

L03: Addendum

EE3109 Applied Electromagnetics

Lecture Notes

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Overview

This addendum ties up loose ends and adds some notes beyond the L03 Lecture Slides

1. Uniform Plane Waves

- A Uniform Plane Wave is characterized by **E** and **H** that are **constant** at all points across an **infinite plane**.
- For simplicity, for now we consider a co-ordinate definition where this plane is the xy-plane
- In this case, both **E** and **H** are not functions of x and y (though they may point in the x - and y - directions)
- Hence, the Helmholtz equations become: (next page)

1. Uniform Plane Waves

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)E_x = 0$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)H_x = 0$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)E_y = 0$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)H_y = 0$$

The z-components of **E** and **H** are zero, because

$$\begin{aligned}E_z &= \mathbf{E} \cdot \hat{\mathbf{z}} = \left(\frac{1}{j\omega\epsilon} \nabla \times \mathbf{H}\right) \cdot \hat{\mathbf{z}} = \frac{1}{j\omega\epsilon} (\nabla \times \mathbf{H}) \cdot \hat{\mathbf{z}} \\&= \frac{1}{j\omega\epsilon} \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}, \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}, \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \cdot \hat{\mathbf{z}} \\&= \frac{1}{j\omega\epsilon} \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = 0\end{aligned}$$

(and the same goes for H_z)

1. Uniform Plane Waves

The E_x equation again: $\left(\frac{\partial^2}{\partial z^2} + k^2\right) E_x = 0$

The solution for E_x is:

Phasor: $(E_0^\pm = |E_0^\pm| \exp(j\phi_0^\pm))$

$$E_x(z) = E_0^+ e^{-jkz} + E_0^- e^{+jkz} = E_x^+(z) + E_x^-(z)$$

Time domain:

$$\begin{aligned} E_x(z, t) &= |E_0^+| \cos(\omega t - kz + \phi_0^+) + |E_0^-| \cos(\omega t + kz + \phi_0^-) \\ &= E_x^+(z, t) + E_x^-(z, t) \end{aligned}$$

A similar form applies for E_y , H_x and H_y .

2a. Travelling Plane Waves (in z-direction)

Consider $E_x^+(z)$ (the same applies to E_y , H_x and H_y):

$$E_x^+(z) = E_0^+ e^{-jkz} \Rightarrow E_x^+(z, t) = |E_0^+| \cos(\omega t - kz + \phi_0^+)$$

Consider the phase of this travelling wave ($\varphi = \omega t - kz + \phi_0^+$):

The wavelength (λ) is the distance through which the wave changes in phase by 2π rad:

$$\lambda \equiv |\Delta z| \text{ s.t. } \Delta\varphi = 2\pi \Rightarrow \lambda = \frac{2\pi}{k}$$

The phase velocity (v_p) of the wave is the rate at which the constant phase point travels:

Rearrange the phase expression to get $z = \frac{\omega}{k}t + \underbrace{\frac{(\phi_0^+ - \varphi)}{k}}_{\text{Constant w.r.t. time}}$

$$\Rightarrow v_p = \frac{dz}{dt} = \frac{\omega}{k} = f\lambda$$

Exercise: Find the velocity of $E_x^-(z)$

Let us consider a forward travelling plane wave, with x denoting the direction of the E-field. i.e. $\mathbf{E} = E_x \hat{\mathbf{x}} = E_0 e^{-jkz} \hat{\mathbf{x}}$. What is the corresponding $\mathbf{H}$?

Apply Faraday's Law:

$$\begin{aligned} H(z) &= -\frac{1}{j\omega\mu} \nabla \times \mathbf{E} = -\frac{1}{j\omega\mu} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (E_0 e^{-jkz}, 0, 0) \\ &= -\frac{1}{j\omega\mu} \frac{\partial}{\partial z} (E_0 e^{-jkz}) \hat{\mathbf{y}} = -E_0 \frac{-jk}{j\omega\mu} e^{-jkz} \hat{\mathbf{y}} = \frac{k}{\omega\mu} E_x \hat{\mathbf{y}} \end{aligned}$$

Recall: $k = \omega\sqrt{\mu\varepsilon}$. Define the **Intrinsic Impedance**:

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Rearrange the above to get:

$$H(z) = \frac{1}{\eta} E_x \hat{\mathbf{y}} = \frac{1}{\eta} E_0 e^{-jkz} \hat{\mathbf{y}}$$

Remarks:

1. **Phase:** E and H travel in phase.
2. **Amplitude:** E and H differ in amplitude by the factor η . i.e.

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{|\mathbf{E}|}{|\mathbf{H}|} \equiv \text{Intrinsic Impedance } [\Omega]$$

3. **Direction:** Define $\mathbf{k} \equiv \text{Wave Vector } [\text{m}^{-1}]$

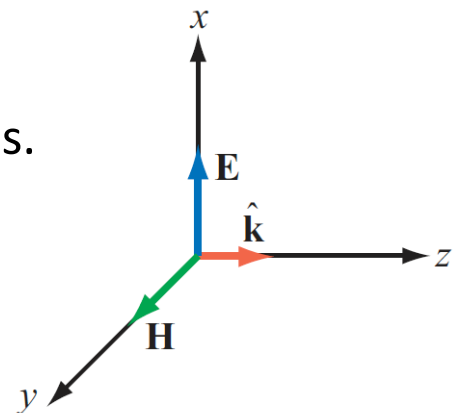
Such that $|\mathbf{k}| = k$; $\hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|}$ points in the direction of phase progression.

