

EE 3114 Systems and Control

8. State-Space Modeling

Semester B, 2017-18

State-Space Representation

Dynamical Relationship

A **state** is a set of **internal variables** that define completely the dynamical behavior of the system

For example, the **states** of a moving object include:

Position

Velocity

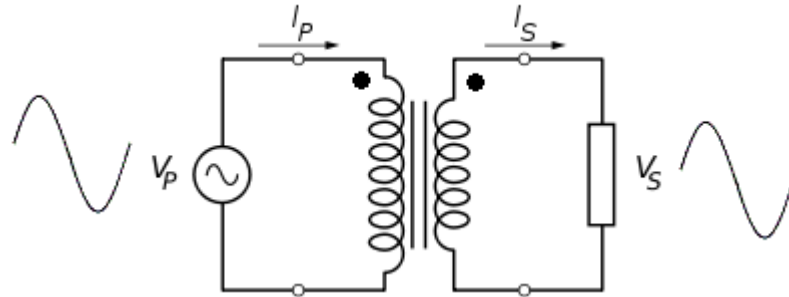
Acceleration

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} p \\ v \\ a \end{bmatrix}$$

State vector State variables (components)

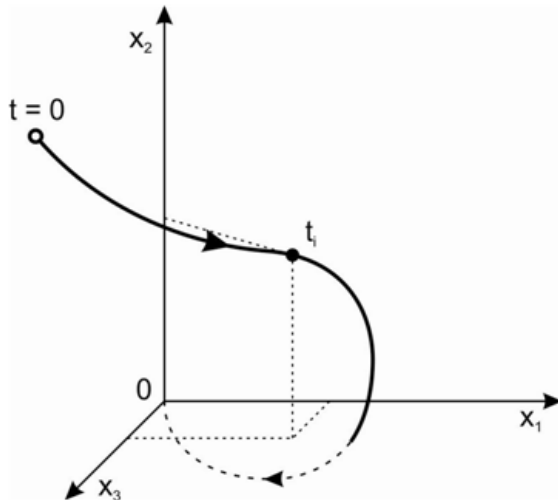
Another example, the **states** of a transformer:

Voltage
Current



$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} V_p \\ I_p \end{bmatrix} \quad \mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} V_s \\ I_s \end{bmatrix}$$

State vector State variables



State and
State Space

Formal Definition of “State”

Definition: A **state** of a dynamic system is a set of variables which, if specified at the time $t = t_0$, provide all the necessary information to find the behavior of the output at any time $t > t_0$, given the input to the system on interval $[t_0, t)$.

Note:

The input-output relationship is **unique**

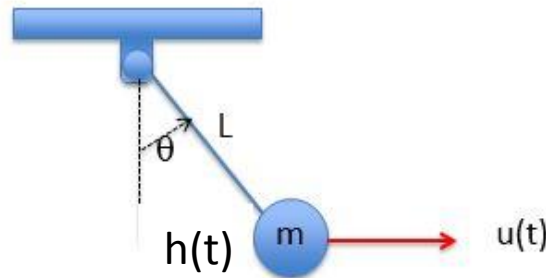
But a definition of the state is **not unique**

Example: Pendulum

Mass: m

Input: force $u(t)$

Output: horizontal displacement $h(t)$ (measurement)



Physical relationship between input and output:

$$mL^2\ddot{\theta} = -mgL \sin \theta + u(t)L \cos \theta$$

Secondary School Physics

$$mL^2\ddot{\theta} = -mgL \sin \theta + u(t)L \cos \theta$$

Representation 1

Define:

$$\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \end{aligned}$$

System dynamics

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{L} \sin x_1 + \frac{1}{mL} u \cos x_1 \end{cases}$$

Output relationship

$$y = L \sin x_1 = h$$

Representation 2

Define:

$$\begin{aligned} x_1 &= \theta \\ x_2 &= mL^2 \dot{\theta} \end{aligned}$$

System dynamics

$$\begin{cases} \dot{x}_1 = \frac{1}{mL^2} x_2 \\ \dot{x}_2 = -mgL \sin x_1 + Lu \cos x_1 \end{cases}$$

