show by direct evaluation that

$$\nabla \cdot \mathbf{D} = 0,$$

$$\nabla \cdot \mathbf{B} = 0,$$

which are the Maxwell's third and fourth equations.

Solution:

Since,

$$\nabla = \frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}}$$

$$\mathbf{D} = \epsilon \mathbf{E}$$

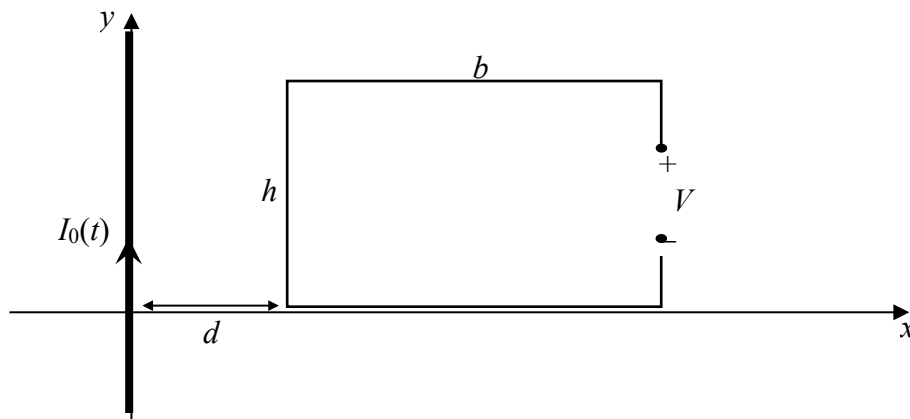
Then we have

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \left(\frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right) \cdot \epsilon_0 \left(E_0 \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \hat{\mathbf{x}} \right) \\ &= \frac{\partial}{\partial x} \left(\epsilon_0 E_0 \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \right) + 0 + 0 = 0 \end{aligned}$$

Similarly,

$$\begin{aligned} \nabla \cdot \mathbf{B} &= \left(\frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right) \cdot \mu_0 \frac{E_0}{c\mu_0} \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \hat{\mathbf{y}} \\ &= 0 + \frac{\partial}{\partial y} \left(\mu_0 \frac{E_0}{c\mu_0} \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \right) + 0 = 0 \end{aligned}$$

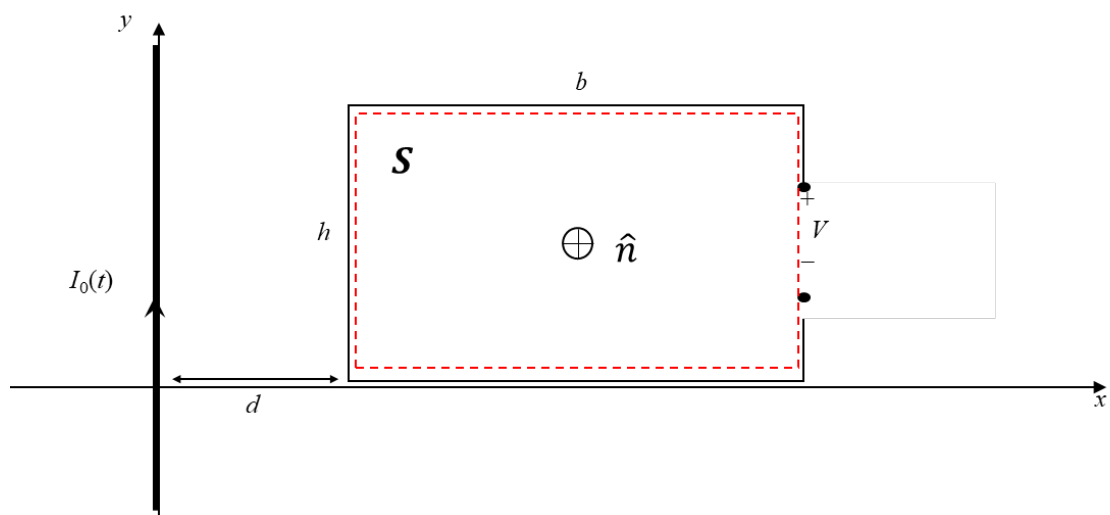
2. On the y -axis, an infinitely long straight wire W carries a current $I_0(t)$ flowing towards the $+y$ -direction. A rectangular open coil C of height h and width b is put at distance d away, as the following figure shows.



