

### Matrix

#### Problem 1

Let  $A = \begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{pmatrix}$ ,  $B = \begin{pmatrix} 4 & -1 & 2 \\ 2 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$  be two  $3 \times 3$  matrices.

- Compute  $AB$  and  $A^2$ .
- Compute  $A(B + I_3)$  where  $I_3$  is identity matrix.

#### Problem 2

Let  $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ ,  $B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}$ ,  $C = \begin{pmatrix} 1 & 0 \\ -2 & 5 \end{pmatrix}$  and  $D = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$  be four matrices.

- Find  $AD$  and  $DA$ .
- Find  $BC$  and  $BD$ .

#### Problem 3

Let  $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$  and  $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  be two matrices.

Compute  $A^3$  and  $B^4$ .

#### Problem 4

Fill in the following table.

|                                                                           | Upper-Triangular | Lower-Triangular | Diagonal | Symmetric | Skew-symmetric |
|---------------------------------------------------------------------------|------------------|------------------|----------|-----------|----------------|
| $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | Yes              | Yes              | Yes      | Yes       | No             |
| $A = \begin{pmatrix} -2 & 1 \\ 0 & 3 \end{pmatrix}$                       |                  |                  |          |           |                |
| $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$   |                  |                  |          |           |                |
| $C = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$  |                  |                  |          |           |                |
| $D = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}^2$                     |                  |                  |          |           |                |

### Determinant

#### Problem 5

Let  $A = \begin{pmatrix} 2 & 0 & -3 \\ 1 & 5 & 1 \\ 0 & 0 & 3 \end{pmatrix}$  be a  $3 \times 3$  matrix. Compute the determinant of  $A$  by

- Expanding along the 1<sup>st</sup> row.
- Expanding along the 2<sup>nd</sup> column.
- Expanding along the 3<sup>rd</sup> row.

(Hint: If your answers are correct, the answers of the three parts must be the same.)

**Problem 6**

Compute each of the following determinants using suitable method

$$(a) \quad \det \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & 0 \\ -1 & 1 & 5 \end{pmatrix}$$

$$(c) \quad \det \begin{pmatrix} 1 & a & 2 \\ a & -1 & 0 \\ 2 & 3 & 1 \end{pmatrix}$$

$$(e) \quad \det \begin{pmatrix} 1 & 2 & 1 & 3 \\ 5 & 0 & 0 & -1 \\ 2 & -1 & -1 & 0 \\ 1 & 0 & 4 & 2 \end{pmatrix}$$

$$(b) \quad \det \begin{pmatrix} 3 & -1 & 2 \\ 2 & 0 & 4 \\ 1 & 1 & 0 \end{pmatrix}$$

$$(d) \quad \det \begin{pmatrix} 1 & 0 & -4 \\ 2 & 1 & 1 \\ 1 & 0 & 5 \end{pmatrix}^5$$

$$(f) \quad \det \begin{pmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & 1 & -1 \\ -3 & 2 & -1 & 0 \\ 5 & 0 & 4 & 2 \end{pmatrix}$$

**Problem 7**

Let  $D$  be a  $3 \times 3$  diagonal matrix, show that the determinant of  $D$  is simply the product of all diagonal entries. Hence, find the determinant of identity matrix  $I_3$ .

(Hint: Let  $D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ , then you need to show  $\det D = abc$ )

**Problem 8**

Let  $A$  and  $B$  be two  $4 \times 4$  matrices with  $\det A = 3$  and  $\det B = 1$ .

Using the properties of determinant, find the value of

- (a)  $\det(A^3)$ ,
- (b)  $\det(A^{-1})$
- (c)  $\det A^{-1}B$
- (d)  $\det B^T A$
- (e)  $\det(2A)$  and  $\det(3A^T B)$ .
- (f)  $\det(2C^2)$  where  $C$  is  $2 \times 2$  matrix with  $\det C = 3$ .

**Inverse of matrix****Problem 9**

Find the inverse (if exist) of the following matrices

$$(a) \quad A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$(b) \quad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(c) \quad C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

$$(d) \quad D = \begin{pmatrix} 2 & -3 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

**Problem 10**

Let  $C = \begin{pmatrix} 1 & a & 1 \\ a & 2 & 0 \\ 2a & 0 & a \end{pmatrix}$  be a  $3 \times 3$  matrix.

- (a) Find all possible  $a$  such that the matrix is *singular*.
- (b) Pick  $a = 2$ , find the inverse of  $C$ .

**Problem 11**

Let  $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -3 & 1 \end{pmatrix}$  be two matrices.

(a) Find  $AB^T$  and  $BA^T$ .

(b) For each of the matrices obtained in (a), determine whether it is invertible and find its inverse if the inverse exists.

### Problem 12

Let  $E = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \\ 3 & 2 & 6 \end{pmatrix}$  be a matrix and suppose  $EX = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 3 & 5 \\ 6 & 0 & 1 \end{pmatrix}$ , find  $X$ .

