

Problem 1

$$(a) \quad AB = \begin{pmatrix} 4 & 1 & 5 \\ 2 & 1 & 7 \\ 2 & 3 & 10 \end{pmatrix} \quad \text{and} \quad A^2 = \begin{pmatrix} 5 & -7 & 1 \\ 2 & -1 & 8 \\ 5 & -6 & 8 \end{pmatrix}$$

$$(b) \quad A(B + I_3) = \begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 4+1 & -1 & 2 \\ 2 & -1+1 & 1 \\ 0 & 1 & 3+1 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 5 & -1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 6 & -1 & 6 \\ 2 & 2 & 9 \\ 3 & 2 & 13 \end{pmatrix}.$$

Problem 2

$$(a) \quad AD = \underbrace{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}}_{1 \times 3} \underbrace{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}_{3 \times 1} = \underbrace{(1(1) + 2(-1) + 3(2))}_{1 \times 1} = (5).$$

$$DA = \underbrace{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}_{3 \times 1} \underbrace{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}}_{1 \times 3} = \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{pmatrix}}_{3 \times 3}$$

$$(b) \quad BC = \underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}}_{2 \times 3} \underbrace{\begin{pmatrix} 1 & 0 \\ -2 & 5 \end{pmatrix}}_{2 \times 2} = \text{undefined}.$$

$$BD = \underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}}_{2 \times 3} \underbrace{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}_{3 \times 1} = \underbrace{\begin{pmatrix} 1 \\ 5 \end{pmatrix}}_{2 \times 1}$$

Problem 3

$$A^3 = \left[\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix} \right] \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 4 \\ 2 & 1 & -4 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix} \\ = \begin{pmatrix} 8 & 8 & 8 \\ -2 & 3 & 12 \\ 4 & -2 & 0 \end{pmatrix}.$$

Since B is diagonal matrix, we then have

$$B^4 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}^4 = \begin{pmatrix} 2^4 & 0 & 0 \\ 0 & (-3)^4 & 0 \\ 0 & 0 & 1^4 \end{pmatrix} = \begin{pmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 4

	Upper-Triangular	Lower-Triangular	Diagonal	Symmetric	Skew-symmetric
$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	Yes	Yes	Yes	Yes	No
$A = \begin{pmatrix} -2 & 1 \\ 0 & 3 \end{pmatrix}$	Yes	No	No	No	No
$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	Yes	Yes	Yes	Yes
$C = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	Yes	No	No	No	No

$D = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}^2$	No	No	No	No	No
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Problem 5

(1) By expanding the determinant along the 1st row, we have

$$\det \begin{pmatrix} 2 & 0 & -3 \\ 1 & 5 & 1 \\ 0 & 0 & 3 \end{pmatrix} = 2 \times \det \begin{pmatrix} 5 & 1 \\ 0 & 3 \end{pmatrix} - 0 \times \det \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} - 3 \times \det \begin{pmatrix} 1 & 5 \\ 0 & 0 \end{pmatrix} = 30.$$

(2) By expanding the determinant along the 2nd column, we have

$$\det \begin{pmatrix} 2 & 0 & -3 \\ 1 & 5 & 1 \\ 0 & 0 & 3 \end{pmatrix} = -0 \times \det \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} + 5 \times \det \begin{pmatrix} 2 & -3 \\ 0 & 3 \end{pmatrix} - 0 \times \det \begin{pmatrix} 2 & -3 \\ 0 & 3 \end{pmatrix} = 30.$$

(3) By expanding the determinant along the 3rd row, we have

$$\det \begin{pmatrix} 2 & 0 & -3 \\ 1 & 5 & 1 \\ 0 & 0 & 3 \end{pmatrix} = 0 \times \det \begin{pmatrix} 0 & -3 \\ 5 & 1 \end{pmatrix} - 0 \times \det \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} + 3 \times \det \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix} = 30.$$

Problem 6

(a) By expanding the determinant along 2nd row, we have

$$\det \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & 0 \\ -1 & 1 & 5 \end{pmatrix} = 2 \times (-1)^{2+2} \det \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix} = 16.$$

(b) By expanding the determinant along 2nd row, we have

$$\det \begin{pmatrix} 3 & -1 & 2 \\ 2 & 0 & 4 \\ 1 & 1 & 0 \end{pmatrix} = 2 \times (-1)^{2+1} \det \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} + 4 \times (-1)^{2+3} \det \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} = -12.$$

(c) By expanding the determinant along 1st row, we have

$$\begin{aligned} \det \begin{pmatrix} 1 & a & 2 \\ a & -1 & 0 \\ 2 & 3 & 1 \end{pmatrix} \\ = 1 \times (-1)^{1+1} \det \begin{pmatrix} -1 & 0 \\ 3 & 1 \end{pmatrix} + a \times (-1)^{1+2} \det \begin{pmatrix} a & 0 \\ 2 & 1 \end{pmatrix} + 2 \times (-1)^{1+3} \det \begin{pmatrix} a & -1 \\ 2 & 3 \end{pmatrix} \\ = -a^2 + 6a + 3. \end{aligned}$$

(d) Note that

$$\det \begin{pmatrix} 1 & 0 & -4 \\ 2 & 1 & 1 \\ 1 & 0 & 5 \end{pmatrix} = 1 \times (-1)^{1+1} \det \begin{pmatrix} 1 & 1 \\ 0 & 5 \end{pmatrix} + (-4) \times (-1)^{1+3} \det \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = 9.$$

Using the properties of determinant, we have

$$\det \begin{pmatrix} 1 & 0 & -4 \\ 2 & 1 & 1 \\ 1 & 0 & 5 \end{pmatrix}^5 = \left[\det \begin{pmatrix} 1 & 0 & -4 \\ 2 & 1 & 1 \\ 1 & 0 & 5 \end{pmatrix} \right]^5 = 9^5 = 59049.$$

(e) By expanding the determinant along 2nd row, we get

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 1 & 3 \\ 5 & 0 & 0 & -1 \\ 2 & -1 & -1 & 0 \\ 1 & 0 & 4 & 2 \end{pmatrix} \\ = 5 \times (-1)^{2+1} \det \begin{pmatrix} 2 & 1 & 3 \\ -1 & -1 & 0 \\ 0 & 4 & 2 \end{pmatrix} + (-1) \times (-1)^{2+4} \det \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ 1 & 0 & 4 \end{pmatrix} \\ = \dots = -5(-14) + (-1)(-21) = 91. \end{aligned}$$

