

MA1201 Calculus and Basic Linear Algebra II

Lecture Note 3

Application of Integration

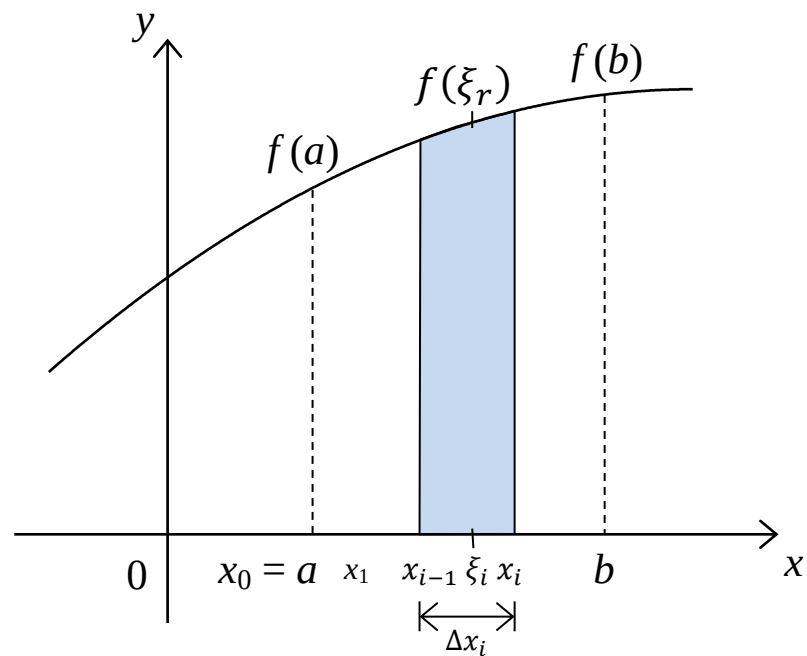
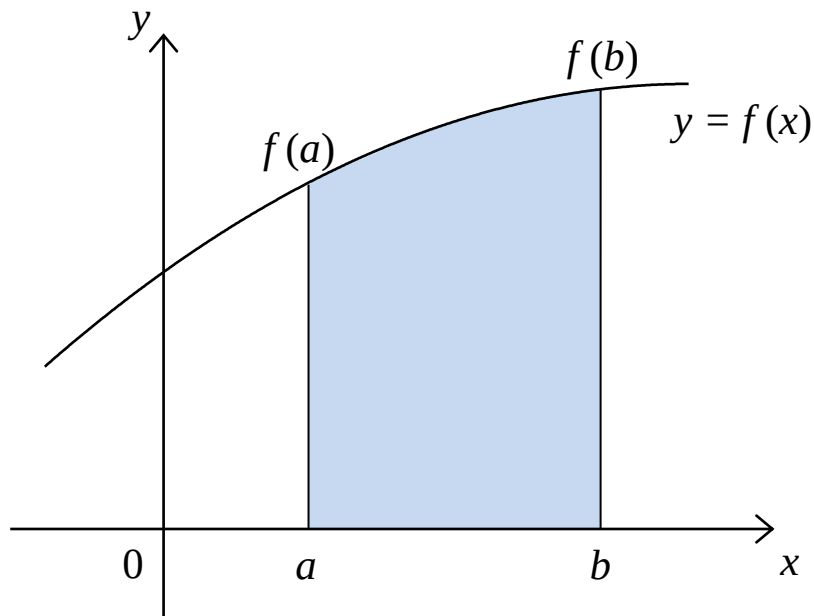
Application of Integration

1. Geometric Application

- Area of region bounded by curves, axes
- Volume of an object formed by the rotation of a curve
- Arc length of a curve
- Surface area (Surface of Revolution)

Area of region bounded by curves, axes

Recall in Chapter 1, the definite integral is defined as the area under the curve $y = f(x)$.



$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(\xi_i)\Delta x_i.$$

As mentioned in Chapter 1 , this integral gives the signed area of the region under the graph:

- **If the curve $y = f(x)$ lies above the x -axis,**

$\int_a^b f(x)dx$ returns the true area of the regions bounded by the curve $y = f(x)$, x -axis, the lines $x = a$ and $x = b$.

