

EF2452: Mathematics for Economics and Finance

Lecture 3

Part I of II

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Main Topics

- Rules for Limits
- Differentiation: Rate of Change of a Function
- Differentiation from First Principles
- Rules: Power, Sum, Product, Quotient

(Intuitive) Rules for Limits

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0$$

$$\lim_{x \rightarrow a} [f(x)]^r = [\lim_{x \rightarrow a} f(x)]^r, \text{ if } [\lim_{x \rightarrow a} f(x)]^r \text{ is defined and } r \text{ is a real number}$$

Limits: Example 1

- Compute $\lim_{x \rightarrow 4} \frac{2x^{3/2} - \sqrt{x}}{x^2 - 15}$.

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{2x^{3/2} - \sqrt{x}}{x^2 - 15} &= \frac{\lim_{x \rightarrow 4} (2x^{3/2} - \sqrt{x})}{\lim_{x \rightarrow 4} (x^2 - 15)} = \frac{\lim_{x \rightarrow 4} 2x^{3/2} - \lim_{x \rightarrow 4} \sqrt{x}}{\lim_{x \rightarrow 4} x^2 - \lim_{x \rightarrow 4} 15} \\ &= \frac{2 \cdot 4^{3/2} - \sqrt{4}}{4^2 - 15} = \frac{2 \cdot 8 - 2}{16 - 15} = 14\end{aligned}$$

Limits: Example 2

- Compute $\lim_{x \rightarrow 2} \frac{3x^2 + 3x - 18}{x - 2}$.
- Note that both the numerator and denominator tend to 0 as x tends to 2. However, see that $3x^2 + 3x - 18$ can be written as

$$3(x^2 + x - 6) = 3(x^2 + 3x - 2x - 6) = 3(x - 2)(x + 3)$$

- For $x \neq 2$, we can cancel $x - 2$ from both numerator and denominator to obtain $3(x + 3)$. Then, we have

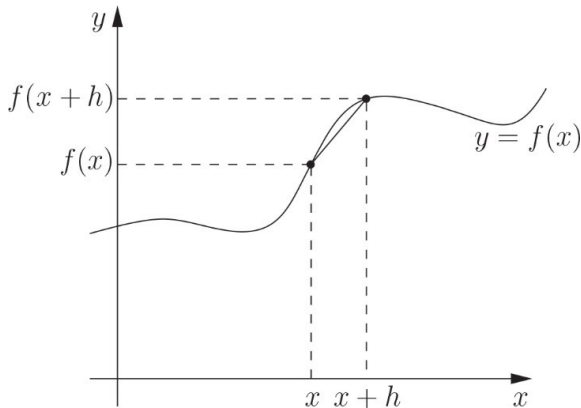
$$\lim_{x \rightarrow 2} \frac{3x^2 + 3x - 18}{x - 2} = \lim_{x \rightarrow 2} 3(x + 3) = 3(2 + 3) = 15$$

The Rate of Change of a Function

- Many problems in economics require us to consider how a function changes with respect to its input.
- For example, how does a small price change affect the quantity of a commodity consumers wish to buy?
- Similarly, governments are concerned about how certain indicators like income (or GDP) or inflation change with respect to time.

The Rate of Change of a Function

We can use the following figure to understand the rate of change of a function $f : \mathbb{R} \rightarrow \mathbb{R}$.



The idea is to compare the value of the function at x with its value at $x + h$, where h is a small quantity.

The Rate of Change of a Function

- The change in the value of f is $f(x+h) - f(x)$, which, when divided by the change h in the 'input', measures the average rate of change.
- $\frac{f(x+h)-f(x)}{h}$ is represented diagrammatically by the gradient of the chord joining the points $(x, f(x))$ and $(x+h, f(x+h))$.
- In the diagram, as h approaches 0, the chord approaches the tangent at $(x, f(x))$, and its gradient approaches a value which we may take to be the 'instantaneous' rate of change of f at x . This leads us to define the gradient of f at x to be the limiting value of $\frac{f(x+h)-f(x)}{h}$, that is,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation from first principles

- The function f' which tells us the rate of change of f is called the **derivative** of f .
- The process of finding the derivative is known as **differentiation**.
- With $y = f(x)$, we often write $f'(x)$ as $\frac{dy}{dx}$. We will interchangeably use these two notations.
- Later on, we will present some important rules of differentiation. However, when we calculate the derivative using the measure $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$, we say that we are **differentiating using first principles**.