Output relationship

$$y = L \sin x_1 = h$$

Not unique

Pendulum: Representation 1

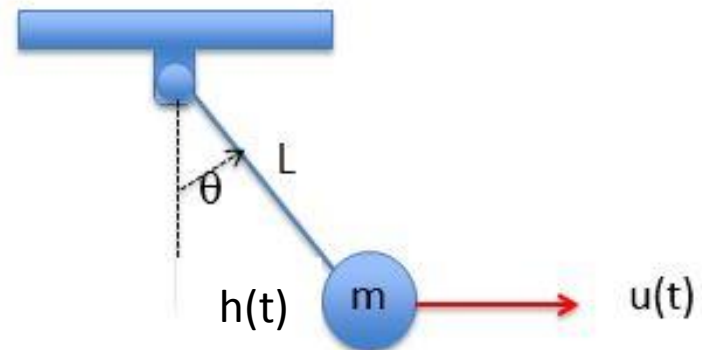
$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

$$\mathbf{y} = g(\mathbf{x})$$

$$f = \begin{bmatrix} x_2(t) \\ -\frac{g}{L} \sin x_1(t) + \frac{1}{mL} \cos x_1 u(t) \end{bmatrix}$$

$$g = L \sin x_1(t)$$

$$x_1 = \theta, \quad x_2 = \dot{\theta}$$

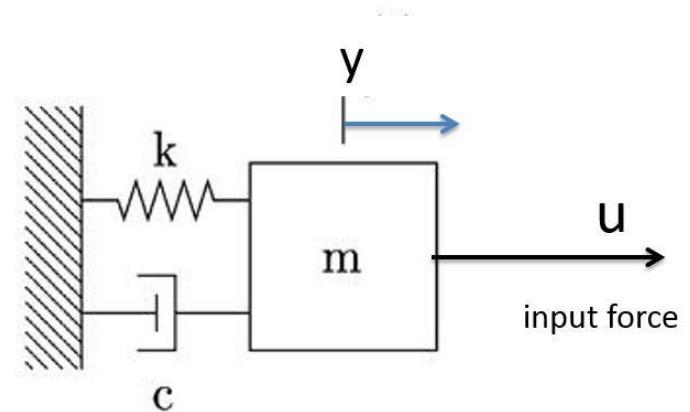


Example: Mass-Spring-Damper System

$$m\ddot{y} + c\dot{y} + ky = u$$



$$\ddot{y} = -\frac{c}{m}\dot{y} - \frac{k}{m}y + \frac{1}{m}u$$



Output: velocity $\dot{y}$

States: $x_1(t) = y(t)$ Displacement

$x_2(t) = \dot{y}(t)$ Velocity

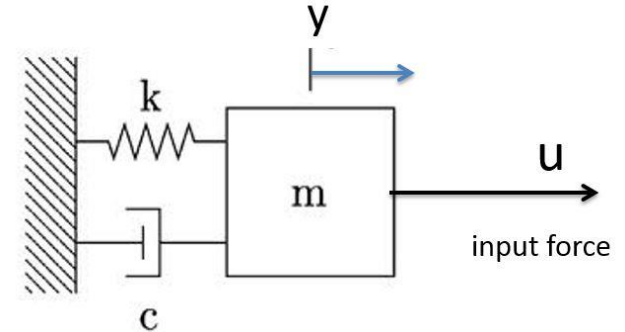
System:
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{c}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}u \end{cases}$$

Output: $z = \dot{y} = x_2$

Output State

State Space Formulation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{c}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}u \end{cases}$$
$$z = \dot{y} = x_2$$

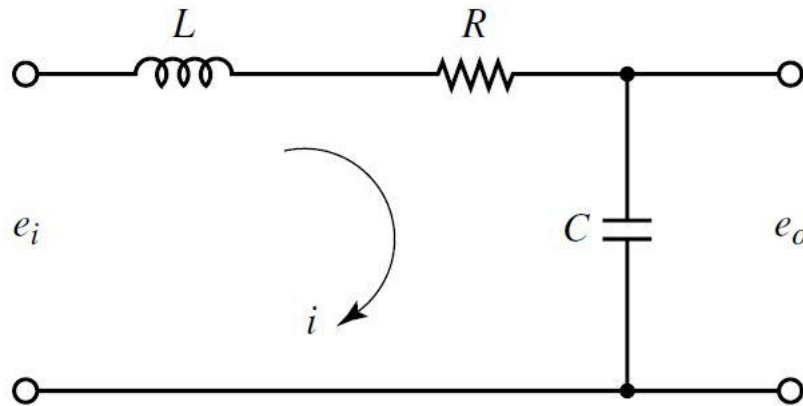


$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + Bu \\ \mathbf{z} = C\mathbf{x} \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$z = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example: LRC Circuit



$$L \frac{di}{dt} + Ri + \frac{1}{C} \int i dt = e_i$$

$$\frac{1}{C} \int i dt = e_o$$



$$\ddot{e}_o + \frac{R}{L} \dot{e}_o + \frac{1}{LC} e_o = \frac{1}{LC} e_i$$

Define:

