

EE2104 Introduction to Electromagnetics
Assignment 3 Solution

Deadline (for uploading): 7:00 pm, 7 Nov 2020

Please put down your name and university ID.

SI units are used unless otherwise specified.

1. Given a field $\mathbf{G} = y\hat{\mathbf{x}} - x\hat{\mathbf{y}}$. There are points $A(1,0)$ and $B(0,1)$.
(a) Find

$$\int_{\text{Path } S} \mathbf{G} \cdot d\mathbf{l}$$

where the path S is a straight line from point A to point B .

- (b) Find

$$\int_{\text{Path } C} \mathbf{G} \cdot d\mathbf{l}$$

where the path C is the circular arc of radius 1 connecting point A and point B .

- (c) Is $\mathbf{G}$ conservative? Why?

Solution:

- (a) When the path is a straight line from point A to point B , we have $y = -x+1$ for the path.
Since $d\mathbf{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy = \hat{\mathbf{x}}dx - \hat{\mathbf{y}}dx = (\hat{\mathbf{x}} - \hat{\mathbf{y}})dx$

$$\int_{\text{Path } S} \mathbf{G} \cdot d\mathbf{l} = \int_{\text{Path } S} ydx - \int_{\text{Path } S} xdy = \int_{x=1}^0 (-x+1)dx - \int_{x=1}^0 x(-dx) = \int_{x=1}^0 dx = -1$$

- (b) When the path is the circular arc of radius 1 connecting point A and point B , we have

$$x = \cos \varphi, \quad y = \sin \varphi \quad \text{for } \varphi: 0 \rightarrow \pi/2$$

Therefore, we have $d\mathbf{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy = -\sin \varphi d\varphi \hat{\mathbf{x}} + \cos \varphi d\varphi \hat{\mathbf{y}}$

So

$$\int_{\text{Path } C} \mathbf{G} \cdot d\mathbf{l} = \int_{\varphi=0}^{\pi/2} [-\sin^2 \varphi - \cos^2 \varphi] d\varphi = \int_{\varphi=0}^{\pi/2} -d\varphi = -\frac{\pi}{2}$$

- (c) $\mathbf{G}$ is not conservative.
Because the integration of $\mathbf{G}$ is dependent on the path.

2. Consider a copper wire with a cross-sectional area $S = 1 \text{ mm}^2$. Copper has $n = 10^{29}$ electrons in every 1 m^3 . It carries a current of $I = 4 \text{ A}$ for driving a light bulb $L = 50 \text{ m}$ away from a voltage source. The conductivity is $\sigma = 10^8 (\Omega^{-1}\text{m}^{-1})$.
- Find the time needed for an electron to drift from the source to the light bulb.
 - Ohm's law in the macroscopic form gives the voltage $V = IR$. Find the resistance R of the copper wire.
 - Find the power dissipated in every 1 m of the wire.

Solution:

- (a) The velocity of electron drifting is

$$V_{\text{drift}} = \frac{I}{Snq} = 250 (\mu\text{m/s})$$

The time needed is

$$\text{Time} = \frac{L}{V_{\text{drift}}} = \frac{50 (m)}{250 (\mu\text{m/s})} = 2 \times 10^5 \text{ s}$$

- (b) Using Ohm's law, we have

$$J = \sigma E \Rightarrow \frac{I}{S} = \sigma E \Rightarrow \frac{I}{S} = \sigma \frac{V}{L}$$

Thus, the resistance R is

$$R = \frac{V}{I} = \frac{L}{\sigma S} = 0.5 \Omega$$

- (c) We know that power

$$P = IV = IEL = \frac{I^2 L}{\sigma S}$$

Thus,

$$\frac{P}{L} = \frac{I^2}{\sigma S} = 0.16 \text{ W/m}$$

which is the power dissipated in every 1 m of the wire.

3. In a dielectric of permittivity ϵ , the potential field is given in spherical coordinates by

$$V(r, \theta, \phi) = V_0 \frac{r}{a} \cos \theta$$

where V_0 is a constant potential and a is a constant length.

(a) Find the electric field.

(b) Find the total charge contained within the region $r < a$.

Solution:

(a) Calculating the gradient of V in spherical coordinates, we have

$$\begin{aligned} \nabla V &= \frac{\partial V}{\partial r} \hat{\mathbf{a}}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\mathbf{a}}_\theta + \frac{1}{r \cos \theta} \frac{\partial V}{\partial \phi} \hat{\mathbf{a}}_\phi \\ &= \frac{V_0}{a} \cos \theta \hat{\mathbf{a}}_r - \frac{V_0}{r} \frac{r}{a} \sin \theta \hat{\mathbf{a}}_\theta \\ &= \frac{V_0}{a} \cos \theta \hat{\mathbf{a}}_r - \frac{V_0}{a} \sin \theta \hat{\mathbf{a}}_\theta \end{aligned}$$

Thus, the electric field is

$$\mathbf{E} = -\nabla V = -\frac{V_0}{a} \cos \theta \hat{\mathbf{a}}_r + \frac{V_0}{a} \sin \theta \hat{\mathbf{a}}_\theta$$

(b) We have

$$\mathbf{D} = \epsilon \mathbf{E} = -\frac{\epsilon V_0}{a} \cos \theta \hat{\mathbf{a}}_r + \frac{\epsilon V_0}{a} \sin \theta \hat{\mathbf{a}}_\theta$$

Consider a spherical Gaussian surface as $r = a$.