(f) By expanding the determinant along 2nd column

$$\begin{aligned} \det \begin{pmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & 1 & -1 \\ -3 & 2 & -1 & 0 \\ 5 & 0 & 4 & 2 \end{pmatrix} \\ = 1 \times (-1)^{1+2} \det \begin{pmatrix} 1 & 1 & -1 \\ -3 & -1 & 0 \\ 5 & 4 & 2 \end{pmatrix} + 2 \times (-1)^{3+2} \det \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & -1 \\ 5 & 4 & 2 \end{pmatrix} \\ = \dots = -1(11) - 2(2) = -15. \end{aligned}$$

Problem 7

We let $D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$. By expanding along 1st row, we get

$$\det D = a \times \det \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix} = abc.$$

Problem 8

(a) $\det(A^3) = (\det A)^3 = 3^3 = 27.$

(b) $\det(A^{-1}) = \frac{1}{\det A} = \frac{1}{3}.$

(c) $\det(A^{-1}B) = \det(A^{-1}) \det B = \frac{1}{3}.$

(d) $\det(B^T A) = \det(B^T) \det A = \det B \det A = 3.$

(e) $\det(2A) = 2^4 \det A = 48,$

$\det(3A^T B) = 3^4 \det(A^T B) = 81 \det A^T \det B = 81 \det A \det B = 243.$

(f) $\det(2C^2) = 2^2 \det(C^2) = 4(\det C)^2 = 36.$

Problem 9

(a) Note that

$$\det A = \det \begin{pmatrix} 2 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix} = \dots = 9 \neq 0.$$

The matrix A is invertible and its inverse is given by

$$\begin{aligned} A^{-1} &= \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = \frac{1}{9} \begin{pmatrix} 5 & -3 & 1 \\ -1 & 6 & -2 \\ 2 & -3 & 4 \end{pmatrix}^T = \begin{pmatrix} 5/9 & -1/9 & 2/9 \\ -1/3 & 2/3 & -1/3 \\ 1/9 & -2/9 & 4/9 \end{pmatrix} \\ \left(\begin{array}{lll} A_{11} = (-1)^{1+1} \det \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} & A_{12} = (-1)^{1+2} \det \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} & A_{13} = (-1)^{1+3} \det \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ A_{21} = (-1)^{2+1} \det \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix} & A_{22} = (-1)^{2+2} \det \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} & A_{23} = (-1)^{2+3} \det \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \\ A_{31} = (-1)^{3+1} \det \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix} & A_{32} = (-1)^{3+2} \det \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} & A_{33} = (-1)^{3+3} \det \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \end{array} \right) \end{aligned}$$

(b) Note that

$$\det B = \det \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \dots = 4 \neq 0.$$

The matrix B is invertible and its inverse is given by

$$B^{-1} = \frac{1}{\det B} \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}^T = \frac{1}{4} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}^T = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} B_{11} = (-1)^{1+1} \det \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} & B_{12} = (-1)^{1+2} \det \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & B_{13} = (-1)^{1+3} \det \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \\ B_{21} = (-1)^{2+1} \det \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & B_{22} = (-1)^{2+2} \det \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} & B_{23} = (-1)^{2+3} \det \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \\ B_{31} = (-1)^{3+1} \det \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} & B_{32} = (-1)^{3+2} \det \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} & B_{33} = (-1)^{3+3} \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \end{pmatrix}$$

(c) Note that

$$\det C = \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} = \dots = 6 \neq 0.$$

The matrix C is invertible and its inverse is given by

$$C^{-1} = \frac{1}{\det C} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{6} \begin{pmatrix} 6 & 0 & 0 \\ -3 & 3 & 0 \\ -1 & -1 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & -1/2 & -1/6 \\ 0 & 1/2 & -1/6 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

$$\begin{pmatrix} C_{11} = (-1)^{1+1} \det \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} & C_{12} = (-1)^{1+2} \det \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} & C_{13} = (-1)^{1+3} \det \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \\ C_{21} = (-1)^{2+1} \det \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} & C_{22} = (-1)^{2+2} \det \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} & C_{23} = (-1)^{2+3} \det \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\ C_{31} = (-1)^{3+1} \det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} & C_{32} = (-1)^{3+2} \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & C_{33} = (-1)^{3+3} \det \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \end{pmatrix}$$

(d) Note that

$$\det D = \det \begin{pmatrix} 2 & -3 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \dots = 0.$$

Thus the matrix D is not invertible and its inverse D^{-1} does not exist.

Problem 10

(a) If the matrix is singular (non-invertible), then we must have

$$\begin{aligned} \det C = 0 &\Rightarrow \det \begin{pmatrix} 2 & 0 \\ 0 & a \end{pmatrix} - a \times \det \begin{pmatrix} a & 0 \\ 2a & a \end{pmatrix} + \det \begin{pmatrix} a & 2 \\ 2a & 0 \end{pmatrix} = 0 \\ &\Rightarrow -a^3 - 2a = 0 \Rightarrow a(a^2 + 2) = 0 \\ &\Rightarrow a = 0. \end{aligned}$$

(b) Take $a = 2$, we have $C = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 4 & 0 & 2 \end{pmatrix}$. Since $\det C = -2^3 - 2(2) = -12 \neq 0$, C is invertible and its inverse is found to be

$$C^{-1} = \frac{1}{\det C} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{-12} \begin{pmatrix} 4 & -4 & -8 \\ -4 & -2 & 8 \\ -2 & 2 & -2 \end{pmatrix}^T = \begin{pmatrix} -1/3 & 1/3 & 1/6 \\ 1/3 & 1/6 & -1/6 \\ 2/3 & -2/3 & 1/6 \end{pmatrix}$$

$$\begin{pmatrix} C_{11} = (-1)^{1+1} \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & C_{12} = (-1)^{1+2} \det \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix} & C_{13} = (-1)^{1+3} \det \begin{pmatrix} 2 & 2 \\ 4 & 0 \end{pmatrix} \\ C_{21} = (-1)^{2+1} \det \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & C_{22} = (-1)^{2+2} \det \begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix} & C_{23} = (-1)^{2+3} \det \begin{pmatrix} 1 & 2 \\ 4 & 0 \end{pmatrix} \\ C_{31} = (-1)^{3+1} \det \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} & C_{32} = (-1)^{3+2} \det \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} & C_{33} = (-1)^{3+3} \det \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \end{pmatrix}$$

Problem 11

(a) $AB^T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 3 & 3 \end{pmatrix}$

$BA^T = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -3 & 3 \end{pmatrix}$

- (b) Note that $\det(AB^T) = \det\begin{pmatrix} 0 & -3 \\ 3 & 3 \end{pmatrix} = 9 \neq 0$ and $\det(BA^T) = \det\begin{pmatrix} 0 & 3 \\ -3 & 3 \end{pmatrix} = 9 \neq 0$, this implies that both matrices are invertible and their inverse is found to be

$$(AB^T)^{-1} = \frac{1}{\det(AB^T)} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}^T = \frac{1}{9} \begin{pmatrix} 3 & -3 \\ 3 & 0 \end{pmatrix}^T = \begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 0 \end{pmatrix}$$

(Here, C_{ij} represents the cofactor of $(i, j)^{\text{th}}$ element of AB^T)

$$(BA^T)^{-1} = \frac{1}{\det(BA^T)} \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}^T = \frac{1}{9} \begin{pmatrix} 3 & 3 \\ -3 & 0 \end{pmatrix}^T = \begin{pmatrix} 1/3 & -1/3 \\ 1/3 & 0 \end{pmatrix}.$$

(Here, D_{ij} represents the cofactor of $(i, j)^{\text{th}}$ element of BA^T .)