$$x_1 = e_o$$

$$x_2 = \dot{e}_o$$

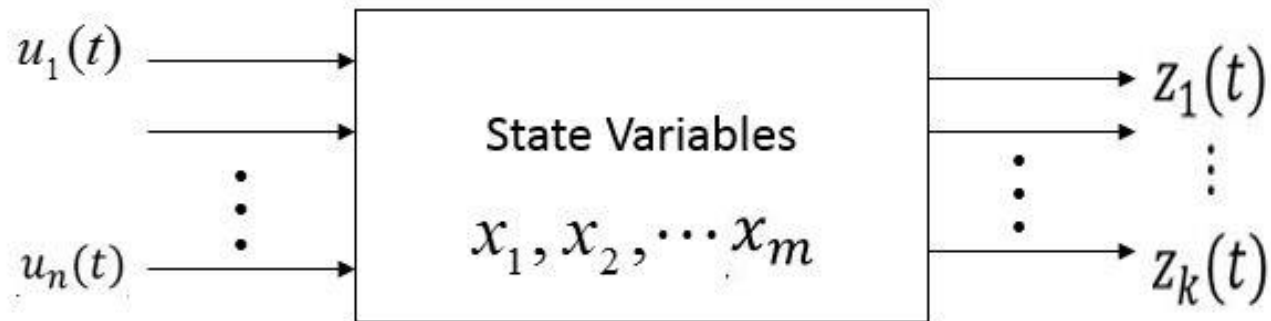
$$u = e_i$$

$$y = e_o = x_1$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



State vector

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix}$$

For example:

Position
Velocity
Acceleration

Control vector

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}$$

For example:

Voltage
Current

Output vector

$$\mathbf{y}(t) = \begin{bmatrix} z_1(t) \\ \vdots \\ z_k(t) \end{bmatrix}$$

For example:

Displacement
Speed

In general, state equations can be written as

State equations

$$\begin{aligned}\dot{x}_1(t) &= f_1(t, x_1(t), \dots, x_m(t), u_1(t), \dots, u_n(t)) \\ \dot{x}_2(t) &= f_2(t, x_1(t), \dots, x_m(t), u_1(t), \dots, u_n(t)) \\ &\vdots \\ \dot{x}_m(t) &= f_m(t, x_1(t), \dots, x_m(t), u_1(t), \dots, u_n(t))\end{aligned}$$

Output equations

$$\begin{aligned}y_1(t) &= g_1(t, x_1(t), \dots, x_m(t), u_1(t), \dots, u_n(t)) \\ &\vdots \\ y_k(t) &= g_k(t, x_1(t), \dots, x_m(t), u_1(t), \dots, u_n(t))\end{aligned}$$

State-Space System

The above OEDs can be written as a state-space system:

State equation: $\dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t), \mathbf{u}(t))$

Output equation: $\mathbf{y}(t) = g(t, \mathbf{x}(t), \mathbf{u}(t))$

where $\mathbf{x} \in \mathbb{R}^m$ is the state vector

$\mathbf{u} \in \mathbb{R}^n$ is the control vector

$\mathbf{y} \in \mathbb{R}^k$ is the output vector

Note:

Only systems described by a **finite number** of states will be considered

The **order** of the system is the **minimal number** of states required to describe its dynamics

This number can be found by determining the number of initial conditions required

General (Nonlinear) State-Space System

Time-Varying:

$$\dot{\mathbf{x}} = f(t, \mathbf{x}, \mathbf{u})$$

$$\mathbf{y} = g(t, \mathbf{x}, \mathbf{u})$$

$$f(t, \mathbf{x}(t), \mathbf{u}(t)) = \begin{bmatrix} f_1(t, \mathbf{x}(t), \mathbf{u}(t)) \\ \vdots \\ f_m(t, \mathbf{x}(t), \mathbf{u}(t)) \end{bmatrix}$$

$$g(t, \mathbf{x}(t), \mathbf{u}(t)) = \begin{bmatrix} g_1(t, \mathbf{x}(t), \mathbf{u}(t)) \\ \vdots \\ g_k(t, \mathbf{x}(t), \mathbf{u}(t)) \end{bmatrix}$$

Time-Invariant:

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

$$\mathbf{y} = g(\mathbf{x}, \mathbf{u})$$

$$f(\mathbf{x}(t), \mathbf{u}(t)) = \begin{bmatrix} f_1(\mathbf{x}(t), \mathbf{u}(t)) \\ \vdots \\ f_m(\mathbf{x}(t), \mathbf{u}(t)) \end{bmatrix}$$

$$g(\mathbf{x}(t), \mathbf{u}(t)) = \begin{bmatrix} g_1(\mathbf{x}(t), \mathbf{u}(t)) \\ \vdots \\ g_k(\mathbf{x}(t), \mathbf{u}(t)) \end{bmatrix}$$

General (Linear) State-Space System

Time-Varying:

