

EE3109 Applied Electromagnetics
2024/25 Semester B
Test 2

Name: Solution

Student #: _____

Date: _____

1. **(22 points)** The electric field of a red laser (free-space wavelength: 633 nm), approximated by the plane wave $\mathbf{E}(\mathbf{r}) = E_0 e^{-jk_0(mx+nz)} \hat{\mathbf{y}}$ [V/m], is incident from free-space and enters a prism ($\epsilon_r = 2, \mu_r = 2$) at $\theta_i = 45^\circ$, as shown in Fig. 1. The plane wave then exits the prism at the output facet, without any change in propagation direction. Assume that the prism is much larger than the wavelength and the laser beamwidth.

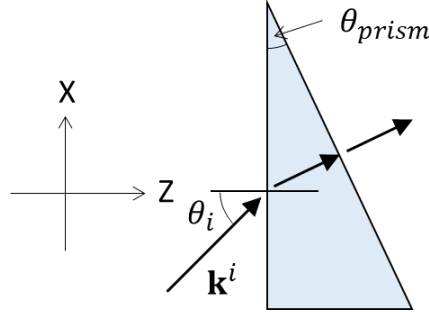


Fig. 1 Diagram for Question 1

- Find the frequency and wavenumber of the red laser. (3)
- Write the incident wave by solving and substituting in as many variables as possible in the equation given in the question. (5)
- Find the reflection and transmission angles (θ_r and θ_t) for the first air-prism interface. (3)
- Find the wedge angle θ_{prism} of the prism. (3)
- Find the reflection and transmission coefficients at the first air-prism interface. (5)
- Write electric field for the wave transmitted into the prism.

You are now given that the incident laser beam has a power density of 1 [mW/m²].

- What is the amplitude of the incident electric field? (3)
- Bonus:** How powerful is the laser beam when it emerges from the prism? (3)

Solution:

- (a) The frequency of the red laser,

$$f = \frac{c}{\lambda_0} = \frac{3 \times 10^8}{633 \times 10^{-9}} = 474 \text{ THz}$$

The wavenumber of the red laser,

$$k_0 = \frac{2\pi}{\lambda_0} = \frac{2\pi}{633 \times 10^{-9}} = 9.926 \times 10^6 \text{ [rad/m]}.$$

- (b) For $\theta_i = 45^\circ$,

$$m = \frac{k_x}{k_0} = \sin\theta_i = 0.7071; \quad n = \frac{k_z}{k_0} = \cos\theta_i = 0.7071$$

Hence the incident wave is

$$\mathbf{E}(\mathbf{r}) = E_0 e^{-j7.017 \times 10^6 (x+z)} \hat{\mathbf{y}} \text{ [V/m]}.$$

- (c) The reflection angle is $\theta_r = \theta_i = 45^\circ$.

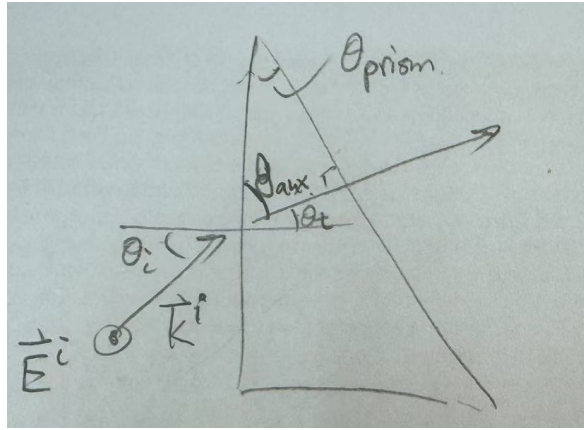
The refractive indices are

$$n_1 = 1; n_2 = \sqrt{\epsilon_r \mu_r} = 2$$

According to Snell's law, $n_1 \sin \theta_i = n_2 \sin \theta_t$, so the transmission angle is,

$$\theta_t = \frac{n_1}{n_2} \sin^{-1} \theta_i = 20.7^\circ$$

- (d) Since the wave exits the prism without a change in direction, we know that it exits the prism at normal incidence. We analyze the angles according to the figure below:



$$\begin{aligned} \theta_{prism} + \theta_{aux} &= \theta_t + \theta_{aux} = 90^\circ \\ \theta_{prism} &= \theta_t = 20.7^\circ \end{aligned}$$

