

1. (a) The auxiliary equation is $\alpha^2 + 3\alpha = \alpha(\alpha + 3) = 0$, its roots are $\alpha = 0, -3$.
Hence the general solution is $y = c_1 + c_2 e^{-3x}$.
- (b) The auxiliary equation is $\alpha^2 - 3\alpha + 2 = (\alpha - 2)(\alpha - 1) = 0$, its roots are $\alpha = 1, 2$.
Hence the general solution is $y = c_1 e^x + c_2 e^{2x}$.
- (c) The auxiliary equation is $\alpha^3 - 8 = (\alpha - 2)(\alpha^2 + 2\alpha + 4) = 0$, its roots are $\alpha = 2, -1 \pm \sqrt{3}i$.
Hence the general solution is $y = e^{-x} (A \cos \sqrt{3}x + B \sin \sqrt{3}x) + c e^{2x}$.
- (d) The auxiliary equation is $\alpha^4 - 10\alpha^2 + 9 = (\alpha^2 - 1)(\alpha^2 - 9) = (\alpha - 1)(\alpha + 1)(\alpha - 3)(\alpha + 3) = 0$, its roots are $\alpha = \pm 1, \pm 3$. Hence the general solution is $y = c_1 e^x + c_2 e^{-x} + c_3 e^{3x} + c_4 e^{-3x}$.
- (e) The auxiliary equation is $\alpha^4 + 2\alpha^2 + 1 = (\alpha^2 + 1)^2 = 0$, its roots are $\alpha = i, i, -i, -i$. Hence the general solution is $y = (A_1 + B_1 x) \cos x + (A_2 + B_2 x) \sin x$.

2. $y'' - 3y' + 2y = 0 \Rightarrow m^2 - 3m + 2 = (m - 2)(m - 1) = 0 \Rightarrow m = 2 \text{ or } m = 1$.

So $y_c = c_1 e^x + c_2 e^{2x}$.

Let a particular integral be given by $y_p = c_1(x) e^x + c_2(x) e^{2x}$.

Then we have $\begin{cases} c_1'(x) e^x + c_2'(x) e^{2x} = 0 \\ c_1'(x) e^x + c_2'(x) 2e^{2x} = (1 + x)e^x \end{cases}$

$$c_1'(x) = \frac{\det \begin{vmatrix} 0 & e^{2x} \\ (1+x)e^x & 2e^{2x} \end{vmatrix}}{\det \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}} = \frac{-(1+x)e^{3x}}{2e^{3x} - e^{3x}} = -(1+x)$$

$$c_2'(x) = \frac{\det \begin{vmatrix} e^x & 0 \\ e^x & (1+x)e^x \end{vmatrix}}{\det \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}} = \frac{(1+x)e^{2x}}{2e^{3x} - e^{3x}} = (1+x)e^{-x}$$

$$c_1'(x) = -(1+x) \Rightarrow c_1(x) = \int -(1+x) dx = -x - \frac{x^2}{2}$$

$$\begin{aligned} c_2'(x) = (1+x)e^{-x} \Rightarrow c_2(x) &= \int (1+x)e^{-x} dx = \int (1+x) d(-e^{-x}) \\ &= -(1+x)e^{-x} - \int -e^{-x} d(1+x) \\ &= -(1+x)e^{-x} + \int e^{-x} dx \\ &= -(1+x)e^{-x} - e^{-x} = -(2+x)e^{-x} \end{aligned}$$

Thus, the particular solution is

$$y_p = \left(-x - \frac{x^2}{2}\right) e^x - (2+x)e^{-x} e^{2x} = -\left(2 + 2x + \frac{x^2}{2}\right) e^x.$$

3. (a) The auxiliary equation is $\alpha^2 + 4\alpha + 13 = 0$, its roots are $\alpha = -2 \pm 3i$.
Hence the complementary function is $y_c = e^{-2x} (A \cos 3x + B \sin 3x)$.

Let $y_p = e^{-2x} (a \cos x + b \sin x)$.