Problem 12

Note that $\det E = \det\begin{pmatrix} 2 & 5 \\ 2 & 6 \end{pmatrix} - \det\begin{pmatrix} 2 & 5 \\ 3 & 6 \end{pmatrix} + 2 \times \det\begin{pmatrix} 2 & 2 \\ 3 & 2 \end{pmatrix} = 1 \neq 0$, then E is invertible and its inverse is given by

$$E^{-1} = \frac{1}{\det E} \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix}^T = \frac{1}{1} \begin{pmatrix} 2 & 3 & -2 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}^T = \begin{pmatrix} 2 & -2 & 1 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{lll} E_{11} = (-1)^{1+1} \det\begin{pmatrix} 2 & 5 \\ 2 & 6 \end{pmatrix} & E_{12} = (-1)^{1+2} \det\begin{pmatrix} 2 & 5 \\ 3 & 6 \end{pmatrix} & E_{13} = (-1)^{1+3} \det\begin{pmatrix} 2 & 2 \\ 3 & 2 \end{pmatrix} \\ E_{21} = (-1)^{2+1} \det\begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix} & E_{22} = (-1)^{2+2} \det\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} & E_{23} = (-1)^{2+3} \det\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \\ E_{31} = (-1)^{3+1} \det\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} & E_{32} = (-1)^{3+2} \det\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} & E_{33} = (-1)^{3+3} \det\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \end{array} \right)$$

Thus the matrix X is found to be

$$X = E^{-1} \begin{pmatrix} 0 & 1 & 2 \\ -1 & 3 & 5 \\ 6 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ -1 & 3 & 5 \\ 6 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & -4 & -5 \\ -6 & 3 & 5 \\ -1 & 1 & 1 \end{pmatrix}.$$

Problem 13

The system can be rewritten as

$$\begin{cases} x - 2y + z = 0 \\ 2x + y - 3z = -5 \\ -x + 4z = 11 \end{cases} \Rightarrow \underbrace{\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ -1 & 0 & 4 \end{pmatrix}}_A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ 11 \end{pmatrix} \dots \dots (*)$$

Note that $\det A = \det\begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix} - (-2) \times \det\begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix} + \det\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = 15 \neq 0$, then A is invertible and its inverse is given by

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = \frac{1}{15} \begin{pmatrix} 4 & -5 & 1 \\ 8 & 5 & 2 \\ 5 & 5 & 5 \end{pmatrix}^T = \begin{pmatrix} 4/15 & 8/15 & 1/3 \\ -1/3 & 1/3 & 1/3 \\ 1/15 & 2/15 & 1/3 \end{pmatrix}$$

$$\left(\begin{array}{lll} A_{11} = (-1)^{1+1} \det\begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix} & A_{12} = (-1)^{1+2} \det\begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix} & A_{13} = (-1)^{1+3} \det\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \\ A_{21} = (-1)^{2+1} \det\begin{pmatrix} -2 & 1 \\ 0 & 4 \end{pmatrix} & A_{22} = (-1)^{2+2} \det\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix} & A_{23} = (-1)^{2+3} \det\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} \\ A_{31} = (-1)^{3+1} \det\begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix} & A_{32} = (-1)^{3+2} \det\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} & A_{33} = (-1)^{3+3} \det\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \end{array} \right)$$

Therefore, the solution of $(*)$ is found to be

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ -1 & 0 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -5 \\ 11 \end{pmatrix} = \begin{pmatrix} 4/15 & 8/15 & 1/3 \\ -1/3 & 1/3 & 1/3 \\ 1/15 & 2/15 & 1/3 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\Rightarrow x = 1, \quad y = 2, \quad z = 3.$$

Problem 14

$$(a) \quad \left(\begin{array}{ccc|c} 1 & -1 & 3 & 15 \\ -3 & 2 & 1 & 4 \\ 2 & -3 & 2 & 9 \end{array} \right) \xrightarrow[R_3-2R_1]{R_2+3R_1} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 15 \\ 0 & -1 & 10 & 49 \\ 0 & -1 & -4 & -21 \end{array} \right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 15 \\ 0 & -1 & 10 & 49 \\ 0 & 0 & -14 & -70 \end{array} \right).$$

Since there is no column with no pivot, the system has a unique solution.

The system can be expressed as
$$\begin{cases} x - y + 3z = 15 \\ -y + 10z = 49 \\ -14z = -70 \end{cases}$$
. Solving the equations backward, we

obtain $z = 5$, $y = 1$, $x = 1$.

$$(b) \quad \left(\begin{array}{ccc|c} 2 & 1 & -3 & 12 \\ 4 & 0 & 1 & 5 \\ 3 & -1 & 2 & 1 \end{array} \right) \xrightarrow{R_1 \div 2} \left(\begin{array}{ccc|c} 1 & 1/2 & -3/2 & 6 \\ 4 & 0 & 1 & 5 \\ 3 & -1 & 2 & 1 \end{array} \right) \xrightarrow[R_3-3R_1]{R_2-4R_1} \left(\begin{array}{ccc|c} 1 & 1/2 & -3/2 & 6 \\ 0 & -2 & 7 & -19 \\ 0 & -5/2 & 13/2 & -17 \end{array} \right)$$

$$\xrightarrow{R_2 \div (-2)} \left(\begin{array}{ccc|c} 1 & 1/2 & -3/2 & 6 \\ 0 & 1 & -7/2 & 19/2 \\ 0 & -5/2 & 13/2 & -17 \end{array} \right) \xrightarrow{R_2 + \frac{5}{2}R_1} \left(\begin{array}{ccc|c} 1 & 1/2 & -3/2 & 6 \\ 0 & 1 & -7/2 & 19/2 \\ 0 & 0 & 61/4 & -163/4 \end{array} \right).$$

Since there is no column with no pivot, the system has a unique solution.