Only the $\hat{\mathbf{a}}_r$ component of $\mathbf{D}$ is perpendicular to the surface.

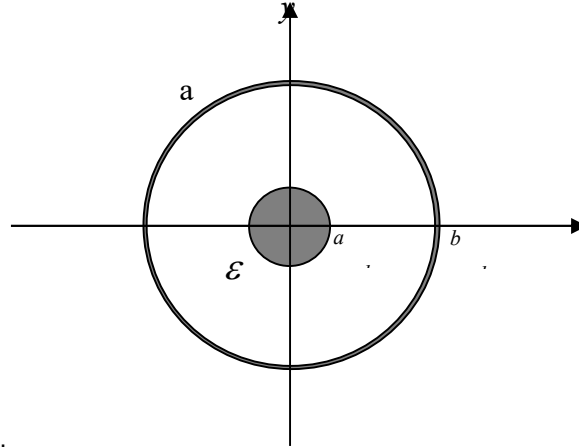
Using Gauss's law, we have

$$Q_{\text{enc}} = \oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left(-\frac{\epsilon V_0}{a} \cos \theta \right) r^2 \sin \theta d\theta d\phi$$

Therefore, the total charge contained within the region $r < a$ is

$$Q_{\text{enc}} = -\frac{\epsilon V_0}{a} a^2 \int_{\theta=0}^{\pi} \cos \theta \sin \theta d\theta \int_{\phi=0}^{2\pi} d\phi = 0$$

4. Consider a coaxial cable below consists of cylinders of perfect conductors of length $l = 7$ m. In the cylindrical coordinate system, the inner conductor is at $0 < \rho < a$ and the outer conductor is at $\rho = b$, where $a = 3$ mm and $b = 8$ mm. The outer conductor is grounded. The region between the conductors is filled with a dielectric with relative permittivity $\epsilon_r = 4$. The inner and outer conductors hold charges of $+Q$ and $-Q$ with $Q = 10$ μC .



- (a) Find the maximal $|\mathbf{E}|$ in the structure.
 (b) Find the capacitance of the whole cable.

Solution:

- (a) Inside the inner and outer conductor, the electric field is zero. For the electric field inside the dielectric, we can consider there is no leaking of the electric field and the charges in the inner/outer conductors are uniformly distributed. Because of the symmetric structure, the electric field can be expressed as

$$\mathbf{E}(\rho, \phi, z) = \mathbf{E}(\rho) = E(\rho)\hat{\rho}$$

Consider a cylindrical Gaussian surface S with a radius of ρ and a height of h , centered along z -axis and placed between the inner and outer conductor.

Using Gauss's law, we have

$$Q_{\text{enc}} = \oint_S \mathbf{D}(\rho, \phi, z) \cdot d\mathbf{S}$$

Thus,

$$Q_{\text{enc}} = \oint_S \epsilon_0 \epsilon_r \mathbf{E}(\rho, \phi, z) \cdot d\mathbf{S} = \oint_S \epsilon_0 \epsilon_r \mathbf{E}(\rho) \cdot d\mathbf{S}$$

Then we have

$$Q_{\text{enc}} = \int_{\text{Top Surf.}} \epsilon_0 \epsilon_r \mathbf{E}(\rho) \cdot d\mathbf{S} + \int_{\text{Curved Surf.}} \epsilon_0 \epsilon_r \mathbf{E}(\rho) \cdot d\mathbf{S} + \int_{\text{Bottom Surf.}} \epsilon_0 \epsilon_r \mathbf{E}(\rho) \cdot d\mathbf{S}$$

Because $\mathbf{E}(\rho) = E(\rho)\hat{\rho}$, we have

$$\int_{\text{Top Surf.}} \epsilon_0 \epsilon_r \mathbf{E}(\rho) \cdot d\mathbf{S} = \int_{\text{Bottom Surf.}} \epsilon_0 \epsilon_r \mathbf{E}(\rho) \cdot d\mathbf{S} = 0$$

Thus,

$$Q_{\text{enc}} = \int_{\text{Curved Surf.}} \epsilon_0 \epsilon_r E(\rho) (\hat{\rho} \cdot \hat{\rho}) dS$$

$$\frac{Qh}{l} = 2\pi\rho h \epsilon_0 \epsilon_r E(\rho)$$

Therefore,

$$E(\rho) = \frac{Q}{2l\pi\epsilon_0\epsilon_r\rho}$$

When $\rho = a$, we have

$$|\mathbf{E}|_{\text{max}} = E_{\text{max}} = \frac{Q}{2al\pi\epsilon_0\epsilon_r} = 2.14 \times 10^6 \text{ (V/m)}$$