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + B(t)\mathbf{u}$$

$$\mathbf{y} = C(t)\mathbf{x} + D(t)\mathbf{u}$$

$$\mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1m}(t) \\ \vdots & \ddots & \vdots \\ a_{m1}(t) & \cdots & a_{mm}(t) \end{bmatrix} \quad \mathbf{C}(t) = \begin{bmatrix} c_{11}(t) & \cdots & c_{1m}(t) \\ \vdots & \ddots & \vdots \\ c_{k1}(t) & \cdots & c_{km}(t) \end{bmatrix}$$
$$\mathbf{B}(t) = \begin{bmatrix} b_{11}(t) & \cdots & b_{1n}(t) \\ \vdots & \ddots & \vdots \\ b_{m1}(t) & \cdots & b_{mn}(t) \end{bmatrix} \quad \mathbf{D}(t) = \begin{bmatrix} d_{11}(t) & \cdots & d_{1n}(t) \\ \vdots & \ddots & \vdots \\ d_{k1}(t) & \cdots & d_{kn}(t) \end{bmatrix}$$

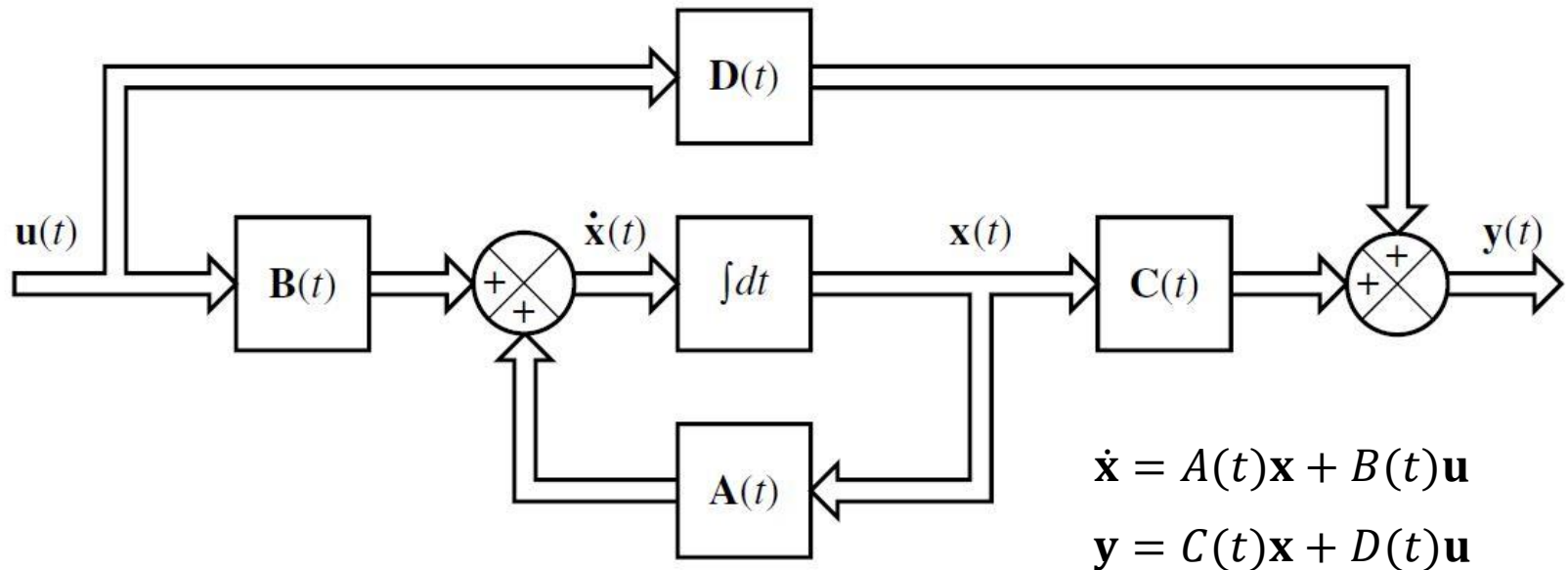
Time-Invariant:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

$$\mathbf{y} = C\mathbf{x} + D\mathbf{u}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & \ddots & \vdots \\ c_{k1} & \cdots & c_{km} \end{bmatrix}$$
$$\mathbf{B} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{k1} & \cdots & d_{kn} \end{bmatrix}$$

Block Diagram of State-Space Systems



$\mathbf{x} \in \mathbb{R}^m$ is the state vector
 $\mathbf{u} \in \mathbb{R}^n$ is the control vector
 $\mathbf{y} \in \mathbb{R}^k$ is the output vector

$A_{m \times m}$ is the system matrix
 $B_{m \times n}$ is the control matrix
 $C_{k \times m}$ is the observation matrix
 $D_{k \times n}$ is the input matrix (often, $D = 0$)