- **If the curve $y = f(x)$ lies below the x -axis,**

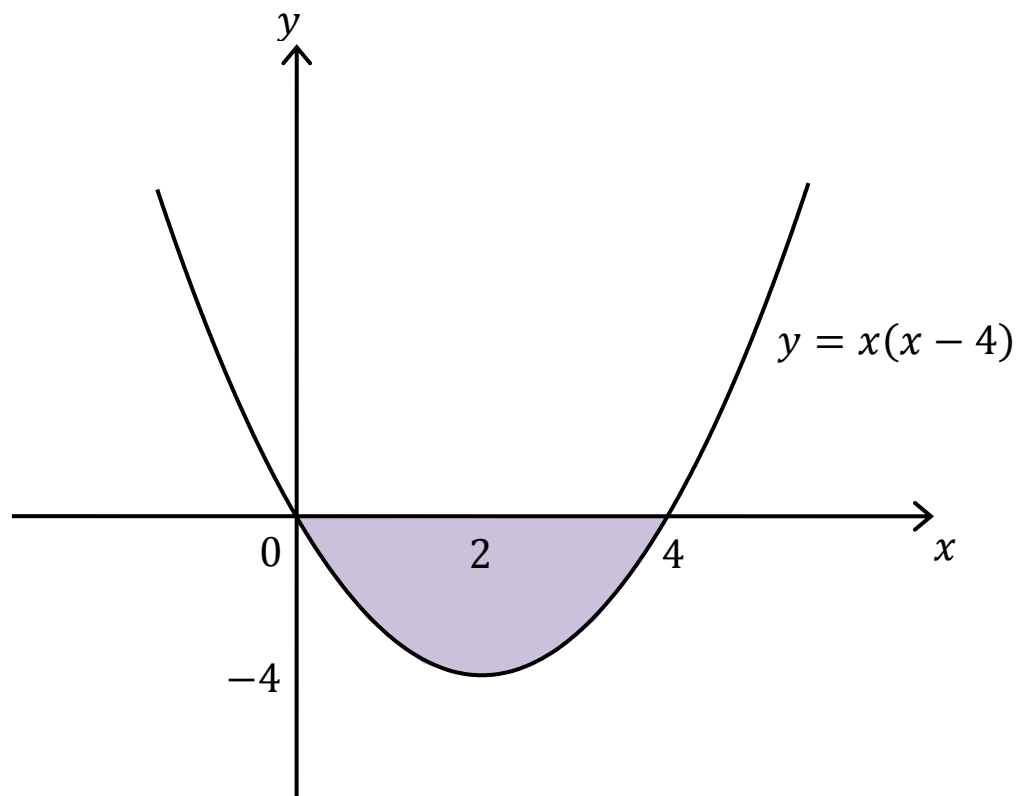
$\int_a^b f(x)dx$ will become a negative number. However, $-\int_a^b f(x)dx$ (which is positive) represents the area of the regions bounded by the curve $y = f(x)$, x -axis, the lines $x = a$ and $x = b$.

Thus, we should use the following formula when finding the area under the curve:

$$\int_a^b |f(x)|dx.$$

Example 1

Calculate the area bounded by the x -axis and the curve $y = x(x - 4)$.



☺Solution:

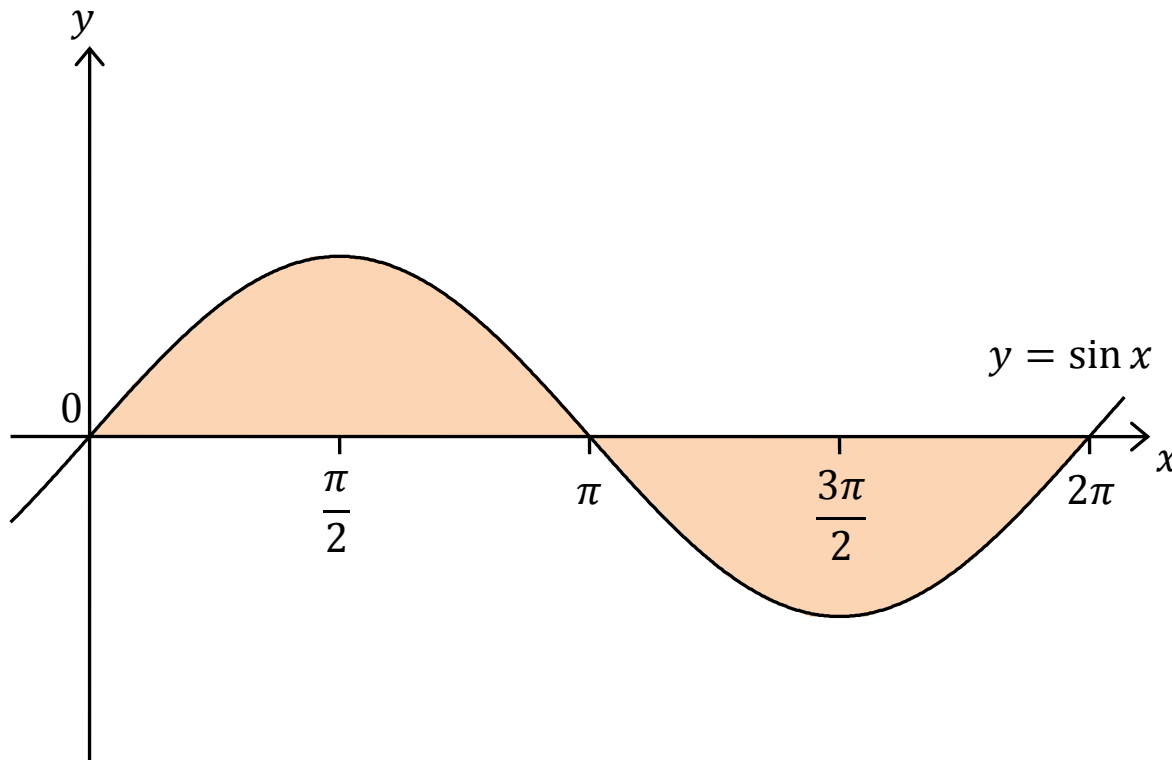
Note that $x(x - 4) = 0 \Rightarrow x = 0$ or $x = 4$,