Then $y_p' = -2e^{-2x} (a \cos x + b \sin x) + e^{-2x} (-a \sin x + b \cos x)$

$$= e^{-2x} [(-2a + b) \cos x + (-a - 2b) \sin x]$$

$$y_p'' = -2e^{-2x} [(-2a + b) \cos x + (-a - 2b) \sin x]$$

$$+ e^{-2x} [(-2a + b) \sin x + (-a - 2b) \cos x]$$

$$= e^{-2x} [(3a - 4b) \cos x + (4a + 3b) \sin x]$$

Substitute these into the differential equation:

$$\{e^{-2x} [(3a - 4b) \cos x + (4a + 3b) \sin x]\} + 4\{e^{-2x} [(-2a + b) \cos x + (-a - 2b) \sin x]\} + 13\{e^{-2x} (a \cos x + b \sin x)\} = e^{-2x} \cos x$$

$$\Rightarrow e^{-2x} [8a \cos x + 8b \sin x] = e^{-2x} \cos x$$

Comparing the coefficients on both sides:

$$\begin{cases} 8a = 1 \\ 8b = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{8} \\ b = 0 \end{cases}$$

$$\therefore y_p = \frac{1}{8} e^{-2x} \cos x$$

Hence the general solution is $y = e^{-2x} \left(A \cos 3x + B \sin 3x + \frac{\cos x}{8} \right)$

- (b) The auxiliary equation is $\alpha^2 + 6\alpha + 13 = 0$, its roots are $\alpha = -3 \pm 2i$

The complementary function is $y_c = e^{-3x} (A \cos 2x + B \sin 2x)$.

We look for a particular integral of the form

$$y_p = h_1(x) e^{-3x} \cos 2x + h_2(x) e^{-3x} \sin 2x.$$

Then we have the system of equations

$$\begin{cases} h_1'(x) e^{-3x} \cos 2x + h_2'(x) e^{-3x} \sin 2x = 0 \end{cases} \dots (1)$$

$$\begin{cases} h_1'(x) e^{-3x} [-3 \cos 2x - 2 \sin 2x] + h_2'(x) e^{-3x} [2 \cos 2x - 3 \sin 2x] = x e^{-3x} \cos 2x \end{cases} \dots (2)$$

$e^{3x} \times (1)$ and $e^{3x} \times (2) + 3e^{3x} \times (1)$ give:

$$\begin{cases} h_1'(x) \cos 2x + h_2'(x) \sin 2x = 0 \end{cases} \dots (3)$$

$$\begin{cases} h_1'(x) (-2 \sin 2x) + h_2'(x) (2 \cos 2x) = x \cos 2x \end{cases} \dots (4)$$

$(2 \sin 2x) \times (3) + (\cos 2x) \times (4)$ gives:

$$2 h_2'(x) (\sin^2 2x + \cos^2 2x) = x \cos^2 2x \Rightarrow h_2'(x) = \frac{1}{2} x \cos^2 2x$$

$$\Rightarrow h_2(x) = \int \frac{1}{2} x \cos^2 2x \, dx = \frac{1}{4} \int x (1 + \cos 4x) \, dx = \frac{x^2}{8} + \frac{1}{16} \int x \, d(\sin 4x)$$

$$= \frac{x^2}{8} + \frac{1}{16} (x \sin 4x - \int \sin 4x \, dx) = \frac{x^2}{8} + \frac{1}{16} x \sin 4x + \frac{1}{64} \cos 4x$$

$$= \frac{x^2}{8} + \frac{1}{8} x \sin 2x \cos 2x + \frac{1}{64} (2 \cos^2 2x - 1)$$

$$= \frac{x^2}{8} + \frac{1}{8} x \sin 2x \cos 2x + \frac{1}{32} \cos^2 2x - \frac{1}{64}$$

