
EE3121

Differential Equations for Engineers

Tutorial 7a



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Tutorial

Problem 1:

Use Laplace transformation, solve the following differential equations:

(a)

$$\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 20y = 3e^{-2t}, \quad y(0) = 0, \quad \dot{y}(0) = 0$$

(b)

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 21y = 1, \quad y(0) = 2, \quad \dot{y}(0) = -4$$

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(a)

$$\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 20y = 3e^{-2t}, \quad y(0) = 0, \quad \dot{y}(0) = 0$$

Solution:

Step 1: Transform the element of the ODE

Define $L\{y(t)\} = Y(s)$

$$L\left\{\frac{d^2y}{dt^2}\right\} = s^2 L\left\{\frac{dy}{dt}\right\} - \frac{dy(0)}{dt} - sy(0) = s^2 Y(s)$$

$$L\left\{\frac{dy}{dt}\right\} = sL\{y\} - y(0) = sY(s)$$

$$L\{3e^{-2t}\} = \frac{3}{s+2}$$

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Step 2: Substitute the transformed elements into the ODEs

$$s^2Y(s) + 9sY(s) + 20Y(s) = \frac{3}{s+2}$$

$$\Rightarrow Y(s) = \frac{3}{(s^2 + 9s + 20)(s + 2)} = \frac{3}{(s + 2)(s + 4)(s + 5)}$$

Partial fraction expansion:

$$\begin{aligned} Y(s) &= \frac{c_1}{s+2} + \frac{c_2}{s+4} + \frac{c_3}{s+5} \\ &= \frac{(c_1 + c_2 + c_3)s^2 + (9c_1 + 7c_2 + 6c_3)s + 20c_1 + 10c_2 + 8c_3}{(s+2)(s+4)(s+5)} \end{aligned}$$

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$$\Rightarrow \begin{cases} c_1 + c_2 + c_3 = 0 \\ 9c_1 + 7c_2 + 6c_3 = 0 \\ 20c_1 + 10c_2 + 8c_3 = 3 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = \frac{1}{2} \\ c_2 = -\frac{3}{2} \\ c_3 = 1 \end{cases}$$

Calculate $Y(s)$ and $y(t)$

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(b)

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 21y = 1, \quad y(0) = 2, \quad \dot{y}(0) = -4$$

Solution:

Step 1: Transform the element of the ODE

Define $L\{y(t)\} = Y(s)$

$$L\left\{\frac{d^2y}{dt^2}\right\} = s^2 L\left\{\frac{dy}{dt}\right\} - \frac{dy(0)}{dt} - sy(0) = s^2 Y(s) + 4 - 2s$$

$$L\left\{\frac{dy}{dt}\right\} = sL\{y\} - y(0) = sY(s) - 2$$

$$L\{1\} = \frac{1}{s}$$

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Step 2: Substitute the transformed elements into the ODEs

$$s^2Y(s) - 2s + 4 + 10(sY(s) - 2) + 21Y(s) = \frac{1}{s}$$

$$\Rightarrow Y(s) = \frac{2s^2 + 16s + 1}{s(s^2 + 10s + 21)} = \frac{2s^2 + 16s + 1}{s(s + 3)(s + 7)}$$

Partial fraction expansion:

$$\begin{aligned} Y(s) &= \frac{c_1}{s} + \frac{c_2}{s + 3} + \frac{c_3}{s + 7} \\ &= \frac{(c_1 + c_2 + c_3)s^2 + (10c_1 + 7c_2 + 3c_3)s + 21c_1}{s(s + 3)(s + 7)} \end{aligned}$$

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$$\Rightarrow \begin{cases} c_1 + c_2 + c_3 = 2 \\ 10c_1 + 7c_2 + 3c_3 = 16 \\ 21c_1 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = \frac{1}{21} \\ c_2 = \frac{29}{12} \\ c_3 = -\frac{13}{28} \end{cases}$$

Calculate $Y(s)$ and $y(t)$

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Problem 2: Prove the following properties of Laplace transform

(a)

$$L \left\{ \frac{d^2 y}{dt^2} \right\} = s^2 Y(s) - sy(0^-) - \dot{y}(0^-)$$

(b)

$$L \left\{ \int_0^t y(\tau) d\tau \right\} = \frac{Y(s)}{s}$$

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Solution:

By the definition of Laplace transform

$$L \left\{ \frac{dy}{dt} \right\} = \int_0^{+\infty} \frac{dy}{dt} e^{-st} dt$$

Use integration by parts

$$\begin{aligned} L \left\{ \frac{dy}{dt} \right\} &= \int_0^{+\infty} e^{-st} dy(t) = e^{-st} y(t) \Big|_0^{\infty} - \int_0^{+\infty} y(t) de^{-st} \\ &= -y(0) + s \int_0^{+\infty} y(t) e^{-st} dt = sY(s) - y(0) \end{aligned} \quad (1)$$

Prove the property (a) by applying the identity (1) to $\dot{y}$.

Let $h(t) = \int_0^t y(\tau) d\tau$, then $h'(t) = y(t)$ and $h(0) = 0$.

Prove the property (b) by applying the identity (1) to $h(t)$.

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