Since the entire shaded region is below the x -axis. The shaded area is given by

$$\begin{aligned} -\int_0^4 x(x - 4)dx &= \int_0^4 (4x - x^2)dx \\ &= 4 \int_0^4 xdx - \int_0^4 x^2 dx \\ &= 4 \left[\frac{x^2}{2} \right]_0^4 - \left[\frac{x^3}{3} \right]_0^4 = 4(8) - \frac{64}{3} = \frac{32}{3} \text{ (square units)}. \end{aligned}$$

Example 2

Find the whole area enclosed by the curve $y = \sin x$ and the x -axis between $x = 0$ and $x = 2\pi$.



☺Solution:

The curve $y = \sin x$ lies above the x -axis when $0 \leq x \leq \pi$ and lies below the x -axis when $\pi \leq x \leq 2\pi$.

The area of shaded region is given by

$$\begin{aligned} &= \int_0^{2\pi} |\sin x| dx \\ &= \int_0^{\pi} \sin x \, dx + \left(- \int_{\pi}^{2\pi} \sin x \, dx \right) = [-\cos x]_0^{\pi} - [-\cos x]_{\pi}^{2\pi} = 4. \end{aligned}$$



lies above the x -axis lies below the x -axis

Example 3

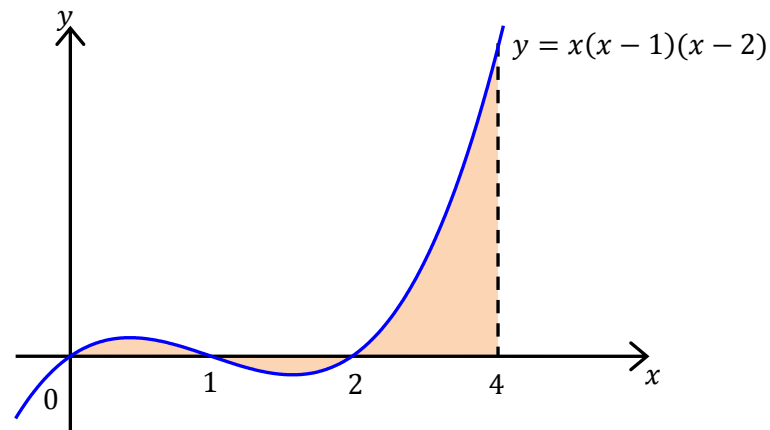
Find the area enclosed by the curve $y = x(x - 1)(x - 2)$ and the x -axis between $x = 0$ and $x = 4$.

☺Solution:

Note that $f(x) = x(x - 1)(x - 2) = 0 \Rightarrow x = 0, x = 1, x = 2$.