Multi-Input Multi-Output (MIMO): $n > 1, k > 1$

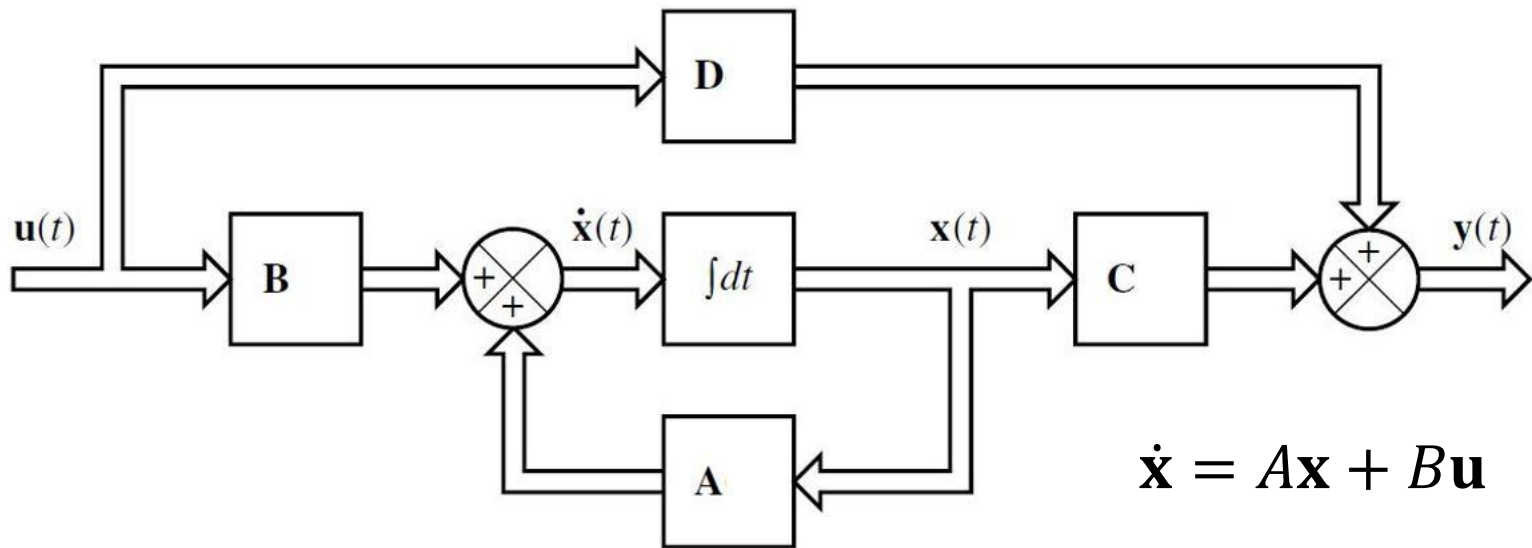
Single-Input Single-Output (SISO): $n = 1, k = 1$

Multi-Input Single-Output (MISO): $n > 1, k = 1$

Single-Input Multi-Output (SIMO): $n = 1, k > 1$

Linear Time-Invariant (LTI) Systems

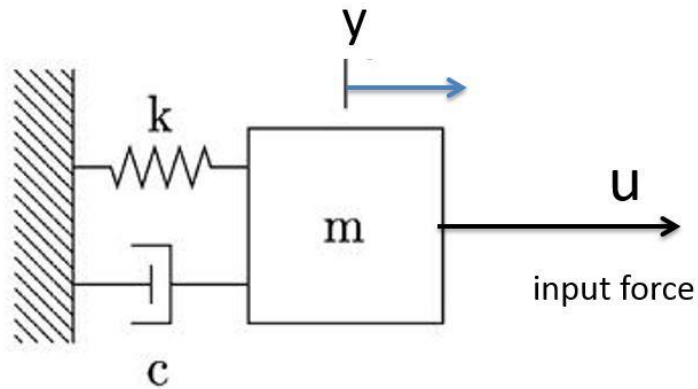
In the following, consider **LTI systems**, unless otherwise indicated



$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

Example: Mass-Spring-Damper System



$$m\ddot{y} + c\dot{y} + ky = u$$

Laplace Transform $\rightarrow$

$$(ms^2 + cs + k)Y(s) = U(s)$$

$$G(s) = \frac{1}{ms^2 + cs + k}$$

Recall: State-Space Formulation

$$m\ddot{y} + c\dot{y} + ky = u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