### Problem 13

Using matrix method, solve the following system of linear equations

$$\begin{cases} x - 2y + z = 0 \\ 2x + y - 3z = -5 \\ -x + 4z = 11 \end{cases}$$

(Hint: Write the system in the matrix form)

### System of Linear Equations

#### Problem 14

Using Gaussian Elimination, solve the following systems

(a) 
$$\begin{cases} x - y + 3z = 15 \\ -3x + 2y + z = 4 \\ 2x - 3y + 2z = 9 \end{cases}$$

(b) 
$$\begin{cases} 2x + y - 3z = 12 \\ 4x + z = 5 \\ 3x - y + 2z = 1 \end{cases}$$

(c) 
$$\begin{cases} x - 2y + 3z = 3 \\ 3x - 5y + z = 4 \end{cases}$$

(d) 
$$\begin{cases} x + y + 2z = 0 \\ -3x + 4y + z = 0 \\ -2x + 5y + 3z = 0 \end{cases}$$

(e) 
$$\begin{cases} x + y + z = 9 \\ 2x + 5y + 7z = 52 \\ 2x + y - z = 0 \end{cases}$$

(f) 
$$\begin{cases} x + 3y - 2z - w = 1 \\ 2x + 5y - z + 3w = 2 \\ -x - y - 3z + 2w = -3 \end{cases}$$

(g) 
$$\begin{cases} x + 2y + 3z + 4w = -2 \\ 2x + 4y + 5z + 9w = 1 \\ -3x - 6y + w = 4 \end{cases}$$

(h) 
$$\begin{cases} x - 3y + 4z + 7w = 1 \\ 2x - 6y - 3z + 5w = 2 \\ 4x - 12y - 17z + w = 4 \end{cases}$$

#### Problem 15

Consider the following system

$$\begin{cases} x - 2y + z = 1 \\ 2x - 4y - 5z = 3 \\ 3x - 6y + 24z = a \end{cases}$$

(a) If  $a = 3$ , does the system have any solutions?

(b) Find all possible values of  $a$  such that the system is consistent.

**Problem 16**

Find all possible values of  $c$  such that the system

$$\begin{cases} 3x - y - z = 1 \\ 2x - 4y + 5z = 1 \\ 4x + 2y - 7z = c \end{cases}$$

is consistent.

**Problem 17**

Consider the system

$$\begin{cases} x + 2y - z = c \\ -x + 4y + z = c^2 \\ x + 8y - z = c^3 \end{cases}$$

Find all possible values of  $c$  such that the system

- (a) has a unique solution.
- (b) has infinitely many solutions.
- (c) has no solution.

**Problem 18**

Consider the system

$$\begin{cases} x - 2y + z = 1 \\ x - y + 2z = 2 \\ y + c^2z = c \end{cases}$$

Find all possible values of  $c$  such that the system

- (a) has a unique solution.
- (b) has infinitely many solutions.
- (c) has no solution.

**Problem 19**

Consider the system

$$\begin{cases} 2x + y - bz = 3 \\ ay - z = 2 \\ -2x + 5y = 1 \end{cases}$$

Find all possible values of  $a$  and  $b$  such that the system

- (a) has a unique solution.
- (b) has infinitely many solutions.
- (c) has no solution.

**Problem 20**

Consider the following system of linear equations

$$\begin{cases} x + 3y - 2z - w = 1 \\ 2x + 5y - z + 3w = 2 \\ -x - y + (a - 3)z + (a^2 - 10)w = b \end{cases}$$

Find all possible values of  $a$  and  $b$  such that the system

- (a) has a unique solution.
- (b) has infinitely many solutions.
- (c) has no solution.

**Problem 21**

Solve the following system of linear equations using Cramer's rule.

$$\begin{cases} x - 2y + 3z = 1 \\ 4x + y - 2z = 5 \\ 2x - y + 3z = 6 \end{cases}$$

**Problem 22**

Find the inverse (if exists) of each of the following matrix using Gauss-Jordan method

(a)  $A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$

(b)  $B = \begin{pmatrix} 3 & 1 \\ -6 & 4 \end{pmatrix}$

(c)  $C = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix}$

(d)  $D = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix}$

(e)  $E = \begin{pmatrix} 0 & -1 & 2 \\ 2 & 1 & 4 \\ 1 & -1 & 5 \end{pmatrix}$

(f)  $F = \begin{pmatrix} -2 & 3 & 4 \\ 1 & 0 & -1 \\ -1 & 5 & 8 \end{pmatrix}$

(g)  $G = \begin{pmatrix} 1 & 1 & 2 & -3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & -1 & 1 & 1 \end{pmatrix}$

(h)  $H = \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 2 & -2 \\ 2 & 0 & -1 & 3 \end{pmatrix}$

**Problem 23**

Determine whether the following set of vectors are linearly independent. (For simplicity, each vector will

be expressed in column vector form. For example:  $\vec{i} + 2\vec{j} + 3\vec{k} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ .)

(a)  $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

(b)  $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

(c)  $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix}$

(d)  $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

**Problem 24**

Find the rank of each of the following matrices

(a)  $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$

(b)  $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix}$

(c)  $\begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{pmatrix}$