	$x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$1 < x < 2$	$x = 2$	$x > 2$
$f(x)$	−	0	+	0	−	0	+

The figure (and required shaded region) is shown below:

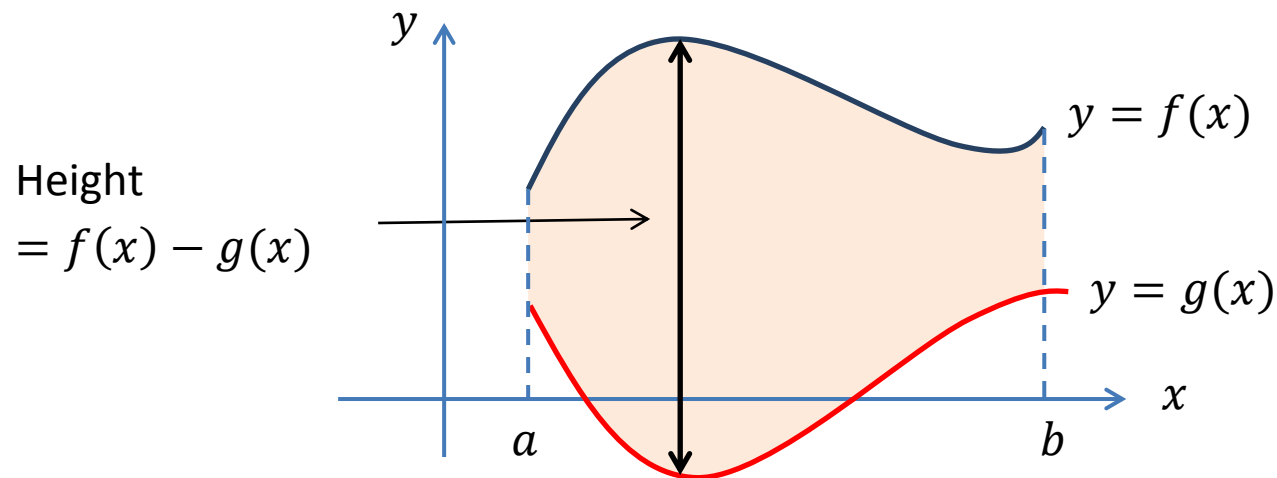


The area of shaded region is given by

$$\begin{aligned} &= \int_0^4 |x(x-1)(x-2)| dx \\ &= \int_0^1 x(x-1)(x-2) dx + \left(- \int_1^2 x(x-1)(x-2) dx \right) + \int_2^4 x(x-1)(x-2) dx \\ &= \left[\frac{x^4}{4} - x^3 + x^2 \right]_0^1 - \left[\frac{x^4}{4} - x^3 + x^2 \right]_1^2 + \left[\frac{x^4}{4} - x^3 + x^2 \right]_2^4 \\ &= 16.5 \text{ (square units).} \end{aligned}$$

Area between curves

Let $y = f(x)$ and $y = g(x)$ be two graphs of functions shown below:



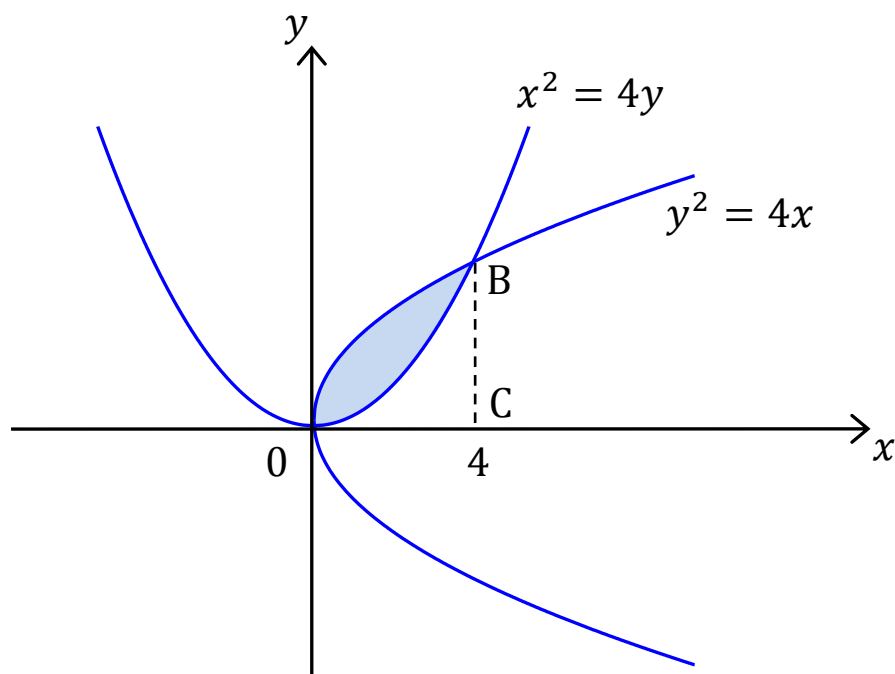
Then the area of the shaded region (area between curves) is given by

$$A = \int_a^b (f(x) - g(x)) dx.$$

- One has to make sure that the upper function is placed in the first term of the formula or otherwise “negative” area will be resulted.

Example 4

Find the area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$.