- Copy the diagram above and then indicate the surface area S enclosed by the coil C . In other words, S has C as its edge.
- Express the differential surface $d\mathbf{S}$ in terms of dx , dy , and/or dz . Remember the direction too. Use the convention that matched the polarity of V as indicated in the diagram above.
- Using the formular sheet, write down the expression of the magnetic field inside any point inside S .
- Find the magnetic flux Φ collected by the coil C .
- Find the mutual inductance L_{wc} from the straight wire W to the coil C .
- Suppose current $I_0(t)$ increases at a rate of 20.23 A/s and $h = b = d = 0.3 \text{ m}$, find the induced voltage V .

Solution:

(a)



(b) $d\mathbf{S} = -dx dy \hat{\mathbf{z}}$

(c) The magnetic field inside $\mathbf{S}$ is

$$\mathbf{B} = -\frac{\mu_0 I}{2\pi x} \hat{\mathbf{a}}_z = -\frac{\mu_0 I_0(t)}{2\pi x} \hat{\mathbf{a}}_z$$

(d) The flux contained by the coil is

$$\Phi = \int_{\text{coil area}} \mathbf{B} \cdot d\mathbf{S} = \int_{x=d}^{x=d+b} \frac{\mu_0 I_0(t)}{2\pi x} (-\hat{\mathbf{a}}_z \cdot -\hat{\mathbf{a}}_z) h dx = \frac{h\mu_0 I_0(t)}{2\pi} \ln\left(\frac{d+b}{d}\right)$$

(e)

$$L_{wc} = \frac{\Phi}{I} = \frac{\frac{h\mu_0 I_0(t)}{2\pi} \ln\left(\frac{d+b}{d}\right)}{I_0(t)} = \frac{h\mu_0}{2\pi} \ln\left(\frac{d+b}{d}\right)$$

(f)

Since current $I_0(t)$ increases at a rate of 20.23 A/s,

$$\frac{dI_0}{dt} = 20.23$$

the induced voltage V is

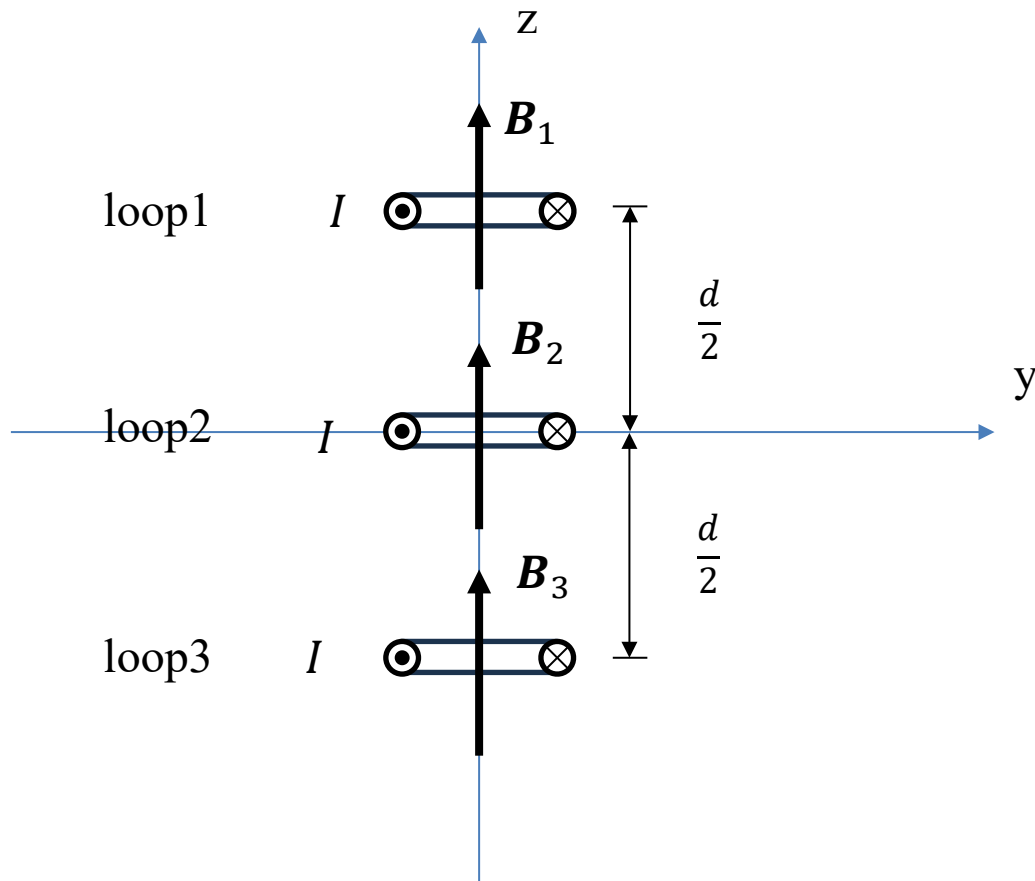
$$V = -\frac{d\Phi}{dt} = -\frac{20.23 h\mu_0 \ln\left(\frac{d+b}{d}\right)}{2\pi} = -841 \text{ (nV)}$$