The system can be expressed as
$$\begin{cases} x + \frac{1}{2}y - \frac{3}{2}z = 6 \\ y - \frac{7}{2}z = \frac{19}{2} \\ \frac{61}{4}z = -\frac{163}{4} \end{cases}$$
. Solving the equations backward, we

obtain $z = -3$, $y = -1$, $x = 2$.

$$(c) \quad \left(\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 3 & -5 & 1 & 4 \end{array} \right) \xrightarrow{R_2-3R_1} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 1 & -8 & -5 \end{array} \right)$$

Since the 3rd column has no pivot, the system has infinitely many solutions and z is a free variable. The corresponding system is given by
$$\begin{cases} x - 2y + 3z = 3 \\ y - 8z = -5 \end{cases}$$
.

Take $z = t$, t is real. By solving the equations backward, we get $y = -5 + 8t$ and $x = -7 + 13t$.

$$(d) \quad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -3 & 4 & 1 & 0 \\ -2 & 5 & 3 & 0 \end{array} \right) \xrightarrow[R_3+2R_1]{R_2+3R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 7 & 7 & 0 \end{array} \right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Since the 3rd column has no pivot, the system has infinitely many solutions and z is a free variable. The corresponding system is given by
$$\begin{cases} x + y + 2z = 0 \\ 7y + 7z = 0 \end{cases}$$
.

Take $z = t$, t is real. By solving the equations backward, we get $y = -t$ and $x = -t$.

$$(e) \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{array} \right) \xrightarrow[R_3-2R_1]{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -1 & -3 & -18 \end{array} \right) \xrightarrow{R_2 \div 3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 5/3 & 34/3 \\ 0 & -1 & -3 & -18 \end{array} \right)$$

$$\xrightarrow{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 5/3 & 34/3 \\ 0 & 0 & -4/3 & -20/3 \end{array} \right)$$

The system can be expressed as
$$\begin{cases} x + y + z = 9 \\ y + \frac{5}{3}z = \frac{34}{3} \\ -\frac{4}{3}z = -\frac{20}{3} \end{cases}$$
. Solving the equations backward, we

obtain $z = 5$, $y = 3$, $x = 1$.

$$(f) \quad \left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 1 \\ 2 & 5 & -1 & 3 & 2 \\ -1 & -1 & -3 & 2 & -3 \end{array} \right) \xrightarrow[R_3+R_1]{R_2-2R_1} \left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 2 & -5 & 1 & -2 \end{array} \right)$$

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \end{array} \left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 2 & -5 & 1 & -2 \end{array} \right) \xrightarrow{R_3 + 2R_2} \left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 11 & -2 \end{array} \right)$$

Since the fourth column has no pivot, the system has infinitely many solutions and w is the free variable. The corresponding system is
$$\begin{cases} x + 3y - 2w - z = 1 \\ -y + 3z + 5w = 0 \\ z + 11w = -2 \end{cases}$$

Take $w = t$, t is real. By solving the equation backward, we obtain $z = -2 - 11t$, $y = -6 - 28t$, $x = 15 + 63t$.

$$(g) \quad \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 2 & 4 & 5 & 9 & 1 \\ -3 & -6 & 0 & 1 & 5 \end{array} \right) \xrightarrow{R_2 - 2R_1, R_3 + 3R_1} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 9 & 13 & -1 \end{array} \right) \xrightarrow{R_3 + 9R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & 22 & 44 \end{array} \right).$$

Since the second column has no pivot, the system has infinitely many solutions and y is the free variable. The corresponding system is
$$\begin{cases} x + 2y + 3z + 4w = -2 \\ -z + w = 5 \\ 22w = 44 \end{cases}$$

Take $y = t$, t is real. By solving the equation backward, we obtain $w = 2$, $z = -3$, $x = -1 - 2t$.

$$(h) \quad \left(\begin{array}{cccc|c} 1 & -3 & 4 & 7 & 1 \\ 2 & -6 & -3 & 5 & 2 \\ 4 & -12 & -17 & 1 & 4 \end{array} \right) \xrightarrow{R_2 - 2R_1, R_3 - 4R_1} \left(\begin{array}{cccc|c} 1 & -3 & 4 & 7 & 1 \\ 0 & 0 & -11 & -9 & 0 \\ 0 & 0 & -33 & -27 & 0 \end{array} \right) \xrightarrow{R_3 - 3R_2} \left(\begin{array}{cccc|c} 1 & -3 & 4 & 7 & 1 \\ 0 & 0 & -11 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

Since the second column and fourth column has pivots, the system has infinitely many solutions. Here, y and w are both free variables.

The corresponding system is
$$\begin{cases} x - 3y + 4z + 7w = 1 \\ -11z - 9w = 0 \end{cases}$$

Take $y = s$ and $w = t$, s, t are real. Solving the equation backward, we get $z = -\frac{9}{11}t$ and $x = 1 + 3s - \frac{41}{11}t$.

Problem 15

Note that

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 2 & -4 & -5 & 3 \\ 3 & -6 & 24 & a \end{array} \right) \xrightarrow{R_2 - 2R_1, R_3 - 3R_1} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 0 & -7 & 1 \\ 0 & 0 & 21 & a - 3 \end{array} \right) \xrightarrow{R_3 + 3R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 0 & -7 & 1 \\ 0 & 0 & 0 & a \end{array} \right)$$

$$(a) \quad \text{If } a = 3, \text{ the system becomes } \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 0 & -7 & 1 \\ 0 & 0 & 0 & 3 \end{array} \right).$$

Since the last row is the row of the type $(0,0,0|b)$, $b \neq 0$, thus the system has no solution.

(b) The system is consistent when there is no row with $(0,0,0|b)$, $b \neq 0$. This happens when $a = 0$.

Problem 16

$$\left(\begin{array}{ccc|c} 3 & -1 & -1 & 1 \\ 2 & -4 & 5 & 1 \\ 4 & 2 & -7 & c \end{array} \right) \xrightarrow{R_1 \div 3} \left(\begin{array}{ccc|c} 1 & -1/3 & -1/3 & 1/3 \\ 2 & -4 & 5 & 1 \\ 4 & 2 & -7 & c \end{array} \right) \xrightarrow{R_2 - 2R_1, R_3 - 4R_1} \left(\begin{array}{ccc|c} 1 & -1/3 & -1/3 & 1/3 \\ 0 & -10/3 & 17/3 & 1/3 \\ 0 & 10/3 & -17/3 & c - 4/3 \end{array} \right) \xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|c} 1 & -1/3 & -1/3 & 1/3 \\ 0 & -10/3 & 17/3 & 1/3 \\ 0 & 0 & 0 & c - 1 \end{array} \right).$$

The system is consistent only when there is no row with $(0,0,0|b)$, $b \neq 0$. This implies that $c - 1 = 0 \Rightarrow c = 1$.

Remark: Since the last column has no pivot, the system has infinitely many solutions if the system is consistent.