- (e) The incident wave is TE-polarized, so

$$\begin{aligned} \eta_1 &= \eta_0; \eta_2 = \eta_0 \sqrt{\frac{\mu_r}{\epsilon_r}} = \eta_0 \\ \Gamma_{TE} &= \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} = \frac{0.7071 - 0.9534}{0.7071 + 0.9534} = -0.139 \\ \tau_{TE} &= 1 + \Gamma_{TE} = 0.861 \end{aligned}$$

- (f) The wavenumbers and its components in the prism are

$$\begin{aligned} k_t &= \frac{n_2}{n_1} k_0 = 1.985 \times 10^7 \text{ rad/m} \\ k_{2z} &= k_t \cos \theta_t = 1.892 \times 10^7 \text{ rad/m} \\ k_{2x} &= k_t \sin \theta_t = 7.017 \times 10^6 \text{ rad/m} \end{aligned}$$

So the transmitted wave is

$$\mathbf{E}^t(\mathbf{r}) = \tau_{TE} E_0 e^{-j(k_{2x}x + k_{2z}z)} \hat{\mathbf{y}} = 0.816 E_0 e^{-j(7.017 \times 10^6 x + 1.892 \times 10^7 z)} \hat{\mathbf{y}} \text{ [V/m]}$$

(g) The power density of the incident plane wave is

$$|S_{av}| = \frac{|E_0|^2}{2\eta_0}.$$

Rearranging gives

$$|E_0| = \sqrt{2\eta_0|S_{av}|} = 0.8683 \text{ [V/m]}.$$

(h) Note that for the prism-to-air interface, all the power is transmitted since $\eta_2 = \eta_1$. So all the power that enters into the prism exits the prism. So the transmitted power is

$$P_t = TP_i = |\tau_{TE}|^2 \frac{\eta_1 \cos \theta_t}{\eta_2 \cos \theta_i} P_i = 0.98 \text{ [W/m}^2\text{]}.$$

2. **(16 Points)** Fig. 2 shows a transmission line circuit connecting a known load to the generator. The transmission line has a characteristic impedance of $100\ \Omega$, a length of 0.7λ (where λ is the wavelength within the transmission line) and a phase velocity of 1.5×10^8 [m/s]. A 500 MHz wave enters the transmission line and gets reflected by the load. (Smith Charts have been included for your use if you need them. If you use them, please clearly indicate the question parts that you are solving and show your steps.)

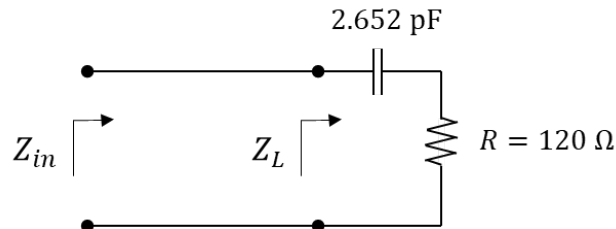


Fig. 2: Diagram for Question 2

- What is the load impedance? (3)
- What is the input impedance? (3)
- What is the voltage standing wave ratio? (3)
- Write the voltage along the line if the current of the incident wave is $I_0^+ = 10$ [mA]. (5)
- Which is closer to the load: the voltage maximum or the current maximum? (2)
- Bonus:** We now change the transmission line's characteristic impedance (Z_0) and the input port's impedance ($Z_{port} = Z_0$) to minimize reflection from the load. What should the optimized Z_0 be? What is the return loss in this case? (3)

Solution:

- (a) The load impedance is

$$Z_L = Z_R + Z_C = R - j \frac{1}{\omega C} = 120 - j \frac{1}{(2\pi \times 500 \times 10^6)(2.652 \times 10^{-12})}$$

$$= 120 - j120\ \Omega$$

- (b) The load impedance $z'_L = Z_L/Z_0 = 1.2 - j1.2$ is plotted on the Smith Chart. We transform toward the generator for $l = 0.7\lambda$ (same impedance as $(0.7 - 0.5)\lambda = 0.2\lambda$). We get $z'_{in} = 0.36 + j0.12$ from the Smith Chart. Unnormalizing gives

$$Z_{in} = z'_{in} Z_0 = 36 + j12\ \Omega$$

(One can also calculate this result using the impedance transformation equation. The value obtained should be similar to the Smith Chart result to within about $1\ \Omega$ for both the real and imaginary parts.)