Substitute $h_2'(x) = \frac{1}{2} x \cos^2 2x$ into (3):

$$h_1'(x) \cos 2x + \frac{1}{2} x \cos^2 2x \sin 2x = 0 \Rightarrow h_1'(x) = -\frac{1}{2} x \cos 2x \sin 2x$$

$$\Rightarrow h_1(x) = -\frac{1}{2} \int x \cos 2x \sin 2x \, dx = -\frac{1}{4} \int x \sin 4x \, dx = \frac{1}{16} \int x \, d(\cos 4x)$$

$$= \frac{1}{16} x \cos 4x - \frac{1}{16} \int \cos 4x \, dx = \frac{1}{16} x \cos 4x - \frac{1}{64} \sin 4x$$

$$= \frac{1}{16} x (1 - 2 \sin^2 2x) - \frac{1}{32} \sin 2x \cos 2x = \frac{x}{16} - \frac{1}{8} x \sin^2 2x - \frac{1}{32} \sin 2x \cos 2x$$

$$\begin{aligned}
\therefore y_p &= \left[\frac{x}{16} - \frac{1}{8}x \sin^2 2x - \frac{1}{32} \sin 2x \cos 2x \right] e^{-3x} \cos 2x \\
&\quad + \left[\frac{x^2}{8} + \frac{1}{8}x \sin 2x \cos 2x + \frac{1}{32} \cos^2 2x - \frac{1}{64} \right] e^{-3x} \sin 2x \\
&= e^{-3x} \left(\frac{1}{16} x \cos 2x - \frac{1}{8} x \sin^2 2x \cos 2x - \frac{1}{32} \sin 2x \cos^2 2x + \frac{x^2}{8} \sin 2x \right. \\
&\quad \left. + \frac{1}{8} x \sin^2 2x \cos 2x + \frac{1}{32} \cos^2 2x \sin 2x - \frac{1}{64} \sin 2x \right) \\
&= e^{-3x} \left(\frac{1}{16} x \cos 2x + \frac{x^2}{8} \sin 2x - \frac{1}{64} \sin 2x \right)
\end{aligned}$$

Hence the general solution is

$$\begin{aligned}
y &= e^{-3x}(A \cos 2x + B \sin 2x) + e^{-3x} \left(\frac{1}{16} x \cos 2x + \frac{x^2}{8} \sin 2x - \frac{1}{64} \sin 2x \right) \\
&= e^{-3x} \left(A \cos 2x + C \sin 2x + \frac{1}{16} x \cos 2x + \frac{x^2}{8} \sin 2x \right) \\
\text{where } C &= B - \frac{1}{64}.
\end{aligned}$$

4. The auxiliary equation is $\alpha^3 + \alpha^2 = 0$, its roots are $\alpha = 0, 0, -1$

Hence the complementary function is $y_c = A + Bx + Ce^{-x}$.

Let $y_p = x(ax^2 + bx + c)e^{-x} = (ax^3 + bx^2 + cx)e^{-x}$.

Then $y_p' = -(ax^3 + bx^2 + cx)e^{-x} + (3ax^2 + 2bx + c)e^{-x}$
 $= [-ax^3 + (3a - b)x^2 + (2b - c)x + c]e^{-x}$,

$y_p'' = -[-ax^3 + (3a - b)x^2 + (2b - c)x + c]e^{-x} + [-3ax^2 + 2(3a - b)x + (2b - c)]e^{-x}$
 $= [ax^3 + (-6a + b)x^2 + (6a - 4b + c)x + (2b - 2c)]e^{-x}$

and

$y_p''' = -[ax^3 + (-6a + b)x^2 + (6a - 4b + c)x + (2b - 2c)]e^{-x}$
 $+ [3ax^2 + 2(-6a + b)x + (6a - 4b + c)]e^{-x}$
 $= [-ax^3 + (9a - b)x^2 + (-18a + 6b - c)x + (6a - 6b + 3c)]e^{-x}$

Substitute these into the differential equation:

$$\begin{aligned}
&[-ax^3 + (9a - b)x^2 + (-18a + 6b - c)x + (6a - 6b + 3c)]e^{-x} \\
&\quad + [ax^3 + (-6a + b)x^2 + (6a - 4b + c)x + (2b - 2c)]e^{-x} = x^2 e^{-x} \\
\Rightarrow [3ax^2 + (-12a + 2b)x + (6a - 4b + c)]e^{-x} &= x^2 e^{-x}
\end{aligned}$$