Problem 17

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & c \\ -1 & 4 & 1 & c^2 \\ 1 & 8 & -1 & c^3 \end{array} \right) \xrightarrow[R_3-R_1]{R_2+R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & c \\ 0 & 6 & 0 & c^2+c \\ 0 & 6 & 0 & c^3-c \end{array} \right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & c \\ 0 & 6 & 0 & c^2+c \\ 0 & 0 & 0 & c^3-c^2-2c \end{array} \right)$$

- (a) Since the third column has no pivot, thus the system cannot have unique solution for any value of c .
- (b) The system has infinitely many solution if there is no row with $(0,0,0|b)$, $b \neq 0$. This happens when
 $c^3 - c^2 - 2c = 0 \Rightarrow c(c-2)(c+1) = 0$
 $\Rightarrow c = 0$ or $c = 2$ or $c = -1$.
- (c) The system has no solution if there is a row with $(0,0,0|b)$, $b \neq 0$. This happens when
 $c^3 - c^2 - 2c \neq 0 \Rightarrow c \neq 0$ and $c \neq 2$ and $c \neq -1$.

Problem 18

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 1 & -1 & 2 & 2 \\ 0 & 1 & c^2 & c \end{array} \right) \xrightarrow{R_2-R_1} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & c^2 & c \end{array} \right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & c^2-1 & c-1 \end{array} \right)$$

- (a) The system has a unique solution if there is no column with no pivot, this happens when
 $c^2 - 1 \neq 0 \Rightarrow c \neq \pm 1$.
- (b) The system has infinitely many solutions if there is a column with no pivot and there is no row with $(0,0,0|b)$, $b \neq 0$. This happens when $c^2 - 1 = 0$ and $c - 1 = 0$. Solving the equations, we get $c = 1$.
- (c) The system has no solutions if there is a row with $(0,0,0|b)$, $b \neq 0$. This happens when
 $c^2 - 1 = 0$ and $c - 1 \neq 0$. Solving the equations, we get $c = -1$.

Problem 19

$$\left(\begin{array}{ccc|c} 2 & 1 & -b & 3 \\ 0 & a & -1 & 2 \\ -2 & 5 & 0 & 1 \end{array} \right) \xrightarrow{R_3+R_1} \left(\begin{array}{ccc|c} 2 & 1 & -b & 3 \\ 0 & a & -1 & 2 \\ 0 & 6 & -b & 4 \end{array} \right) \xrightarrow{R_3 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 2 & 1 & -b & 3 \\ 0 & 6 & -b & 4 \\ 0 & a & -1 & 2 \end{array} \right)$$

$$\xrightarrow{R_2 \div 6} \left(\begin{array}{ccc|c} 2 & 1 & -b & 3 \\ 0 & 1 & -b/6 & 2/3 \\ 0 & a & -1 & 2 \end{array} \right) \xrightarrow{R_3 - aR_2} \left(\begin{array}{ccc|c} 2 & 1 & -b & 3 \\ 0 & 1 & -b/6 & 2/3 \\ 0 & 0 & ab/6 - 1 & 2 - 2a/3 \end{array} \right)$$

- (a) The system has a unique solution if there is no column with no pivot. This happens when
 $\frac{ab}{6} - 1 \neq 0 \Rightarrow ab \neq 6$.
- (b) The system has infinitely many solutions if there is a column with no pivot and there is no row with $(0,0,0|b)$, $b \neq 0$. This happens when $\begin{cases} \frac{ab}{6} - 1 = 0 \\ 2 - \frac{2a}{3} = 0 \end{cases} \Rightarrow a = 3, b = 2$.
- (c) The system has no solution if there is a row with $(0,0,0|b)$, $b \neq 0$. This happens when
 $\begin{cases} \frac{ab}{6} - 1 = 0 \\ 2 - \frac{2a}{3} \neq 0 \end{cases} \Rightarrow ab = 6, a \neq 3$.

Problem 20

$$\begin{pmatrix} 1 & 3 & -2 & -1 & | & 1 \\ 2 & 5 & -1 & 3 & | & 2 \\ -1 & -1 & a-3 & a^2-10 & | & b \end{pmatrix} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{pmatrix} 1 & 3 & -2 & -1 & | & 1 \\ 0 & -1 & 3 & 5 & | & 0 \\ 0 & 2 & a-5 & a^2-11 & | & b+1 \end{pmatrix} \xrightarrow{R_3+2R_2} \begin{pmatrix} 1 & 3 & -2 & -1 & | & 1 \\ 0 & -1 & 3 & 5 & | & 0 \\ 0 & 0 & a+1 & a^2-1 & | & b+1 \end{pmatrix}$$

- (a) Since the matrix has 3 rows only, there are at most 3 pivots in the matrix. Thus at least one of the 3rd column and 4th column has no pivot, the system cannot have a unique solution for any values of a and b .
- (b) The system has infinitely many solutions when there is no row with $(0,0,0,0|b)$, $b \neq 0$. This happens when $b+1=0 \Rightarrow b=-1$.
*(*Remark: There is no restriction on a since at least one of the columns has no pivot according to the discussion in (a).)*
- (c) The system has no solution if there is a row with $(0,0,0,0|b)$, $b \neq 0$. This happens when $a+1=0$ and $a^2-1=0$ and $b+1 \neq 0$. Solving these equations, we get $a=-1$ and $b \neq -1$.

Problem 21

The given system can be expressed as $\begin{pmatrix} 1 & -2 & 3 \\ 4 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$.

Using Cramer's rule, the solution of the system is found to be

$$x = \frac{\det \begin{pmatrix} 1 & -2 & 3 \\ 5 & 1 & -2 \\ 6 & -1 & 3 \end{pmatrix}}{\det \begin{pmatrix} 1 & -2 & 3 \\ 4 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix}} = \frac{22}{15}, \quad y = \frac{\det \begin{pmatrix} 1 & 1 & 3 \\ 4 & 5 & -2 \\ 2 & 6 & 3 \end{pmatrix}}{\det \begin{pmatrix} 1 & -2 & 3 \\ 4 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix}} = \frac{53}{15}, \quad z = \frac{\det \begin{pmatrix} 1 & -2 & 1 \\ 4 & 1 & 5 \\ 2 & -1 & 6 \end{pmatrix}}{\det \begin{pmatrix} 1 & -2 & 3 \\ 4 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix}} = \frac{11}{5}.$$

Problem 22

(a) $\begin{pmatrix} 1 & 2 & | & 1 & 0 \\ -3 & 4 & | & 0 & 1 \end{pmatrix} \xrightarrow{R_2+3R_1} \begin{pmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 10 & | & 3 & 1 \end{pmatrix} \xrightarrow{R_2 \div 10} \begin{pmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 1 & | & 3/10 & 1/10 \end{pmatrix} \xrightarrow{R_1-2R_2} \begin{pmatrix} 1 & 0 & | & 4/10 & -2/10 \\ 0 & 1 & | & 3/10 & 1/10 \end{pmatrix}$

Thus we conclude that $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{4}{10} & -\frac{2}{10} \\ \frac{3}{10} & \frac{1}{10} \end{pmatrix}$.