- (c) The reflection coefficient is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{120 - j120 - 100}{120 - j120 + 100} = 0.2994 - j0.3822 = 0.4855 \angle -51.93^\circ.$$

The voltage standing wave ratio is,

$$S = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = 2.887.$$

(d) The voltage along the transmission line is

$$V(z) = V_0^+ e^{-j\beta z} (1 + \Gamma_L e^{+j2\beta z})$$

where

$$V_0^+ = I_0^+ Z_0 = 1 \text{ [V]}$$

$$\beta = \frac{\omega}{v_p} = \frac{2\pi \times 500 \times 10^6}{1.5 \times 10^8} = 20.94 \text{ [rad/m]} .$$

Therefore,

$$V(z) = V_0^+ e^{-j\beta z} (1 + \Gamma_L e^{+j2\beta z})$$

$$= e^{-j(20.94)z} (1 + 0.4855 \angle -51.93^\circ e^{+j(41.89)z}) \text{ [V]} .$$

(e) From the Smith Chart, adding the transmission line length will make the impedance first reach a point where the phase of the reflection coefficient is 180 degrees. Therefore, we will first reach a current maximum.

(f) Let $z'_L = Z_L/Z_0 = (120 - j120)/Z_0 = a - ja$

Smith Chart method:

We plot z'_L for different values of a on the Smith Chart. We can see that a circular curve is traced out. The point on the curve closest to the center of the Smith Chart is

$$z'_{L,opt} = 0.7 - j0.7 \Rightarrow Z_{0,opt} = \frac{Z_L}{z'_{L,opt}} = 171 \Omega .$$

From the Smith Chart, $|\Gamma_{new}| = \frac{32 \text{ mm}}{78 \text{ mm}} = 0.41 \Rightarrow RL = -20 \log |\Gamma_{new}| = 7.74 \text{ dB} .$

Calculation Method:

The reflection coefficient corresponding to $z'_L = a - ja$ is:

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{z'_L - 1}{z'_L + 1} = \frac{a - 1 - ja}{a + 1 - ja} ;$$

$$|\Gamma|^2 = \frac{(a - 1)^2 + a^2}{(a + 1)^2 + a^2} = \frac{2a^2 - 2a + 1}{2a^2 + 2a + 1} = 1 - \frac{4a}{2a^2 + 2a + 1} = 1 - F .$$