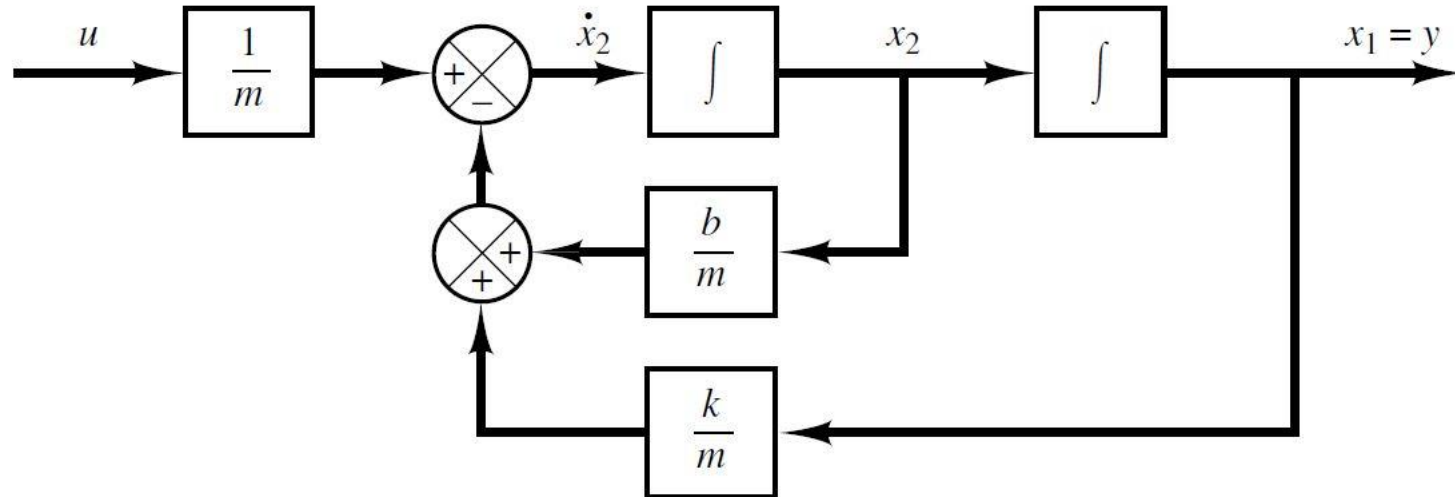
$$z = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0$$

The matrices A, B, C, D are time-invariant

This is an SISO linear time-invariant (LTI) system

Block Diagram: Mass-Spring-Damper System



$$m\ddot{y} + c\dot{y} + ky = u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

$$z = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Transfer Function and State-Space Equation

$$G(s) = \frac{Y(s)}{U(s)} \quad \longleftrightarrow \quad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$$\text{Laplace Transform} \rightarrow \begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

$$(sI - A)X(s) = BU(s)$$

$$X(s) = (sI - A)^{-1}BU(s)$$

$$Y(s) = C(sI - A)^{-1}BU(s) + DU(s)$$

$$\longrightarrow G(s) = C(sI - A)^{-1}B + D$$

Consider: $\dot{\mathbf{x}} = A\mathbf{x}$ ($B = 0$)

$$\begin{aligned}\text{Laplace Transform} &\rightarrow sX(s) - X(0) = AX(s) \\ &\rightarrow (sI - A)X(s) = X(0) \\ &\rightarrow X(s) = (sI - A)^{-1} X(0)\end{aligned}$$

Inverse Laplace Transform

$$\begin{aligned}\rightarrow x(t) &= L^{-1}\{(sI - A)^{-1}\}X(0) \\ &:= \Phi(t)X(0)\end{aligned}$$

Here, $\Phi(t)$ is called the **State Transition Matrix**

$$\begin{aligned}\Phi(t) &= L^{-1}\{(sI - A)^{-1}\} = e^{At} \quad \leftarrow \text{(next page)} \\ &= \left[I + At + \frac{A^2}{2}t^2 + \frac{A^3}{3!}t^3 + \dots \right]\end{aligned}$$

$$(I - C)^{-1} = I + C + C^2 + C^3 + \dots \text{ (if series converges)}$$

Recall: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

$$(sI - A)^{-1} = (1/s)(I - A/s)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \dots$$

(valid for $|s|$ large enough) so

$$\Phi(t) = \mathcal{L}^{-1}((sI - A)^{-1}) = I + tA + \frac{(tA)^2}{2!} + \dots = e^{At}$$

Recall: $e^{at} = 1 + ta + \frac{(ta)^2}{2!} + \dots$

%MatLab code

ss2tf(A,B,C,D) %state space to transfer function

tf2ss(num,den) %transfer function to state space

Consider: $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$

Laplace Transform $\rightarrow$

$$sX(s) - X(0) = AX(s) + BU(s)$$

$$\rightarrow (sI - A)X(s) = X(0) + BU(s)$$

$$\rightarrow X(s) = (sI - A)^{-1} X(0) + (sI - A)^{-1} BU(s)$$

Inverse Laplace Transform $\rightarrow$

$$\begin{aligned}\mathbf{x}(t) &= L^{-1}\{(sI - A)^{-1}\}X(0) + L^{-1}\{(sI - A)^{-1}\}BU(s) \\ &= \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t - \tau)B\mathbf{u}(\tau)d\tau\end{aligned}$$