(b) $\begin{pmatrix} 3 & 1 & | & 1 & 0 \\ -6 & 4 & | & 0 & 1 \end{pmatrix} \xrightarrow{R_2+2R_1} \begin{pmatrix} 3 & 1 & | & 1 & 0 \\ 0 & 6 & | & 2 & 1 \end{pmatrix} \xrightarrow{R_2 \div 6} \begin{pmatrix} 3 & 1 & | & 1 & 0 \\ 0 & 1 & | & 1/3 & 1/6 \end{pmatrix} \xrightarrow{R_1-\frac{1}{3}R_2} \begin{pmatrix} 3 & 0 & | & 2/9 & -1/18 \\ 0 & 1 & | & 1/3 & 1/6 \end{pmatrix}$

Thus we deduce that $\begin{pmatrix} 3 & 1 \\ -6 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{2}{9} & -\frac{1}{18} \\ \frac{1}{3} & \frac{1}{6} \end{pmatrix}$.

(c) $\begin{pmatrix} 1 & 1 & 2 & | & 1 & 0 & 0 \\ 3 & 2 & 3 & | & 0 & 1 & 0 \\ 4 & 2 & 1 & | & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_3-4R_1]{R_2-3R_1} \begin{pmatrix} 1 & 1 & 2 & | & 1 & 0 & 0 \\ 0 & -1 & -3 & | & -3 & 1 & 0 \\ 0 & -2 & -7 & | & -4 & 0 & 1 \end{pmatrix}$

$$\begin{array}{c} R_3 - 2R_2 \\ \approx \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 1 & 0 \\ 0 & 0 & -1 & 2 & -2 & 1 \end{array} \right) \begin{array}{c} R_2 \div (-1) \\ R_3 \div (-1) \\ \approx \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & -1 & 0 \\ 0 & 0 & 1 & -2 & 2 & -1 \end{array} \right)$$

$$\begin{array}{c} R_1 - 2R_3 \\ R_2 - 3R_3 \\ \approx \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 5 & -4 & 2 \\ 0 & 1 & 0 & 9 & -7 & 3 \\ 0 & 0 & 1 & -2 & 2 & -1 \end{array} \right) \begin{array}{c} R_1 - R_2 \\ \approx \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 3 & -1 \\ 0 & 1 & 0 & 9 & -7 & 3 \\ 0 & 0 & 1 & -2 & 2 & -1 \end{array} \right)$$

Thus we conclude that $\begin{pmatrix} 1 & 1 & 2 \\ 3 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -4 & 3 & -1 \\ 9 & -7 & 3 \\ -2 & 2 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{array}{c} R_3 - 5R_1 \\ \approx \end{array} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{pmatrix}$

$$\begin{array}{c} R_2 \div 2 \\ R_3 \div (-4) \\ \approx \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array} \right) \begin{array}{c} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \\ \approx \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1/4 & 0 & 1/4 \\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array} \right)$$

$$\begin{array}{c} R_1 - R_2 \\ \approx \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 13/8 & -1/2 & -1/8 \\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array} \right).$$

Thus we conclude that $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 13/8 & -1/2 & -1/8 \\ -15/8 & 1/2 & 3/8 \\ 5/4 & 0 & -1/4 \end{pmatrix}.$

(e) $\begin{pmatrix} 0 & -1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 1 & 0 \\ 1 & -1 & 5 & 0 & 0 & 1 \end{pmatrix} \begin{array}{c} R_3 \leftrightarrow R_1 \\ \approx \end{array} \begin{pmatrix} 1 & -1 & 5 & 0 & 0 & 1 \\ 2 & 1 & 4 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & 0 & 0 \end{pmatrix}$

$$\begin{array}{c} R_2 - 2R_1 \\ \approx \end{array} \left(\begin{array}{ccc|ccc} 1 & -1 & 5 & 0 & 0 & 1 \\ 0 & 3 & -6 & 0 & 1 & -2 \\ 0 & -1 & 2 & 1 & 0 & 0 \end{array} \right) \begin{array}{c} R_2 \div 3 \\ \approx \end{array} \left(\begin{array}{ccc|ccc} 1 & -1 & 5 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1/3 & -2/3 \\ 0 & -1 & 2 & 1 & 0 & 0 \end{array} \right)$$

$$\begin{array}{c} R_3 + R_2 \\ \approx \end{array} \left(\begin{array}{ccc|ccc} 1 & -1 & 5 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1/3 & -2/3 \\ 0 & 0 & 0 & 1 & 1/3 & -2/3 \end{array} \right)$$

Since the last row is of the form $(0,0,0|a,b,c)$, $a, b, c \neq 0$, thus the matrix is not invertible.

(f) $\begin{pmatrix} -2 & 3 & 4 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 5 & 8 & 0 & 0 & 1 \end{pmatrix} \begin{array}{c} R_2 \leftrightarrow R_1 \\ \approx \end{array} \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ -2 & 3 & 4 & 1 & 0 & 0 \\ -1 & 5 & 8 & 0 & 0 & 1 \end{pmatrix}$

$$\begin{array}{c} R_2 + 2R_1 \\ R_3 + R_1 \\ \approx \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & 2 & 0 \\ 0 & 5 & 7 & 0 & 1 & 1 \end{array} \right) \begin{array}{c} R_2 \div 3 \\ \approx \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2/3 & 1/3 & 2/3 & 0 \\ 0 & 5 & 7 & 0 & 1 & 1 \end{array} \right)$$

$$\begin{array}{c} R_3 - 5R_2 \\ \approx \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2/3 & 1/3 & 2/3 & 0 \\ 0 & 0 & 11/3 & -5/3 & -7/3 & 1 \end{array} \right) \begin{array}{c} R_3 \div \frac{11}{3} \\ \approx \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2/3 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1 & -5/11 & -7/11 & 3/11 \end{array} \right)$$

$$\begin{array}{c} R_1 + R_3 \\ R_2 - \frac{2}{3}R_3 \\ \approx \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/11 & 4/11 & 3/11 \\ 0 & 1 & 0 & 7/11 & 12/11 & -2/11 \\ 0 & 0 & 1 & -5/11 & -7/11 & 3/11 \end{array} \right)$$

Thus we conclude that $\begin{pmatrix} -2 & 3 & 4 \\ 1 & 0 & -1 \\ -1 & 5 & 8 \end{pmatrix}^{-1} = \begin{pmatrix} -5/11 & 4/11 & 3/11 \\ 7/11 & 12/11 & -2/11 \\ -5/11 & -7/11 & 3/11 \end{pmatrix}.$