Comparing coefficients:

$$\begin{cases} 3a = 1 \\ -12a + 2b = 0 \\ 6a - 4b + c = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{3} \\ b = 2 \\ c = 6 \end{cases}$$

$$\therefore y_p = \left(\frac{1}{3}x^3 + 2x^2 + 6x \right) e^{-x}$$

The general solution is

$$y = A + Bx + Ce^{-x} + \left(\frac{1}{3}x^3 + 2x^2 + 6x \right) e^{-x}.$$

Then $y' = B - Ce^{-x} - \left(\frac{1}{3}x^3 + 2x^2 + 6x \right) e^{-x} + (x^2 + 4x + 6)e^{-x}$
 $= B - Ce^{-x} + \left(-\frac{1}{3}x^3 - x^2 - 2x + 6 \right) e^{-x}$

$y'' = Ce^{-x} - \left(-\frac{1}{3}x^3 - x^2 - 2x + 6 \right) e^{-x} + (-x^2 - 2x - 2)e^{-x}$
 $= Ce^{-x} + \left(\frac{1}{3}x^3 - 8 \right) e^{-x}$

From the initial conditions $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$, we have

$$\begin{cases} A + C = 0 \\ B - C + 6 = 1 \\ C - 8 = 0 \end{cases} \Rightarrow \begin{cases} A = -8 \\ B = 3 \\ C = 8 \end{cases}$$

Hence the particular solution is

$$y = -8 + 3x + 8e^{-x} + \left(\frac{1}{3}x^3 + 2x^2 + 6x\right)e^{-x}.$$

5. The auxiliary equation is $\alpha^2 - 9 = 0 \Rightarrow \alpha^2 = 9 \Rightarrow \alpha = \pm 3$.
The complementary function is $y_c = C_1 e^{-3x} + C_2 e^{3x}$.

Note that $\cosh 3x = \frac{1}{2}(e^{3x} + e^{-3x}) = \frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}$.

For a particular integral y_p , we try $y_p = ax^2 + bx + c + Axe^{-3x} + Bxe^{3x}$, where a, b, c, A, B are unknown constants to be determined.

Then $y_p' = 2ax + b - 3Axe^{-3x} + Ae^{-3x} + 3Bxe^{3x} + Be^{3x}$

and $y_p'' = 2a + 9Axe^{-3x} - 3Ae^{-3x} - 3Ae^{-3x} + 9Bxe^{3x} + 3Be^{3x} + 3Be^{3x}$
 $= 2a + 9Axe^{-3x} - 6Ae^{-3x} + 9Bxe^{3x} + 6Be^{3x}$

Substituting these into the differential equation yields

$$2a + 9Axe^{-3x} - 6Ae^{-3x} + 9Bxe^{3x} + 6Be^{3x} - 9(ax^2 + bx + c + Axe^{-3x} + Bxe^{3x}) = x^2 + \frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}$$

$$\Rightarrow -9ax^2 - 9bx + (2a - 9c) - 6Ae^{-3x} + 6Be^{3x} = x^2 + \frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}$$

$$\Rightarrow \begin{cases} -9a = 1 \\ -9b = 0 \\ 2a - 9c = 0 \\ -6A = 1/2 \\ 6B = 1/2 \end{cases} \Rightarrow \begin{cases} a = -1/9 \\ b = 0 \\ c = -2/81 \\ A = -1/12 \\ B = 1/12 \end{cases}$$

Therefore $y_p = -\frac{x^2}{9} - \frac{2}{81} - \frac{xe^{-3x}}{12} + \frac{xe^{3x}}{12} = -\frac{1}{9}\left(x^2 + \frac{2}{9}\right) + \frac{1}{6}x \sinh x$.