Convolution

Example: (MIMO LTI System)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 3 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

%MatLab code

```
A=[0 1 0; 0 -4 3; -1 -1 -2];
```

```
B=[0 0 ; 1 0; 0 1];
```

```
C=[1 0 0; 0 0 1];
```

```
D=[0 0; 0 0];
```

```
G=ss(A,B,C,D);
```

```
G2tf=tf(G)
```

$$(sI - A)^{-1} = \frac{\text{adj}(sI - A)}{|sI - A|}$$

$$= \frac{1}{s^3 + 6s^2 + 11s + 3} \begin{bmatrix} s^2 + 6s + 11 & s + 2 & 3 \\ -3 & s^2 + 2 & 3s \\ s + 4 & -s - 1 & s^2 + 4s \end{bmatrix}$$

$$G(s) = C(sI - A)^{-1}B + D$$

$$\Phi(t) = L^{-1}[(sI - A)^{-1}]$$

State-Space Systems: Canonical Form (1)

Recall: $m\ddot{y} + c\dot{y} + ky = u$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

$$z = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

General form: $\ddot{y} + a_1\dot{y} + a_2y = u$

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} = \dot{x}_1 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$z = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State-Space Systems: Canonical Form (1)

In general:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} \dot{y} + a_n y = u$$

Define:

$$\begin{array}{ccc} x_1 = y & & \dot{x}_1 = x_2 \\ x_2 = \dot{y} & & \dot{x}_2 = x_3 \\ \vdots & \Rightarrow & \vdots \\ \vdots & & \vdots \\ x_n = y^{(n-1)} & & \dot{x}_{n-1} = x_n \\ & & \dot{x}_n = -a_n x_1 - \cdots - a_1 x_n + u \end{array}$$

State-Space Systems: Canonical Form (1)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Exactly

same $y \rightarrow$

$$y = [1 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$



$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

State-Space Systems: Canonical Form (2)

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} \dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1} \dot{u} + b_n u$$

Define:

$$x_1 = y - \beta_0 u$$

$$x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$

$$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$

.

.

.

$$x_n = y^{(n-1)} - \beta_0 u^{(n-1)} - \beta_1 u^{(n-2)} - \cdots - \beta_{n-2} \dot{u} - \beta_{n-1} u = \dot{x}_{n-1} - \beta_{n-1} u$$

with

$$\beta_0 = b_0$$

$$\beta_1 = b_1 - a_1 \beta_0$$

$$\beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0$$

$$\beta_3 = b_3 - a_1 \beta_2 - a_2 \beta_1 - a_3 \beta_0$$

.

.

.

$$\beta_{n-1} = b_{n-1} - a_1 \beta_{n-2} - \cdots - a_{n-2} \beta_1 - a_{n-1} \beta_0$$

Also, define:

$$\beta_n = b_n - a_1 \beta_{n-1} - \cdots - a_{n-1} \beta_1 - a_n \beta_0$$

State-Space Systems: Canonical Form (2)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u$$

$$y = [1 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \beta_0 u$$

Here: $\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$

State-Space Systems: Canonical Form (2)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$
$$\mathbf{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad \cdots \quad 0], \quad D = \beta_0 = b_0$$

Transfer Function → State-Space

Example:

$$\frac{Y(s)}{U(s)} = \frac{s}{s^3 + 14s^2 + 56s + 160}$$

Canonical Form →

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -14 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

Transfer Function → State-Space

Example:

$$\frac{Y(s)}{U(s)} = \frac{s}{s^3 + 14s^2 + 56s + 160}$$

```
%MatLab code  
num = [1 0];  
den = [1 14 56 160];  
[A,B,C,D] = tf2ss(num,den)
```

MatLab

→ Another Form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

State-Space → Transfer Function

Recall:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad \longrightarrow \quad G(s) = C(sI - A)^{-1}B + D$$

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -25 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 25 \\ -120 \end{bmatrix} u \quad \longrightarrow \quad \frac{Y(s)}{U(s)} = \frac{25s + 5}{s^3 + 5s^2 + 25s + 5}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

%MatLab code

```
A = [0 1 0; 0 0 1; -5 -25 -5];
```

```
B = [0; 25; -120];
```

```
C = [1 0 0];
```

```
D = [0];
```

```
[num,den] = ss2tf(A,B,C,D)
```