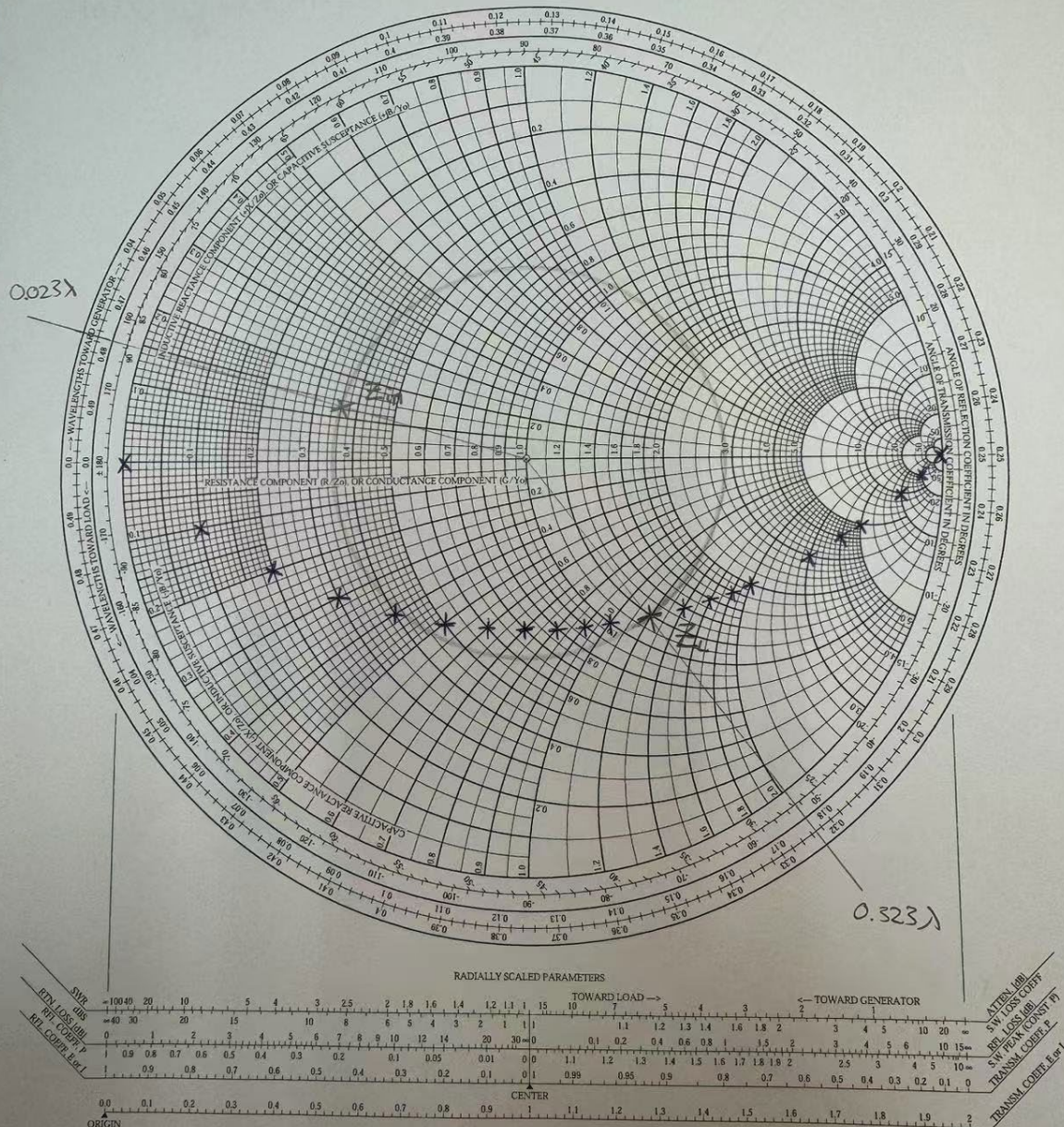
We wish find a that minimizes $|\Gamma|$. This is equivalent to finding a that minimizes $|\Gamma|^2$, which in turn is equivalent to finding a that maximizes F .

$$\frac{dF}{da} = \frac{4(2a^2 + 2a + 1) - (16a^2 + 8a)}{(2a^2 + 2a + 1)^2}$$

$$= \frac{8a^2 + 8a + 4 - 16a^2 - 8a}{(2a^2 + 2a + 1)^2} = \frac{-8a^2 + 4}{(2a^2 + 2a + 1)^2} .$$

$$\frac{dF}{da} = 0 \Rightarrow -8a^2 + 4 = 0 \Rightarrow a = \frac{1}{\sqrt{2}}$$

The negative root is excluded because the resistance should be a positive number; one can check that this is a maximum for F because extreme values like $a = 0$ and $a = \infty$ results in $F = 0$, while $a = 1/\sqrt{2}$ results in $F > 0$. Hence,



3. (10 Points) Plot the following points on the Smith Chart below (2 Points each). Operate the Smith Chart in impedance mode unless otherwise instructed.
- Point A: The open circuit point
 - Point B: The matched point
 - Point C: $Z_C = 100 - j75 \Omega$ (when $Z_0 = 50 \Omega$)
 - Point D: $Y_D = 0.02 + j0.1 \text{ S}$ (when $Z_0 = 50 \Omega$) (please plot in admittance mode)
 - Point E: $\Gamma_E = 0.5 \angle -60^\circ$

Solution:

