

- Submission Due Date: April 16, Tuesday, Midnight, 11:59 PM
- Please submit the answer sheet to Canvas before the due expires.
- Once it is past the due date & time, the submission page will automatically close.
- Make sure to write the name and student ID on top of the first page.

1. (20 points) (a) Solve the following DE and find the highest point on the curve $y = y(x)$.

$$y'' + 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 6$$

- (b) Solve the following initial value problem (IVP).

$$3y''' + 2y'' = 0, \quad y(0) = -1, \quad y'(0) = 0, \quad y''(0) = 1$$

(Note that it is a third order DE.)

2. (20 points) Let us consider the following DE with a general solution $y = y(x)$.

$$\frac{d^2y}{dx^2} + 3 \cdot \frac{dy}{dx} + 2y = 0 \tag{1}$$

- (a) Find two complete solutions $y_1(x)$ and $y_2(x)$ of Eq. (1) that satisfy the following IVPs.

$$\begin{aligned} y_1(0) &= 3, & y_1'(0) &= 1 \\ y_2(0) &= 0, & y_2'(0) &= 1 \end{aligned}$$

- (b) Find the interaction point: i.e., the coordinate of $y_1(x) = y_2(x)$.

3. (20 points) Solve the following DEs.

- (a) Find the complete solution $y = y(x)$.

$$\frac{d^2y}{dx^2} - 2 \cdot \frac{dy}{dx} + 2y = 0, \quad y(0) = 0, \quad y'(0) = 5$$

- (b) Find the general solution $y = y(x)$.

$$9 \cdot \frac{d^2y}{dx^2} - 12 \cdot \frac{dy}{dx} + 4y = 0$$

4. (20 points) Find the general solution of the DEs. For IVPs, determine the complete solution.

(a)

$$\frac{d^2y}{dt^2} + 4 \cdot \frac{dy}{dt} + 3y = 2 \sin(2t)$$

(b)

$$\frac{d^2y}{dt^2} + 7 \cdot \frac{dy}{dt} + 10y = 6e^{-2t}$$

(c)

$$\frac{d^2y}{dt^2} - 10 \cdot \frac{dy}{dt} + 25y = 17 \cdot te^{5t}$$

(d)

$$t^2 \cdot \frac{d^2y}{dt^2} - t \cdot \frac{dy}{dt} + y = 0, \quad y(1) = 3, \quad y'(1) = -2$$

5. (20 points) Find a power series solution centered at $x = 0$, where $y(0) = a_0$, $y'(0) = a_1$.

(a)

$$\frac{d^2y}{dx^2} - 2x \cdot \frac{dy}{dx} + 2y = 0$$

(b)

$$(1 - x^2) \cdot \frac{d^2y}{dx^2} - 2x \cdot \frac{dy}{dx} + 20y = 0$$

6. (20 points) Find the general solutions of the following differential equations.

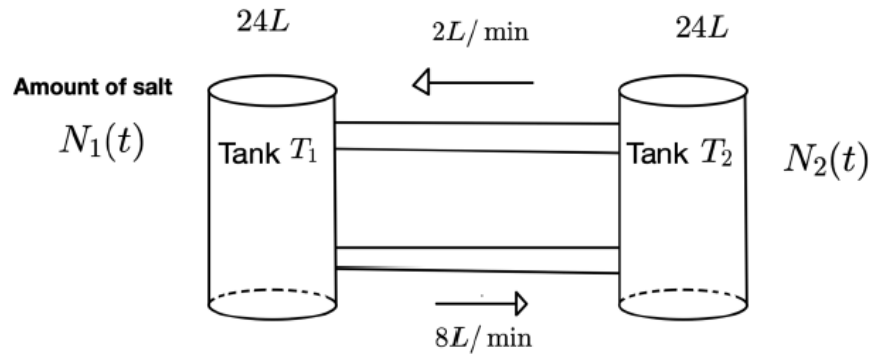
(a)

$$\frac{d^2y}{dx^2} + 2 \cdot \frac{dy}{dx} + 2y = 2e^{2x} + e^{-3x}$$

(b)