The general solution is

$$y = C_1 e^{-3x} + C_2 e^{3x} - \frac{1}{9}\left(x^2 + \frac{2}{9}\right) + \frac{1}{6}x \sinh x.$$

6. $y'' + 4y = \sec 2x \Rightarrow m^2 + 4 = 0 \Leftrightarrow m = \pm 2i$. Thus, $y_c = c_1 \cos 2x + c_2 \sin 2x$.

Let $y_p = \varphi_1 \cos 2x + \varphi_2 \sin 2x$. Then we have the system

$$\begin{cases} \varphi_1' \cos 2x + \varphi_2' \sin 2x = 0 \\ -2\varphi_1' \sin 2x + 2\varphi_2' \cos 2x = \sec 2x \end{cases} \Rightarrow \begin{cases} 2\varphi_1' \sin 2x \cos 2x + 2\varphi_2' \sin^2 2x = 0 \\ -2\varphi_1' \sin 2x \cos 2x + 2\varphi_2' \cos^2 2x = 1 \end{cases}$$

$$\Rightarrow 2\varphi_2' = 1 \Rightarrow \varphi_2 = \frac{1}{2}x$$

$$\varphi_1' \cos 2x + \frac{1}{2} \sin 2x = 0 \Rightarrow \varphi_1' = -\frac{1}{2} \frac{\sin 2x}{\cos 2x}$$

$$\Rightarrow \varphi_1 = -\frac{1}{2} \int \frac{\sin 2x}{\cos 2x} dx = \frac{1}{4} \int \frac{1}{\cos 2x} d(\cos 2x) = \frac{1}{4} \ln |\cos 2x|$$

$\therefore y_p = \frac{1}{4} \ln |\cos 2x| \cos 2x + \frac{1}{2} x \sin 2x$.

Thus, $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4} \cos 2x \ln |\cos 2x| + \frac{1}{2} x \sin 2x$

7. Consider $P(\alpha) = \alpha^3 - 7\alpha^2 + 15\alpha - 9$.

Observe that $P(1) = 0$ so $P(\alpha) = \alpha^3 - 7\alpha^2 + 15\alpha - 9 = (\alpha - 1)(\alpha^2 - 6\alpha + 9) = (\alpha - 1)(\alpha - 3)^2$.

Thus, the roots of the auxiliary equation $\alpha^3 - 7\alpha^2 + 15\alpha - 9 = 0$ are $\alpha = 1, 3, 3$.

In particular, $\alpha = 3$ is a root of multiplicity 2 of the auxiliary equation.

The complementary function is $x_c = Ae^t + Be^{3t} + Cte^{3t}$.

For a particular integral x_p , let $x_p = t^2(at^3 + bt^2 + ct + d)e^{3t} = (at^5 + bt^4 + ct^3 + dt^2)e^{3t}$.

Then $x_p' = (5at^4 + 4bt^3 + 3ct^2 + 2dt)e^{3t} + 3(at^5 + bt^4 + ct^3 + dt^2)e^{3t}$

$$= [3at^5 + (5a + 3b)t^4 + (4b + 3c)t^3 + (3c + 3d)t^2 + 2dt]e^{3t},$$

$$x_p'' = [15at^4 + 4(5a + 3b)t^3 + 3(4b + 3c)t^2 + 2(3c + 3d)t + 2d]e^{3t}$$

$$+ 3[3at^5 + (5a + 3b)t^4 + (4b + 3c)t^3 + (3c + 3d)t^2 + 2dt]e^{3t}$$

$$= [9at^5 + (30a + 9b)t^4 + (20a + 24b + 9c)t^3 + (12b + 18c + 9d)t^2 + (6c + 12d)t + 2d]e^{3t},$$

$$x_p''' = [27at^5 + (135a + 27b)t^4 + (180a + 108b + 27c)t^3 + (60a + 108b + 81c + 27d)t^2 + (24b + 54c + 54d)t + (6c + 18d)]e^{3t}.$$