3. Given the expression of the B -field at distance z directly above the center of a circular coil of radius a carrying current I is

$$B = \frac{\mu_0}{2} \frac{Ia^2}{(z^2 + a^2)^{3/2}}$$

Suppose there are three coils each with current I flowing in the direction of increasing ϕ . All of them are parallel to the xy -plane, but the centers are at $z = -d/2, 0$ and $+d/2$, respectively. Find B at the $(x, y, z) = (0, 0, 0)$ in terms of (a) direction and (b) magnitude.

Solution:



The magnetic field due to a loop current along its center axis is normal to the loop surface as shown in the picture. Based on the superposition principle, the B field at the point $(x, y, z) = (0, 0, 0)$ can be obtained by:

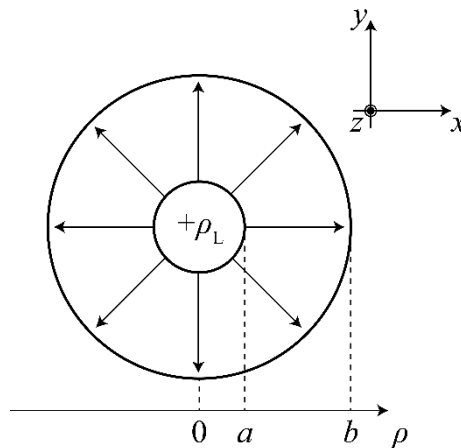
$$\begin{aligned} \mathbf{B} &= \mathbf{B}_1 \left(0, 0, -\frac{d}{2}\right) + \mathbf{B}_2(0, 0, 0) + \mathbf{B}_3 \left(0, 0, \frac{d}{2}\right) \\ &= \frac{\mu_0}{2} \frac{Ia^2}{\left(\left(-\frac{d}{2}\right)^2 + a^2\right)^{3/2}} \hat{\mathbf{a}}_z + \frac{\mu_0}{2} \frac{Ia^2}{((0)^2 + a^2)^{3/2}} \hat{\mathbf{a}}_z + \frac{\mu_0}{2} \frac{Ia^2}{\left(\left(\frac{d}{2}\right)^2 + a^2\right)^{3/2}} \hat{\mathbf{a}}_z \\ &= \hat{\mathbf{a}}_z \frac{\mu_0 Ia^2}{2} \left(\frac{2}{\left(\left(\frac{d}{2}\right)^2 + a^2\right)^{3/2}} + \frac{1}{a^3} \right) \end{aligned}$$

4. Consider a long coaxial cable of inner and outer radii $a = 2$ mm and $b = 7$ mm, respectively. The gap is filled with air. The length of the cable is $L = 10$ m. Its center is along the z -axis
- Find the capacitance C using Gauss's law. Be detailed in the showing the steps. Refer to example(s) in Chapter 4.
 - Find the inductance L using Ampere's law. Be detailed in showing the steps. Refer to example(s) in Chapters 9 and 10.
 - Suppose there is no net charge in the core nor the shielding. A current of $I = 23$ A flows through the core (and returning through the shielding).
 - Using any equation in the formular sheet, or using the results in part (b), write down the magnetic flux density $B(\rho, \phi, z)$.
 - Using the results in part (b) and page 11 in Chapter 11, write down total magnetic energy W_m stored in the cable
 - Using the results in part (b), find the following integration over the air gap:

$$\frac{1}{2} \int_{\text{Volume of the air gap}} \mu_0 |\mathbf{H}|^2 dV$$

Note: The volume of the air gap means for $z = 0$ to L , $\phi = 0$ to 2π , and $\rho = a$ to b . The answers in (ii) and (iii) should be the same.

Solution:



- Assume the inner conduction is charged with a charge density of $+\rho_L$ (C/m).