(b) The potential difference between the conductors is

$$V = - \int_{\rho=b}^a \mathbf{E} \cdot d\mathbf{L} = - \int_{\rho=b}^a E(\rho) (\hat{\rho} \cdot \hat{\rho}) d\rho = - \int_{\rho=b}^a \frac{Q}{2l\pi\epsilon_0\epsilon_r\rho} d\rho = \frac{Q}{2l\pi\epsilon_0\epsilon_r} \ln\left(\frac{b}{a}\right)$$

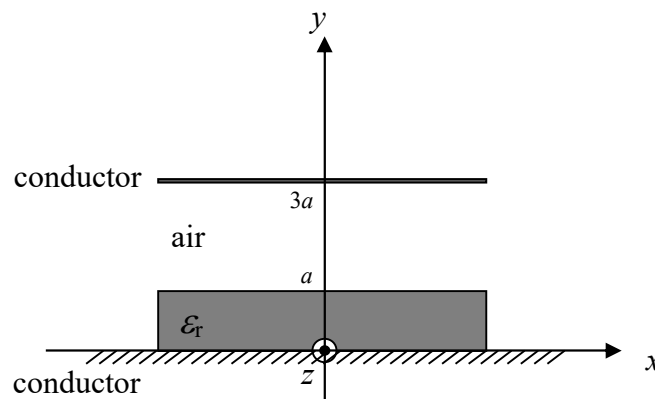
So the capacitance of the cable is

$$C = \frac{Q}{V} = \frac{2l\pi\epsilon_0\epsilon_r}{\ln\left(\frac{b}{a}\right)} = 1.59 \text{ (nF)}$$

5. The figure below shows a planar structure of infinite length. The structure is along z -axis. The different layers contain the following materials:
- (i) Lower conductor for $y < 0$,
 - (ii) Dielectric with relative permittivity ϵ_r for $0 < y < a$,
 - (iii) Air gap for $a < y < 3a$,
 - (iv) Upper conductor for $y = 3a$.

The potential of the upper conductor is maintained at $+V_0$ while the potential of the lower conductor is grounded to zero. Perform the following tasks to solve for the potential V in or between the two conductors.

- (a) Using the Laplace's equation, find the expressions for the potential V for $0 < y < a$ and $a < y < 3a$. There should be a total of 4 unknown constants.
- (b) Using the boundary condition of V at the air-dielectric boundary, reduce the number of unknown constants to 3.
- (c) Using the boundary condition for the electric field at the air-dielectric boundary, reduce the number of unknown constants to 2.
- (d) Using the boundary conditions for V at the conductors, solve for V .



Solution:

- (a) Apply Laplace's equation for $0 < y < a$ first. Then we have $\nabla^2 V = 0$. According to the symmetry of the structure, we have $V(x, y) = V(y)$. Thus,

$$\nabla^2 V = 0 \Rightarrow \frac{\partial^2 V}{\partial^2 y} = 0 \Rightarrow V(y) = Ay + B$$

Similarly, for $a < y < 3a$, we have

$$V(y) = Cy + D$$

where A , B , C , and D are unknown constants.

- (b) At the air-dielectric boundary, the potential is continuous. Thus, at $y = a$, we have

$$Aa + B = Ca + D \Rightarrow D = (A - C)a + B$$

Then we have

$$\begin{aligned} V(y) &= Ay + B, & \text{for } 0 < y \leq a \\ V(y) &= Cy + (A - C)a + B, & \text{for } a < y \leq 3a \end{aligned}$$

(c) D is continuous at the air-dielectric boundary.

Thus,

$$D_y(a^-) = D_y(a^+) \Rightarrow \epsilon_0 \epsilon_r E_y(a^-) = \epsilon_0 E_y(a^+)$$

So we have

$$-\epsilon_0 \epsilon_r \frac{\partial V(a^-)}{\partial y} = \epsilon_0 \frac{\partial V(a^+)}{\partial y} \Rightarrow \epsilon_r A = C$$

Therefore, after reducing the unknown constants to 2, we have

$$\begin{aligned} V(y) &= Ay + B, & \text{for } 0 < y \leq a \\ V(y) &= \epsilon_r Ay + (A - \epsilon_r A)a + B, & \text{for } a < y \leq 3a \end{aligned}$$

(d) Using the boundary conditions for V at the conductor as

$$V(y = 0) = 0 \text{ and } V(y = 3a) = +V_0$$

We have

$$0 = B \text{ and } V_0 = 3a\epsilon_r A + (A - \epsilon_r A)a + B$$

Hence,

$$A = \frac{V_0}{2\epsilon_r a + a}$$

So

$$V = \frac{V_0}{2\epsilon_r a + a} y \quad \text{for } 0 < y \leq a$$

$$V = \frac{\epsilon_r V_0}{2\epsilon_r a + a} y + \frac{V_0(1 - \epsilon_r)}{2\epsilon_r + 1} \quad \text{for } a < y \leq 3a$$