$$\frac{d^2y}{dx^2} + 4 \cdot \frac{dy}{dx} + 4y = \frac{e^{-2x}}{x^2}$$

7. (20 points) Consider two tanks T_1 and T_2 , each containing $24L$ of water and it is assumed that the water does not overflow. Initially, we dissolved $N_1(0)$ amount of salt in T_1 and $N_2(0)$ in T_2 . After connecting the tanks, the liquid is circulated at a different rate. Specifically, liquid is leaving from tank T_1 at a rate of $8L$ per minute, whereas liquid is leaving from tank T_2 at a rate of $2L$ per minute. Calculate the amount of salt at time t in each tank, i.e., $N_1(t)$ in tank T_1 and $N_2(t)$ in tank T_2 , respectively. Please provide derivations to receive full marks.



8. (20 points) Convert the following DE into a first-order Matrix form.

(a)

$$\frac{d^2 y}{dt^2} - 3 \cdot \frac{dy}{dt} + 6y = \sin(t)$$

(b)

$$\frac{d^2 y}{dt^2} + 4y = t^2 - e^{3t}$$

(c)

$$\frac{d^3 y}{dt^3} - 8 \cdot \frac{d^2 y}{dt^2} + 5 \cdot \frac{dy}{dt} + 2y = 7t$$

9. (20 points) Use the Matrix exponential to solve the following initial value problem (IVP).

$$\mathbf{y}'(t) = A\mathbf{y}(t), \quad A = \begin{bmatrix} 6 & -3 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

10. (20 points) Solve the following system of DEs.

$$\begin{cases} x' &= 3x + 4y \\ y' &= 3x + 2y \end{cases}$$

11. (20 points) Find the general solution of the following system of DEs.

(a)

$$\frac{dx_1}{dt} = x_1 + x_2, \quad \frac{dx_2}{dt} = 4x_1 + x_2$$

(b)

$$\frac{dx_1}{dt} = -\frac{1}{2}x_1 + x_2, \quad \frac{dx_2}{dt} = -x_1 - \frac{1}{2}x_2$$

(c)

$$\frac{dx_1}{dt} = x_1 - x_2, \quad \frac{dx_2}{dt} = x_1 + 3x_2$$

12. (20 points) Find the general solution of the following system of DEs using Substitution Method.

(a)

$$\frac{dx}{dt} = x + 2y, \quad \frac{dy}{dt} = 2x - 2y, \quad x(0) = 1, \quad y(0) = 2$$

(b)

$$\frac{d^2x_1}{dt^2} = -5x_1 + 4x_2, \quad \frac{d^2x_2}{dt^2} = 4x_1 - 5x_2$$

13. (20 points) Find the general solution of the following DEs using Laplace Transforms.

(a)

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 0, \quad x(0) = 2, \quad x'(0) = -1$$

(b)

$$\frac{d^2x}{dt^2} + 4x = \sin(3t), \quad x(0) = 0, \quad x'(0) = 0$$

14. (20 points) Find the general solution of the following DEs using Laplace Transforms.

(a)

$$\frac{dy}{dt} = 2y - 3z, \quad \frac{dz}{dt} = -2y + z, \quad y(0) = 8, \quad z(0) = 3$$

(b)

$$y'' + 2y' + y = 1, \quad y(0) = 2, \quad y'(0) = -2$$

(c)

$$y'' + y = \sin(3t), \quad y(0) = y'(0) = 0$$

15. (20 points) Find the solution of the following DEs using Laplace Transforms.

(a)

$$y'' + 7y' + 12y = 3e^{-2t}, \quad y(0) = y'(0) = 0$$

(b)

$$y'' + 8y' + 15y = -4\delta(t - 7), \quad y(0) = y'(0) = 0$$

(c)

$$y'' + 8y' + 20y = 4\delta(t - 3) - 12\delta(t - 5), \quad y(0) = -2, \quad y'(0) = 6$$

(d)

$$x' = x + 2y, \quad y' = 2x - 2y, \quad x(0) = 1, \quad y(0) = 0$$