Gauss' law is applied for a cylindrical surface S of radius ρ and length L , we have the E-field derived as follows:

- By symmetry, $\mathbf{D} = D(\rho)\hat{\mathbf{a}}_\rho$

ii) Then,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \rho_L L$$

$$\int_{S_{top}} (D \hat{\mathbf{a}}_\rho) \cdot (dS \hat{\mathbf{a}}_z) + \int_{S_{bottom}} (D \hat{\mathbf{a}}_\rho) \cdot (dS (-\hat{\mathbf{a}}_z)) + \int_{S_{curved}} (D \hat{\mathbf{a}}_\rho) \cdot (dS \hat{\mathbf{a}}_\rho) = \rho_L L$$

$$\int_{S_{curved}} D(\rho) dS = \rho_L L$$

$$D(\rho) \int_{S_{curved}} dS = \rho_L L$$

iii) Therefore,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho} \hat{\mathbf{a}}_\rho$$

iv) The voltage can be calculated as:

v)

$$V_{ba} = - \int_{\rho=a}^b \mathbf{E} \cdot d\boldsymbol{\rho} = - \int_{\rho=a}^b \frac{\rho_L}{2\pi\epsilon_0\rho} d\rho = - \frac{\rho_L}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)$$

We can obtain the capacitance (as the inner conductor carries a total charge of $\rho_L L$)

$$C = \frac{\rho_L L}{|V|} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)}$$

Since $a=2\text{mm}=0.002\text{m}$; $b=7\text{mm}=0.007\text{m}$ and $L=10\text{m}$,

$$C = \frac{2\pi\epsilon_0 \cdot 10}{\ln(3.5)} = 7.9824 \cdot 2\pi\epsilon_0 = 0.444 \text{ (nF)}$$

- (b) Assume a current $I_0 \hat{\mathbf{a}}_z$ is flowing along the positive z direction via the inner conductor and a current $-I_0 \hat{\mathbf{a}}_z$ along the negative z -direction via the outer conductor.

Ampere's law is applied over a closed integration loop C of radius ρ on a plane perpendicular to the z -axis. We have the B -field derived as follows:

- i) By symmetry, $\mathbf{H} = H(\rho) \hat{\mathbf{a}}_\phi$
 ii) Using the Ampere's Law, we have in between the two conductors:

$$\oint_C \mathbf{H} \cdot d\mathbf{L} = I_0$$

$$\oint_C H dL = I_0$$

$$H(\rho) \oint_C dL = I_0$$

$$H(\rho) 2\pi\rho = I_0$$

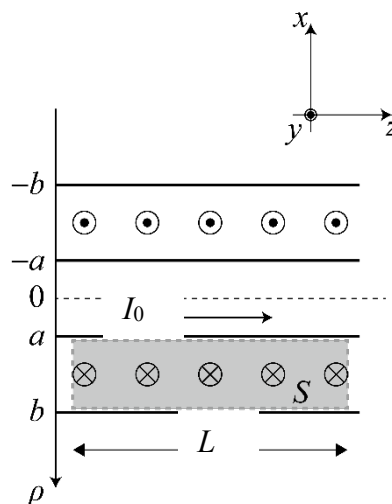
$$\therefore \mathbf{B} = \frac{\mu_0 I_0}{2\pi\rho} \hat{\mathbf{a}}_\phi$$

- iii) The magnetic flux through a surface S in the figure below can be written as:

$$\Phi = \int_S \frac{\mu_0 I_0}{2\pi\rho} dS = \int_{z=0}^L \int_{\rho=a}^b \frac{\mu_0 I}{2\pi\rho} d\rho dz = L \frac{\mu_0 I_0}{2\pi} \ln\left(\frac{b}{a}\right)$$

- iv) Finally, the inductance is given by:

$$L = \frac{\Phi}{I} = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) L = 2.505 \text{ } (\mu H)$$



(c)

i) The magnitude of the magnetic flux density is

$$B = \frac{\mu_0 11.5}{\pi \rho} = \frac{4.6 \times 10^{-6}}{\rho} (T)$$

ii) The total magnetic energy is, from the circuital perspective,

$$\frac{1}{2} L I_0^2 = \frac{1}{2} 2.505 \times 10^{-6} \times 23^2 = 663 \text{ (}\mu\text{J)}$$

iii) The total magnetic energy is, from the electromagnetic perspective,

$$\begin{aligned} & \frac{1}{2} \int_{\text{Volume of the air gap}} \mu_0 |\mathbf{H}|^2 dV \\ &= \frac{1}{2} \int_{\text{Volume of the air gap}} \mu_0 \left| \frac{I_0}{2\pi\rho} \right|^2 dV \\ &= \frac{\mu_0}{2} \int_0^{2\pi} \int_a^b \int_0^L \frac{I_0^2}{(2\pi\rho)^2} \rho d\rho d\varphi dz \\ &= \frac{\mu_0 I_0^2}{8\pi^2} \cdot 2\pi L \ln\left(\frac{b}{a}\right) \\ &= 663 \text{ }\mu\text{J} \end{aligned}$$