
EE3121

Differential Equations for Engineers

Tutorial 3-1



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Tutorial

Problem 1:

Solve the following nonlinear differential equations:

(i)

$$\frac{dy}{dt} + (\cos 2t)y^3 = 0, \quad y(0) = 1.$$

(ii)

$$\frac{dy}{dt} - 10y - y^2 = 21, \quad y(0) = 0.$$

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Solution:

(i) Step 1: Separate variables

$$\frac{dy}{y^3} = -\cos 2t dt.$$

Step 2: Integrate

$$\int_{y(0)}^{y(t)} \frac{dy}{y^3} = - \int_0^t \cos 2\tau d\tau.$$

$$\begin{aligned} \int_{y(0)}^{y(t)} \frac{dy}{y^3} &= \int_{y(0)}^{y(t)} d\left(-\frac{1}{2}y^{-2}\right) = -\frac{1}{2} \left[\frac{1}{y^2(t)} - \frac{1}{y^2(0)} \right] = -\frac{1}{2} \left[\frac{1}{y^2(t)} - 1 \right], \\ \int_0^t \cos 2\tau d\tau &= \int_0^t \frac{1}{2} d(\sin 2\tau) = \frac{1}{2} (\sin 2t - 0) = \frac{1}{2} \sin 2t, \end{aligned}$$

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$$\Rightarrow -\frac{1}{2} \left[\frac{1}{y^2(t)} - 1 \right] = \frac{1}{2} \sin 2t,$$
$$\Rightarrow y(t) = \frac{1}{\sqrt{1 + \sin 2t}}, y(0) = 1$$

(ii)

$$\frac{dy}{dt} - 10y - y^2 = 21, y(0) = 0$$

$$\Rightarrow \frac{dy}{dt} = y^2 + 10y + 21,$$

$$\Rightarrow \frac{dy}{y^2 + 10y + 21} = dt.$$

$$\int_{y(0)}^{y(t)} \frac{dy}{y^2 + 10y + 21} = \int_0^t d\tau.$$

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$\frac{1}{y^2+10y+21}$: Rational Function $\Rightarrow$ Partial Fraction Expansion

$$\frac{1}{y^2 + 10y + 21} = \frac{1}{(y + 3)(y + 7)} = \frac{A_1}{y + 3} + \frac{A_2}{y + 7}.$$

$$\frac{1}{(y + 3)(y + 7)} = \frac{A_1y + 7A_1 + A_2y + 3A_2}{(y + 3)(y + 7)} = \frac{(A_1 + A_2)y + (7A_1 + 3A_2)}{(y + 3)(y + 7)}$$

Hence,

$$A_1 + A_2 = 0$$

$$7A_1 + 3A_2 = 1,$$

$$\Rightarrow A_1 = -\frac{1}{4}, A_2 = \frac{1}{4}.$$

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$$\begin{aligned}\int_{y(0)}^{y(t)} \frac{dy}{y^2 + 10y + 21} &= \frac{1}{4} \int_{y(0)}^{y(t)} \frac{dy}{y + 3} - \frac{1}{4} \int_{y(0)}^{y(t)} \frac{dy}{y + 7} \\&= \frac{1}{4} \ln \frac{y(t) + 3}{y(0) + 3} - \frac{1}{4} \ln \frac{y(t) + 7}{y(0) + 7} \\&= \frac{1}{4} \ln \frac{y(t) + 3}{y(0) + 3} \cdot \frac{y(0) + 7}{y(t) + 7}.\end{aligned}$$

$$y(0) = 0,$$

$$\begin{aligned}\Rightarrow \frac{1}{4} \ln \frac{y(t) + 3}{y(t) + 7} \cdot \frac{7}{3} &= t \\ \Rightarrow \ln \frac{y(t) + 3}{y(t) + 7} \cdot \frac{7}{3} &= 4t\end{aligned}$$

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Problem 2:

Find the second successive approximation $y_2(t)$ to the solution

$$\frac{dy}{dt} + \left(\frac{2}{t}\right) y + \varepsilon y^2 = t, \quad y(1) = \frac{1}{4}.$$

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Solution:

Step 1: Set $\varepsilon = 0$

$$\Rightarrow \frac{dy_1}{dt} + \left(\frac{2}{t}\right) y_1 = t, y_0 = y(1) = \frac{1}{4}$$

$$y_1(t) = y_0 \exp \left(- \int_{t_0}^t p(\tau) d\tau \right) + \int_{t_0}^t \exp \left(- \int_{\tau}^t p(\tau) d\tau \right) g(\tau) d\tau$$

$$\Rightarrow \int_{\tau}^t p(\tau) d\tau = \int_{\tau}^t \frac{2}{\tau} d\tau = 2 \ln \left(\frac{t}{\tau} \right)$$

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$$\begin{aligned} y_1(t) &= y_1 \exp \left(-2 \ln \left(\frac{t}{t_0} \right) \right) + \int_1^t \exp \left(-2 \ln \left(\frac{t}{\tau} \right) \right) \tau d\tau \\ &= y(1) \cdot \frac{1}{t^2} + \int_1^t \left(\frac{\tau}{t} \right)^2 \tau d\tau = \frac{1}{4t^2} + \frac{1}{t^2} \cdot \left(\frac{1}{4} t^4 - \frac{1}{4} \right) = \frac{1}{4} t^2 \end{aligned}$$

For $y_2(t)$:

$$\frac{dy_2}{dt} + \left(\frac{2}{t} \right) y_2 = t - \varepsilon y_1^2(t) = t - \varepsilon \cdot \frac{1}{16} t^4 = g_1(t)$$

$$y_2(t) = y(1) \cdot \frac{1}{t^2} + \int_1^t \left(\frac{\tau}{t} \right)^2 \left(\tau - \varepsilon \cdot \frac{1}{16} \tau^4 \right) d\tau$$

END