Differentiation from first principles: An Example

- Suppose $f(x) = x^3$. Calculate the derivative of this function from first principles.
- We have

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h} = \frac{(x^3 + h^3 + 3xh(x+h)) - x^3}{h}$$

the last equality follows from the formula $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$.

- Then, we have

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{h^3 + 3xh(x+h)}{h} = 3x^2 + 3xh + h^2 \\ \implies \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2\end{aligned}$$

Constants

- If $f(x) = k$, where k is a constant, then

$$f'(x) = 0$$

Differentiation of a constant is zero: a constant doesn't change when there is a small change in x .

- If $g(x) = kf(x)$, where k is a constant, then

$$g'(x) = kf'(x)$$

Power Rule

- If $f(x) = x^n$ where n is any real number, then

$$f'(x) = nx^{n-1}$$

- An example: Suppose the demand set for a commodity is given by

$$D = \{(q, p) | p^3 q = 8000\}$$

What will be the expected fall in sales (approximately) when the price is increased slightly from \$20?

- To answer the question, we first need to write q as a function of p and then find the derivative of the demand function.

Power Rule

- The demand function is given by

$$q^D(p) = 8000p^{-3}$$

- Using the power rule, we have

$$q^{D'}(p) = 8000 \times (-3)p^{-3-1} = \frac{-24000}{p^4}$$

- Note that as the price increases, the quantity demanded falls. Moreover, to answer the question, we have

$$q^{D'}(20) = \frac{-24000}{20^4} = -0.15$$

Sum and Product Rule

- The Sum Rule: If $h(x) = f(x) + g(x)$, then

$$h'(x) = f'(x) + g'(x)$$

- The Product Rule: If $h(x) = f(x)g(x)$, then

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

Sum and Product Rule: An Example

- Find the derivative of the following function

$$h(x) = (x^3 + 1)(x^2 + 5x)$$

- First, note that $f(x) = x^3 + 1$, $g(x) = x^2 + 5x$.
- Use the sum rule to obtain

$$f'(x) = \frac{d}{dx}(x^3) + \frac{d}{dx}(1) = 3x^2$$

The last expression follows from the power rule and the fact that the differentiation of a constant is zero.

- Similarly,

$$g'(x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(5x) = 2x + 5$$

Sum and Product Rule: An Example

- From the product rule, we have

$$\begin{aligned}h'(x) &= f'(x)g(x) + f(x)g'(x) \\&= (3x^2)(x^2 + 5x) + (x^3 + 1)(2x + 5) \\&= 3x^4 + 15x^3 + 2x^4 + 5x^3 + 2x + 5 \\&= 5x^4 + 20x^3 + 2x + 5\end{aligned}$$

- An alternative method would be to multiply out the original expression and then find the derivative of each term.

Quotient Rule

- If $h(x) = \frac{f(x)}{g(x)}$ and $g(x) \neq 0$, then

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

- In fact, you can find the above formula (quotient rule) using the product rule. Note that because $h(x) = \frac{f(x)}{g(x)}$, we have

$$f(x) = h(x)g(x)$$

- Using product rule, we have

$$f'(x) = h'(x)g(x) + g'(x)h(x) = h'(x)g(x) + g'(x)\frac{f(x)}{g(x)}$$

where we obtain the last equality by substituting $h(x) = \frac{f(x)}{g(x)}$.

Quotient Rule

$$f'(x) = h'(x)g(x) + g'(x)\frac{f(x)}{g(x)}$$

$$\implies f'(x) = \frac{h'(x)[g(x)]^2 + g'(x)f(x)}{g(x)}$$

$$\implies f'(x)g(x) = h'(x)[g(x)]^2 + g'(x)f(x)$$

$$\implies h'(x)[g(x)]^2 = f'(x)g(x) - g'(x)f(x)$$

$$\implies h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Quotient Rule: Example

- Suppose $h(x) = \frac{3x-5}{x-2}$. Calculate $h'(x)$ and $h'(4)$.
- We have $f(x) = 3x - 5$ and $g(x) = x - 2$. Then, $f'(x) = 3$ and $g'(x) = 1$. For $x \neq 2$, we have

$$\begin{aligned}h'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} = \frac{(3)(x-2) - (3x-5)(1)}{(x-2)^2} \\&= \frac{3x - 6 - 3x + 5}{(x-2)^2} = \frac{-1}{(x-2)^2} \\h'(4) &= \frac{-1}{(4-2)^2} = \frac{-1}{4}\end{aligned}$$

References

Following chapter of the textbook:

6 Introduction to Calculus