

Homework #1

Solutions:

Course: EE3121 Differential Equations for Electrical Engineering

Date:

Problem 1

Solution. (i)

$$\frac{dy}{dt} - \frac{1}{\sqrt{1-t}}y = 0, y(0) = 5.$$

Step 1: Separate variables

$$\frac{dy}{y} = \frac{dt}{\sqrt{1-t}}.$$

Step 2: Integrate

$$\begin{aligned}\int_{y(0)}^{y(t)} \frac{dy}{y} &= \int_0^t \frac{d\tau}{\sqrt{1-\tau}}, \\ \ln y(t) - \ln y(0) &= \int_0^t d(-2\sqrt{1-\tau}) = -2\sqrt{1-t} + 2, \\ \ln \frac{y(t)}{y(0)} &= 2(1 - \sqrt{1-t}).\end{aligned}$$

Fill rest of the steps.

$$y(t) = 5e^{2(1-\sqrt{1-t})}.$$

(ii)

$$\begin{aligned}\frac{dy}{dt} + (t \ln t)y &= 0, y(1) = 7, \\ \Rightarrow \frac{dy}{y} &= -t \ln t dt, \\ \int_{y(1)}^{y(t)} \frac{dy}{y} &= - \int_1^t \tau \ln \tau d\tau.\end{aligned}$$

Hint: Use integration by parts to find $\int_1^t \tau \ln \tau d\tau$.

$$\begin{aligned}\int_1^t \tau \ln \tau d\tau &= \frac{1}{2} \int_1^t \ln \tau d\tau^2 = \frac{1}{2} \left(\tau^2 \ln \tau \Big|_1^t - \int_1^t \tau^2 d \ln \tau \right) = \frac{1}{2} t^2 \ln t - \frac{1}{2} \int_1^t \tau^2 \cdot \frac{1}{\tau} d\tau \\ &= \frac{1}{2} t^2 \ln t - \frac{1}{4} t^2 + \frac{1}{4}.\end{aligned}$$

$$\Rightarrow \ln \frac{y(t)}{y(1)} = -\frac{1}{2} t^2 \ln t + \frac{1}{4} t^2 - \frac{1}{4}.$$

Therefore,

$$y(t) = 7e^{-\frac{1}{2}t^2 \ln t + \frac{1}{4}t^2 - \frac{1}{4}} = 7t^{-\frac{1}{2}t^2} e^{\frac{1}{4}t^2 - \frac{1}{4}}$$

Problem 2**Solution.** (i)

$$\frac{dy}{dt} + 4y = 7e^{-2t}, y(0) = 3.$$

Step 1: Evaluate

$$\int_0^t p(\tau) d\tau = \int_0^t 4 d\tau = 4t.$$

Step 2: Evaluate

$$\begin{aligned} \int_0^t e^{\int_t^\tau p(\tau) d\tau} g(\tau) d\tau &= \int_0^t e^{4(\tau-t)} \cdot 7e^{-2\tau} d\tau = 7 \int_0^t e^{2\tau} \cdot e^{-4t} d\tau = 7e^{-4t} \int_0^t e^{2\tau} d\tau \\ &= 7e^{-4t} \cdot \frac{1}{2} e^{2\tau} \Big|_0^t = \frac{7}{2} e^{-4t} (e^{2t} - 1), \end{aligned}$$

$$\Rightarrow y(t) = 3e^{-4t} + \frac{7}{2} e^{-4t} (e^{2t} - 1) = -\frac{1}{2} e^{-4t} + \frac{7}{2} e^{-2t}.$$

(ii)

$$\frac{dy}{dt} + \frac{1}{t}y = 6 \sin 2\pi t, y(1) = 1,$$

$$\int_t^\tau p(\tau) d\tau = \int_t^\tau \frac{1}{\tau} d\tau = \ln \frac{\tau}{t},$$

$$\Rightarrow \int_1^t p(\tau) d\tau = \ln t,$$

$$y(t) = \frac{1}{t} + \int_1^t e^{\ln \frac{\tau}{t}} 6 \sin 2\pi \tau d\tau = \frac{1}{t} + \int_1^t \frac{\tau}{t} 6 \sin 2\pi \tau d\tau = \frac{1}{t} + \frac{6}{t} \int_1^t \tau \sin 2\pi \tau d\tau.$$