(g) $\begin{pmatrix} 1 & 1 & 2 & -3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{c} R_4 - R_1 \\ \approx \end{array} \begin{pmatrix} 1 & 1 & 2 & -3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2 & -1 & 4 & -1 & 0 & 0 & 1 \end{pmatrix}$

$$\begin{aligned}
& \begin{matrix} R_4+R_2 \\ \sim \end{matrix} \begin{pmatrix} 1 & 1 & 2 & -3 & | & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & | & -1 & 1 & 0 & 1 \end{pmatrix} \begin{matrix} R_2 \div 2 \\ R_3 \div 3 \\ R_4 \div 4 \\ \sim \end{matrix} \begin{pmatrix} 1 & 1 & 2 & -3 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & | & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & | & -1/4 & 1/4 & 0 & 1/4 \end{pmatrix} \\
& \begin{matrix} R_1+3R_4 \\ \sim \end{matrix} \begin{pmatrix} 1 & 1 & 2 & 0 & | & 1/4 & 3/4 & 0 & 3/4 \\ 0 & 1 & 1/2 & 0 & | & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & | & -1/4 & 1/4 & 0 & 1/4 \end{pmatrix} \\
& \begin{matrix} R_1-2R_3 \\ R_2-\frac{1}{2}R_3 \\ \sim \end{matrix} \begin{pmatrix} 1 & 1 & 0 & 0 & | & 1/4 & 3/4 & -2/3 & 3/4 \\ 0 & 1 & 0 & 0 & | & 0 & 1/2 & -1/6 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & | & -1/4 & 1/4 & 0 & 1/4 \end{pmatrix} \\
& \begin{matrix} R_1-R_2 \\ \sim \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 & | & 1/4 & 1/4 & -1/2 & 3/4 \\ 0 & 1 & 0 & 0 & | & 0 & 1/2 & -1/6 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & | & -1/4 & 1/4 & 0 & 1/4 \end{pmatrix}
\end{aligned}$$

Thus the inverse is given by $\begin{pmatrix} 1 & 1 & 2 & -3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & -1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/4 & 1/4 & -1/2 & 3/4 \\ 0 & 1/2 & -1/6 & 0 \\ 0 & 0 & 1/3 & 0 \\ -1/4 & 1/4 & 0 & 1/4 \end{pmatrix}.$

(h) $\begin{pmatrix} 1 & 3 & -1 & 1 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & | & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & -2 & | & 0 & 0 & 1 & 0 \\ 2 & 0 & -1 & 3 & | & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} R_3-R_1 \\ R_4-2R_1 \\ \sim \end{matrix} \begin{pmatrix} 1 & 3 & -1 & 1 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & | & 0 & 1 & 0 & 0 \\ 0 & -2 & 3 & -3 & | & -1 & 0 & 1 & 0 \\ 0 & -6 & 1 & 1 & | & -2 & 0 & 0 & 1 \end{pmatrix}$

$$\begin{aligned}
& \begin{matrix} R_3+2R_2 \\ R_4+6R_2 \\ \sim \end{matrix} \begin{pmatrix} 1 & 3 & -1 & 1 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 3 & | & -1 & 2 & 1 & 0 \\ 0 & 0 & 7 & 19 & | & -2 & 6 & 0 & 1 \end{pmatrix} \begin{matrix} R_3 \div 5 \\ \sim \end{matrix} \begin{pmatrix} 1 & 3 & -1 & 1 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/5 & | & -1/5 & 2/5 & 1/5 & 0 \\ 0 & 0 & 7 & 19 & | & -2 & 6 & 0 & 1 \end{pmatrix} \\
& \begin{matrix} R_4-7R_3 \\ \sim \end{matrix} \begin{pmatrix} 1 & 3 & -1 & 1 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/5 & | & -1/5 & 2/5 & 1/5 & 0 \\ 0 & 0 & 0 & 74/5 & | & -3/5 & 16/5 & -7/5 & 1 \end{pmatrix} \\
& \begin{matrix} R_4 \div \frac{74}{5} R_3 \\ \sim \end{matrix} \begin{pmatrix} 1 & 3 & -1 & 1 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/5 & | & -1/5 & 2/5 & 1/5 & 0 \\ 0 & 0 & 0 & 1 & | & -3/74 & 8/37 & -7/74 & 5/74 \end{pmatrix} \\
& \begin{matrix} R_1-R_4 \\ R_2-3R_4 \\ R_3-\frac{3}{5}R_4 \\ \sim \end{matrix} \begin{pmatrix} 1 & 3 & -1 & 0 & | & 77/74 & -8/37 & 7/74 & -5/74 \\ 0 & 1 & 1 & 0 & | & 9/74 & 13/37 & 21/74 & -15/74 \\ 0 & 0 & 1 & 0 & | & -13/74 & 10/37 & 19/74 & -3/74 \\ 0 & 0 & 0 & 1 & | & -3/74 & 8/37 & -7/74 & 5/74 \end{pmatrix} \\
& \begin{matrix} R_1+R_3 \\ R_2-R_3 \\ \sim \end{matrix} \begin{pmatrix} 1 & 3 & 0 & 0 & | & 32/37 & 2/37 & 13/37 & -4/37 \\ 0 & 1 & 0 & 0 & | & 11/37 & 3/37 & 1/37 & -6/37 \\ 0 & 0 & 1 & 0 & | & -13/74 & 10/37 & 19/74 & -3/74 \\ 0 & 0 & 0 & 1 & | & -3/74 & 8/37 & -7/74 & 5/74 \end{pmatrix} \\
& \begin{matrix} R_1-3R_2 \\ \sim \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 & | & -1/37 & -7/37 & 10/37 & 14/37 \\ 0 & 1 & 0 & 0 & | & 11/37 & 3/37 & 1/37 & -6/37 \\ 0 & 0 & 1 & 0 & | & -13/74 & 10/37 & 19/74 & -3/74 \\ 0 & 0 & 0 & 1 & | & -3/74 & 8/37 & -7/74 & 5/74 \end{pmatrix}
\end{aligned}$$

The inverse is found to be $\begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 2 & -2 \\ 2 & 0 & -1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -1/37 & -7/37 & 10/37 & 14/37 \\ 11/37 & 3/37 & 1/37 & -6/37 \\ -13/74 & 10/37 & 19/74 & -3/74 \\ -3/74 & 8/37 & -7/74 & 5/74 \end{pmatrix}.$

Problem 23

(a) We let $\vec{a} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. Then we consider the following equation:

$$x_1\vec{a} + x_2\vec{b} + x_3\vec{c} = \vec{0}$$

$$\Rightarrow x_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_2 - x_3 \\ x_1 + 2x_2 + x_3 \\ x_1 + x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Comparing the components, we obtain $\begin{cases} x_2 - x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \\ x_1 + x_3 = 0 \end{cases}$.