Substituting these into the differential equation, we have

$$\begin{aligned} & [27at^5 + (135a + 27b)t^4 + (180a + 108b + 27c)t^3 + (60a + 108b + 81c + 27d)t^2 + (24b + 54c + 54d)t + (6c + 18d)]e^{3t} \\ & - 7[9at^5 + (30a + 9b)t^4 + (20a + 24b + 9c)t^3 + (12b + 18c + 9d)t^2 + (6c + 12d)t + 2d]e^{3t} \\ & + 15[3at^5 + (5a + 3b)t^4 + (4b + 3c)t^3 + (3c + 3d)t^2 + 2dt]e^{3t} - 9(at^5 + bt^4 + ct^3 + dt^2)e^{3t} = t^3 e^{3t} \\ \Rightarrow & [40at^3 + (60a + 24b)t^2 + (24b + 12c)t + (6c + 4d)]e^{3t} = t^3 e^{3t} \end{aligned}$$

Comparing the coefficients, we have

$$\begin{cases} 40a = 1 \\ 60a + 24b = 0 \\ 24b + 12c = 0 \\ 6c + 4d = 0 \end{cases} \Rightarrow \begin{cases} a = 1/40 \\ b = -1/16 \\ c = 1/8 \\ d = -3/16 \end{cases}$$

Thus, a particular integral is given by $x_p = \left(\frac{t^5}{40} - \frac{t^4}{16} + \frac{t^3}{8} - \frac{3t^2}{16} \right) e^{3t}$.

8. Let $x = e^t$ then $\frac{dx}{dt} = e^t = x$. By chain rule, $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = x \frac{dy}{dx}$.

And

$$\frac{d^2y}{dt^2} = \frac{d}{dt} \left(\frac{dy}{dt} \right) = \frac{d}{dx} \left(x \frac{dy}{dx} \right) \frac{dx}{dt} = \left(x \frac{d^2y}{dx^2} + \frac{dy}{dx} \right) x = x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx}$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - x \frac{dy}{dx} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Putting $x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$, $x \frac{dy}{dx} = \frac{dy}{dt}$, $\frac{1}{x} = e^{-t}$ into $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{1}{x}$, we have

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + 3 \frac{dy}{dt} + y = e^{-t} \Leftrightarrow \frac{d^2y}{dt^2} + 2 \frac{dy}{dt} + y = e^{-t}$$

The roots of the auxiliary equation $\alpha^2 + 2\alpha + 1 = 0$ are $\alpha = -1, -1$

The complementary function is $y_c = (c_1 t + c_2)e^{-t}$.

For a particular integral, we try $y_p = at^2 e^{-t}$.

Then $y_p' = a(-t^2 + 2t)e^{-t}$

and $y_p'' = a(t^2 - 4t + 2)e^{-t}$.

Substituting these into the differential equation $\frac{d^2 y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t}$:

$$a(t^2 - 4t + 2)e^{-t} + 2a(-t^2 + 2t)e^{-t} + at^2 e^{-t} = e^{-t}$$

$$\Rightarrow 2ae^{-t} = e^{-t}$$

$$\Rightarrow a = \frac{1}{2}$$

$$\therefore y_p = \frac{1}{2}t^2 e^{-t}$$

Hence the general solution is $y = (c_1 t + c_2)e^{-t} + \frac{1}{2}t^2 e^{-t}$.

Restoring the original variables, the general solution of the Euler equation is

$$y = (c_1 \ln x + c_2) \frac{1}{x} + \frac{1}{x} \frac{(\ln x)^2}{2} = \frac{1}{x} \left(\frac{(\ln x)^2}{2} + c_1 \ln x + c_2 \right).$$

With the boundary conditions $y(1) = 0$ and $y(2) = 0$,

$$0 = y(1) = \frac{1}{1} \left(\frac{(\ln 1)^2}{2} + c_1 \ln 1 + c_2 \right), \quad 0 = y(2) = \frac{1}{2} \left(\frac{(\ln 2)^2}{2} + c_1 \ln 2 + c_2 \right) \Rightarrow \begin{cases} c_2 = 0 \\ c_1 = -\frac{\ln 2}{2} \end{cases}$$

$$\text{Hence the particular solution is } y = \frac{\ln x}{2x} (\ln x - \ln 2) \Rightarrow y = \frac{\ln x \ln \frac{x}{2}}{2x}.$$