Hint: Use integration by parts.

$$\begin{aligned} \int_1^t \tau \sin 2\pi \tau d\tau &= -\frac{1}{2\pi} \int_1^t \tau d \cos 2\pi \tau = -\frac{1}{2\pi} \left(\tau \cos 2\pi \tau \Big|_1^t - \int_1^t \cos 2\pi \tau d\tau \right) \\ &= -\frac{1}{2\pi} (t \cos 2\pi t - \cos 2\pi) + \frac{1}{2\pi} \cdot \frac{1}{2\pi} \sin 2\pi \tau \Big|_1^t \\ &= -\frac{1}{2\pi} t \cos 2\pi t + \frac{1}{2\pi} + \frac{1}{4\pi^2} \sin 2\pi t, \end{aligned}$$

$$\Rightarrow y(t) = \frac{1}{t} + \frac{3}{\pi t} - \frac{3}{\pi} \cos 2\pi t + \frac{3}{2\pi^2 t} \sin 2\pi t.$$

Problem 3**Solution.** According to Kirchhoff loop law,

$$L \frac{dI}{dt} + RI = V(t) = -V_0 \sin \omega t.$$

Put this equation in the standard form:

$$\frac{dI}{dt} + \frac{R}{L} I = -\frac{V_0}{L} \sin \omega t.$$

Assume $I(0) = I_0$.

Hint: Use integration by parts.

$$\int_t^\tau p(\tau) d\tau = \int_t^\tau \frac{R}{L} d\tau = \frac{R}{L} (\tau - t),$$

$$\Rightarrow \int_0^t p(\tau) d\tau = \frac{R}{L}t,$$

$$y(t) = I_0 e^{-\frac{R}{L}t} - \int_0^t e^{\frac{R}{L}(\tau-t)} \frac{V_0}{L} \sin \omega \tau d\tau = I_0 e^{-\frac{R}{L}t} - e^{-\frac{R}{L}t} \cdot \frac{V_0}{L} \int_0^t e^{\frac{R}{L}\tau} \sin \omega \tau d\tau.$$

$$\begin{aligned} \int_0^t e^{\frac{R}{L}\tau} \sin \omega \tau d\tau &= \frac{1}{\omega} \int_0^t e^{\frac{R}{L}\tau} d \cos \omega \tau = -\frac{1}{\omega} \left(e^{\frac{R}{L}\tau} \cos \omega \tau \Big|_0^t - \int_0^t \cos \omega \tau d e^{\frac{R}{L}\tau} \right) \\ &= -\frac{1}{\omega} e^{\frac{R}{L}t} \cos \omega t + \frac{1}{\omega} + \frac{R}{L\omega^2} \int_0^t e^{\frac{R}{L}\tau} d \sin \omega \tau \\ &= -\frac{1}{\omega} e^{\frac{R}{L}t} \cos \omega t + \frac{1}{\omega} + \frac{R}{L\omega^2} \left(e^{\frac{R}{L}\tau} \sin \omega \tau \Big|_0^t - \int_0^t \sin \omega \tau d e^{\frac{R}{L}\tau} \right) \\ &= -\frac{1}{\omega} e^{\frac{R}{L}t} \cos \omega t + \frac{1}{\omega} + \frac{R}{L\omega^2} e^{\frac{R}{L}t} \sin \omega t - \frac{R^2}{L^2\omega^2} \int_0^t e^{\frac{R}{L}\tau} \sin \omega \tau d\tau, \\ \Rightarrow \int_0^t e^{\frac{R}{L}\tau} \sin \omega \tau d\tau &= \frac{-\frac{1}{\omega} e^{\frac{R}{L}t} \cos \omega t + \frac{1}{\omega} + \frac{R}{L\omega^2} e^{\frac{R}{L}t} \sin \omega t}{1 + \frac{R^2}{L^2\omega^2}}. \end{aligned}$$

Therefore,

$$I(t) = I_0 e^{-\frac{R}{L}t} - \frac{L\omega V_0}{L^2\omega^2 + R^2} \left(e^{\frac{R}{L}t} + \frac{R}{L\omega} \sin \omega t - \cos \omega t \right)$$