(E, H, k) form a right-handed triplet: $\hat{\mathbf{E}} \times \hat{\mathbf{H}} = \hat{\mathbf{k}}$

These are true when μ and ε both have positive real parts.

Hence, for plane waves in this case:

$$\begin{aligned}\mathbf{E} &= \eta \mathbf{H} \times \hat{\mathbf{k}} \\ \mathbf{H} &= \frac{1}{\eta} \hat{\mathbf{k}} \times \mathbf{E}\end{aligned}$$



2b. Travelling Plane Waves (in general direction)

We have described mathematically a plane wave for which the E-field is in the x-direction, the H-field is in the y-direction and the direction of propagation (travelling) is in the z-direction.

One can transform this plane wave (e.g. using a co-ordinate transformation) to obtain the equation for a wave travelling in an arbitrary direction. The time-domain and phasor expressions (for the E-field) are (in the lossless case):

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= |E_0| \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi_0) \hat{\mathbf{e}} \\ \mathbf{E}(\mathbf{r}) &= |E_0| e^{j\phi_0} e^{-j\mathbf{k} \cdot \mathbf{r}} \hat{\mathbf{e}}\end{aligned}$$

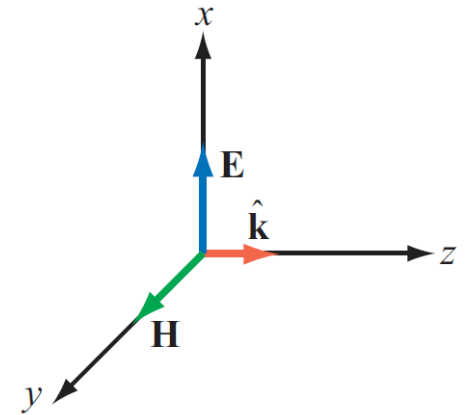
Where $\mathbf{k} \cdot \mathbf{r}$ is the inner product between $\mathbf{k}$ and $\mathbf{r}$. In Cartesian co-ordinates $\mathbf{r} = (x, y, z) = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, and

$$\mathbf{k} \cdot \mathbf{r} = (k_x, k_y, k_z) \cdot (x, y, z) = k_x x + k_y y + k_z z$$

The corresponding magnetic field is $\mathbf{H}(\mathbf{r}) = \frac{|E_0|}{\eta} e^{j\phi_0} e^{-j\mathbf{k} \cdot \mathbf{r}} \hat{\mathbf{h}}$, where $\hat{\mathbf{e}}$, $\hat{\mathbf{h}}$ and $\hat{\mathbf{k}}$ form a right-hand triplet.

3. Right-Hand Triplet

- (x, y, z) and $(\mathbf{E}, \mathbf{H}, \mathbf{k})$ both form right-hand triplets (can apply the right-hand rule to show their directional relationship)



Dir

$$\hat{\mathbf{k}} = \hat{\mathbf{E}} \times \hat{\mathbf{H}}$$

$$\hat{\mathbf{k}} = -\hat{\mathbf{H}} \times \hat{\mathbf{E}}$$

Amp:

$$\eta = \frac{|\mathbf{E}|}{|\mathbf{H}|}$$

$$\hat{\mathbf{H}} = \hat{\mathbf{k}} \times \hat{\mathbf{E}}$$

$$\hat{\mathbf{H}} = -\hat{\mathbf{E}} \times \hat{\mathbf{k}}$$

$$\hat{\mathbf{E}} = \hat{\mathbf{H}} \times \hat{\mathbf{k}}$$

$$\hat{\mathbf{E}} = -\hat{\mathbf{k}} \times \hat{\mathbf{H}}$$

$$\vec{\mathbf{H}} = \frac{1}{\eta} \hat{\mathbf{k}} \times \vec{\mathbf{E}}$$

$$\vec{\mathbf{E}} = \eta \vec{\mathbf{H}} \times \hat{\mathbf{k}}$$

4. On Nepers and dB

- α is usually [Np/m], where Np, or Neper, is unitless
 - 1 [Np/m] means the E/H-field amplitude decays to e^{-1} of its original value after 1 [m].
- It is useful to also quote the decay in [dB/m]
 - 1 [Bell/m] means the plane wave's power decays to 1/10 of its original value after 1 [m].
 - 1 [dB/m] means the plane wave's power decays to 1/10 of its original value after 10 [m].
- (Will be shown in Lect. 6: A plane wave's power density is proportional to the square of the E-field.)

4. On Nepers and dB

-1 Neper is the logarithmic measure when the wave amplitude decays to e^{-1} of its original value; **-10 dB** is logarithmic measure when the wave power decays to **1/10** of its original value.

Converting -1 Np into dB: Consider $E(z_1) = E(z_0)e^{-1}$ (i.e. the E-field decays by 1 Np from z_0 to z_1). The decay in dB is:

$$20 \log \left(\frac{E(z_1)}{E(z_0)} \right) = 20 \log(e^{-1}) = -\frac{20 \ln e}{\ln 10} = -\frac{20}{\ln 10} \text{ [dB]}$$
$$\cong -8.686 \text{ [dB]}$$

Hence **1 Np = 8.686 dB** or **1 dB = 0.1151 Np**