9. Let $Q(t)$ and $I(t)$ be the instantaneous charge and current at time t . By Kirchhoff's laws:

$$2 \frac{dI}{dt} + 12I + \frac{Q}{0.02} = 100 \sin 4t \quad \text{or written as} \quad \frac{d^2 Q}{dt^2} + 6 \frac{dQ}{dt} + 25Q = 50 \sin 4t.$$

The roots of the auxiliary equation $\alpha^2 + 6\alpha + 25 = 0$ are $\alpha = -3 \pm 4i$.

The complementary function is $Q_c = e^{-3t}(A \cos 4t + B \sin 4t)$.

For a particular integral Q_p , we try $Q_p = a \sin 4t + b \cos 4t$, where a, b are unknown constants to be determined. Then $Q_p' = 4a \cos 4t - 4b \sin 4t$, $Q_p'' = -16a \sin 4t - 16b \cos 4t$.

Substituting these into the differential equation yields

$$-16a \sin 4t - 16b \cos 4t + 6(4a \cos 4t - 4b \sin 4t) + 25(a \sin 4t + b \cos 4t) = 50 \sin 4t$$

$$\Rightarrow (-16a - 24b + 25a) \sin 4t + (-16b + 24a + 25b) \cos 4t = 50 \sin 4t$$

$$\Rightarrow (9a - 24b) \sin 4t + (24a + 9b) \cos 4t = 50 \sin 4t$$

$$\Rightarrow \begin{cases} 9a - 24b = 50 \\ 24a + 9b = 0 \end{cases} \Rightarrow \begin{cases} 72a - 192b = 400 \\ 72a + 27b = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{50}{73} \\ b = -\frac{400}{219} \end{cases}$$

A particular integral is given by $Q_p = \frac{50}{73} \sin 4t - \frac{400}{219} \cos 4t$.

The general solution is

$$Q(t) = Q_c(t) + Q_p(t) = e^{-3t}(A \cos 4t + B \sin 4t) + \frac{50}{73} \sin 4t - \frac{400}{219} \cos 4t.$$

Observe that

$$Q(t) = e^{-3t} (A \cos 4t + B \sin 4t) + \frac{50}{73} \sin 4t - \frac{400}{219} \cos 4t$$

$$\Rightarrow \frac{dQ}{dt}(t) = -3e^{-3t} (A \cos 4t + B \sin 4t) + e^{-3t} (-4A \sin 4t + 4B \cos 4t) + \frac{200}{73} \cos 4t + \frac{1600}{219} \sin 4t$$

With the boundary conditions $Q(0) = 0$, $I(0) = \frac{dQ}{dt}(0) = 0$, we have

$$\begin{cases} A - \frac{400}{219} = 0 \\ -3A + 4B + \frac{200}{73} = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{400}{219} \\ B = \frac{50}{73} \end{cases}.$$

The charge for $t > 0$, $Q(t) = e^{-3t} \left(\frac{400 \cos 4t}{219} + \frac{50 \sin 4t}{73} \right) + \frac{50 \sin 4t}{73} - \frac{400 \cos 4t}{219}$.

The current for $t > 0$,

$$\begin{aligned} I(t) &= \frac{dQ}{dt} = -3e^{-3t} \left(\frac{400 \cos 4t}{219} + \frac{50 \sin 4t}{73} \right) + e^{-3t} \left(\frac{-1600 \sin 4t}{219} + \frac{200 \cos 4t}{73} \right) + \frac{200 \cos 4t}{73} + \frac{1600 \sin 4t}{219} \\ &= \frac{200}{219} (3 \cos 4t + 8 \sin 4t) - e^{-3t} \left(\frac{2050 \sin 4t}{219} + \frac{200 \cos 4t}{73} \right) \end{aligned}$$

The first term is the steady-state current and the second term which tends to zero to 0 as t tends to ∞ is called the transient current.

-End-