Using Gaussian Elimination, we have

$$\begin{pmatrix} 0 & 1 & -1 & | & 0 \\ 1 & 2 & 1 & | & 0 \\ 1 & 0 & 1 & | & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 1 & 0 & 1 & | & 0 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & -2 & 0 & | & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 + 2R_2} \begin{pmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{pmatrix}$$

The system corresponds to $\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_2 - x_3 = 0 \\ -2x_3 = 0 \end{cases}$. Solving the equations backward, we have

$$x_3 = 0, x_2 = 0, x_1 = 0.$$

Since $(x_1, x_2, x_3) = (0, 0, 0)$ is the only solution, thus the vectors are linearly independent.

(b) We let $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. Then we consider the following equation:

$$x_1\vec{a} + x_2\vec{b} + x_3\vec{c} = \vec{0}$$

$$\Rightarrow x_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 - 2x_2 + 2x_3 \\ 2x_1 + x_2 + x_3 \\ x_1 + x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Comparing the components, we obtain $\begin{cases} x_1 - 2x_2 + 2x_3 = 0 \\ 2x_1 + x_2 + x_3 = 0 \\ x_1 + x_3 = 0 \end{cases}$.

Using Gaussian Elimination, we have

$$\begin{pmatrix} 1 & -2 & 2 & | & 0 \\ 2 & 1 & 1 & | & 0 \\ 1 & 0 & 1 & | & 0 \end{pmatrix} \xrightarrow{R_2 - 2R_1, R_3 - R_1} \begin{pmatrix} 1 & -2 & 2 & | & 0 \\ 0 & 5 & -3 & | & 0 \\ 0 & 2 & -1 & | & 0 \end{pmatrix} \xrightarrow{R_2 \div 5} \begin{pmatrix} 1 & -2 & 2 & | & 0 \\ 0 & 1 & -3/5 & | & 0 \\ 0 & 2 & -1 & | & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & -2 & 2 & | & 0 \\ 0 & 1 & -3/5 & | & 0 \\ 0 & 0 & 1/5 & | & 0 \end{pmatrix}$$

The system corresponds to $\begin{cases} x_1 - 2x_2 + 2x_3 = 0 \\ x_2 - \frac{3}{5}x_3 = 0 \\ \frac{1}{5}x_3 = 0 \end{cases}$. Solving the equations backward, we have

$$x_3 = 0, x_2 = 0, x_1 = 0.$$

Since $(x_1, x_2, x_3) = (0, 0, 0)$ is the only solution, thus the vectors are linearly independent.

(c) We let $\vec{a} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix}$. Then we consider the following equation:

$$x_1\vec{a} + x_2\vec{b} + x_3\vec{c} = \vec{0}$$

$$\Rightarrow x_1 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 + 4x_3 \\ 3x_1 + 2x_2 + 6x_3 \\ x_2 - 3x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Comparing the components, we obtain
$$\begin{cases} x_1 + 4x_3 = 0 \\ 3x_1 + 2x_2 + 6x_3 = 0 \\ x_2 - 3x_3 = 0 \end{cases}$$

Using Gaussian Elimination, we have

$$\begin{pmatrix} 1 & 0 & 4 & | & 0 \\ 3 & 2 & 6 & | & 0 \\ 0 & 1 & -3 & | & 0 \end{pmatrix} \xrightarrow{R_2-3R_1} \begin{pmatrix} 1 & 0 & 4 & | & 0 \\ 0 & 2 & -6 & | & 0 \\ 0 & 1 & -3 & | & 0 \end{pmatrix} \xrightarrow{R_2 \div 2} \begin{pmatrix} 1 & 0 & 4 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 1 & -3 & | & 0 \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} 1 & 0 & 4 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

The 3rd column has no pivot and the corresponding variable x_3 is a free variable. Take $x_3 = t$, t is real. By solving the corresponding system
$$\begin{cases} x_1 + 4x_3 = 0 \\ x_2 - 3x_3 = 0 \end{cases}$$
 backward, we have

$$x_1 = -4t \text{ and } x_2 = 3t.$$

Since there is non-trivial solutions for the equation (for example, take $t = 1$, we get $x_1 = -4$, $x_2 = 3$, $x_3 = 1$), the vectors are linearly dependent.

(d) We let $\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{d} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Then we consider the following equation:

$$x_1\vec{a} + x_2\vec{b} + x_3\vec{c} + x_4\vec{d} = \vec{0}$$

$$\Rightarrow x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 + x_3 + x_4 \\ x_2 + 2x_3 + 2x_4 \\ x_1 + x_2 + 3x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Comparing the components, we obtain
$$\begin{cases} x_1 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 0 \\ x_1 + x_2 + 3x_4 = 0 \end{cases}$$

Using Gaussian Elimination, we have

$$\begin{pmatrix} 1 & 0 & 1 & 1 & | & 0 \\ 0 & 1 & 2 & 2 & | & 0 \\ 1 & 1 & 0 & 3 & | & 0 \end{pmatrix} \xrightarrow{R_3-R_1} \begin{pmatrix} 1 & 0 & 1 & 1 & | & 0 \\ 0 & 1 & 2 & 2 & | & 0 \\ 0 & 1 & -1 & 2 & | & 0 \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} 1 & 0 & 1 & 1 & | & 0 \\ 0 & 1 & 2 & 2 & | & 0 \\ 0 & 0 & -3 & 0 & | & 0 \end{pmatrix}$$

The 4th column has no pivot and the corresponding variable x_4 is a free variable. Take $x_4 = t$, t is real. By solving the corresponding system
$$\begin{cases} x_1 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 0 \\ -3x_3 = 0 \end{cases}$$
 backward, we have

$$x_3 = 0, x_2 = -2t \text{ and } x_1 = -t.$$

Since there is a non-trivial solution for the equation (for example, take $t = 1$, we get $x_1 = -1$, $x_2 = -2$, $x_3 = 0$, $x_4 = 1$), the vectors are linearly dependent.

Problem 24

(a)
$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} \xrightarrow{R_2-2R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

Since the number of pivots is 2, thus the rank of the matrix is 2.

(b)
$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix} \xrightarrow{R_2-3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & 3 & 4 \end{pmatrix} \xrightarrow{R_2 \div (-4)} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 4 \end{pmatrix} \xrightarrow{R_3-3R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{pmatrix}.$$

Since the number of pivots is 3, thus the rank of the matrix is 3.

(c)
$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{pmatrix} \xrightarrow{R_2-2R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 3 & 3 & 6 \end{pmatrix} \xrightarrow{R_3-3R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the number of pivots is 1, thus the rank of the matrix is 1.