State-Space Forms are not unique, but
their Transfer Functions are the same

$$\begin{array}{lcl}
 \dot{x}(t) = Ax(t) + Bu(t) & \longleftrightarrow & \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \\
 y(t) = Cx(t) + Du(t) & x(t) = p\hat{x}(t) & y(t) = \hat{C}\hat{x}(t) + \hat{D}u(t)
 \end{array}$$

p is nonsingular

$$\begin{aligned}
 p\dot{\hat{x}} &= Ap\hat{x} + Bu \Rightarrow \dot{\hat{x}} = p^{-1}Ap\hat{x} + p^{-1}Bu \\
 y &= Cp\hat{x} + Du
 \end{aligned}$$

$$\begin{array}{lcl}
 \hat{A} = p^{-1}Ap & & A = p\hat{A}p^{-1} \\
 \hat{B} = p^{-1}B & & B = p\hat{B} \\
 \hat{C} = Cp & \longleftrightarrow & C = \hat{C}p^{-1} \\
 \hat{D} = D & & D = \hat{D}
 \end{array}$$

$$\begin{array}{lll}
 \dot{x}(t) = Ax(t) + Bu(t) & \longleftrightarrow & \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \\
 y(t) = Cx(t) + Du(t) & x(t) = p\hat{x}(t) & y(t) = \hat{C}\hat{x}(t) + \hat{D}u(t)
 \end{array}$$

p is nonsingular

$$\begin{aligned}
 \hat{G}(s) &= \hat{C}(sI - \hat{A})^{-1} \hat{B} + \hat{D} \\
 &= Cp(sp^{-1}Ip - p^{-1}Ap)^{-1} p^{-1}B + D \\
 &= Cp[p^{-1}(sI - A)p]^{-1} p^{-1}B + D \\
 &= Cpp^{-1}(sI - A)^{-1} pp^{-1}B + D \\
 &= C(sI - A)^{-1} B + D \\
 &= G(s)
 \end{aligned}$$

These two LTI systems are **equivalent**

Zero-Pole Cancellation and Stability

Consider an LTI system with the Transfer Function

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

All poles are on LHP → system is stable

$N(s)$ and $D(s)$ may have zero-pole cancellations

Example:

$$H(s) = \frac{N(s)}{D(s)} = \frac{s - 1}{s^2 + 2s - 3} = \frac{s - 1}{\underbrace{(s + 2)}_{\text{unstable}} \underbrace{(s - 1)}_{\text{stable}}} = \frac{1}{s + 2}$$

Important: Determine stability **before** pole-zero cancellation !

Recall: Second-order Transfer Function: $\frac{K}{as^2+bs+c}$ ($a \neq 0$)

Characteristic equation: $as^2 + bs + c = 0$

$b^2 - 4ac > 0$:

Two distinct real poles: $r_1 \neq r_2 \rightarrow y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

Both poles $r_1 < 0, r_2 < 0 \rightarrow y(t) \rightarrow 0$ as $t \rightarrow \infty \rightarrow$ system **stable**

$b^2 - 4ac < 0$:

Two complex conjugate poles: $r_{1,2} = \sigma \pm j\omega_d$

$\rightarrow y(t) = C_0 e^{\sigma t} (C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t))$

Both poles have $\sigma < 0 \rightarrow y(t) \rightarrow 0$ as $t \rightarrow \infty \rightarrow$ system **stable**

$b^2 - 4ac = 0$:

Two same real poles: $r_1 = r_2 = r \rightarrow y(t) = C_1 e^{rt} + C_2 t e^{rt}$

Both poles $r_1 = r_2 < 0 \rightarrow y(t) \rightarrow 0$ as $t \rightarrow \infty \rightarrow$ system **stable**

Example: $\ddot{y} + 2\dot{y} - 3y = \dot{u} - u$

Laplace Transform with Initial Conditions →

$$s^2Y(s) - sy(0) - \dot{y}(0) + 2sY(s) - 2y(0) - 3Y(s) = sU(s) - u(0) - U(s)$$



$$Y(s) = \underbrace{\frac{s-1}{s^2+2s-3}}_{H(s)} U(s) + \frac{s+2}{s^2+2s-3}y(0) + \frac{1}{s^2+2s-3}\dot{y}(0) - \frac{1}{s^2+2s-3}u(0)$$

Assume: $y(0) = y_0 \neq 0$, $\dot{y}(0) = 0$, $u(0) = 0$ →

$$Y(s) = \frac{s+2}{s^2+2s-3}y_0 = \frac{y_0}{4} \left(\frac{1}{s+3} + \frac{1}{s-1} \right) \Rightarrow \frac{y_0}{4} \left(\underset{\rightarrow 0}{e^{-3t}} + \underset{\rightarrow \infty}{e^t} \right)$$

But $\frac{s-1}{s^2+2s-3} = \frac{s-1}{(s+2)(s-1)} = \frac{1}{s+2} \Rightarrow y_0 e^{-2t} \rightarrow 0 \text{ as } t \rightarrow \infty$

END