☺Solution:

We first find the intersection points of these 2 curves.

$$\begin{cases} y^2 = 4x \\ x^2 = 4y \end{cases} \Rightarrow \left(\frac{y^2}{4}\right)^2 = 4y \Rightarrow y^4 - 64y = 0 \Rightarrow y(y^3 - 64) = 0$$
$$\Rightarrow y(y - 4)(y^2 + 4y + 16) = 0 \Rightarrow y = 0 \ (x = 0) \text{ or } y = 4 \ (x = 4).$$

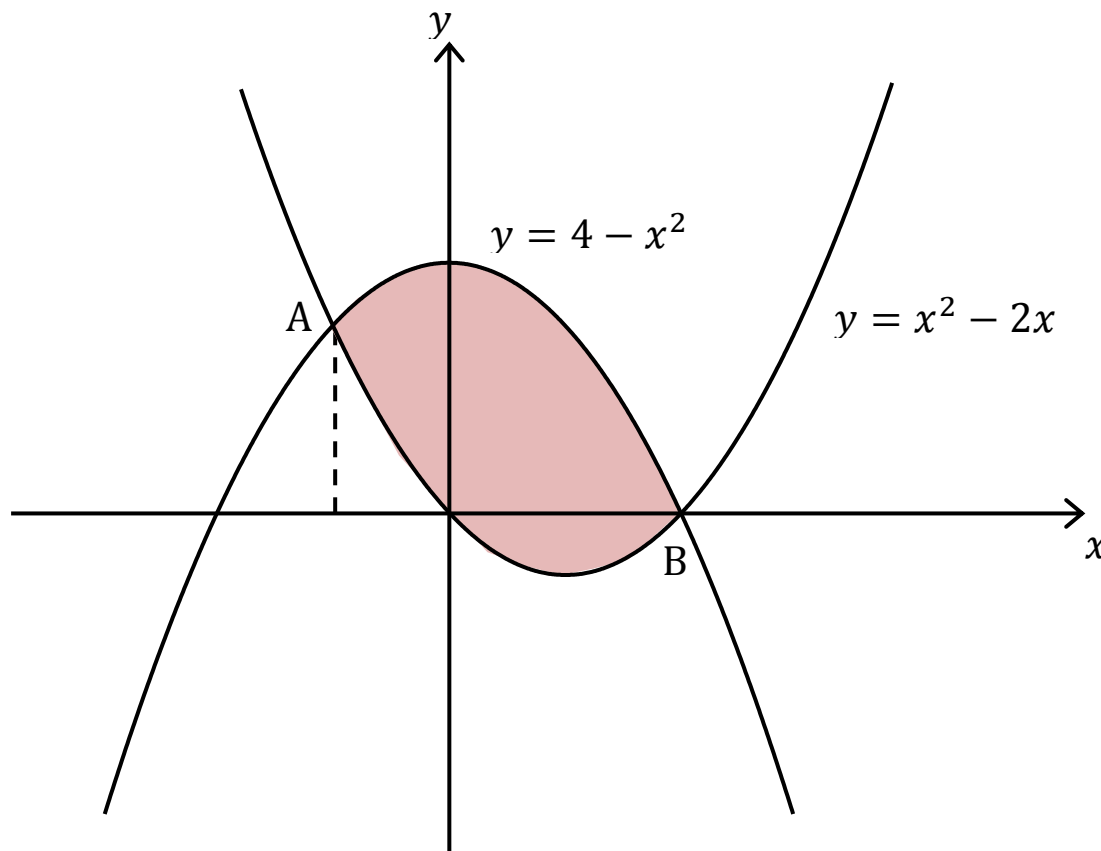
So $O = (0,0)$ and $B = (4,4)$.

The area of shaded region is then given by

$$\begin{aligned} &= \int_0^4 \left(\sqrt{4x} - \frac{x^2}{4}\right) dx = 2 \int_0^4 \sqrt{x} dx - \frac{1}{4} \int_0^4 x^2 dx = 2 \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^4 - \frac{1}{4} \left[\frac{x^3}{3} \right]_0^4 \\ &= \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{ (square units)} \end{aligned}$$

Example 5

Find the area of the region bounded by $y = 4 - x^2$ and $y = x^2 - 2x$.



☺Solution

We first find the coordinates of the intersection between two curves.

$$\begin{cases} y = 4 - x^2 \\ y = x^2 - 2x \end{cases} \Rightarrow x^2 - 2x = 4 - x^2 \Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x - 2)(x + 1) = 0 \Rightarrow x = 2 \ (y = 0) \text{ or } x = -1 \ (y = 3).$$

Then the area of the region is given by

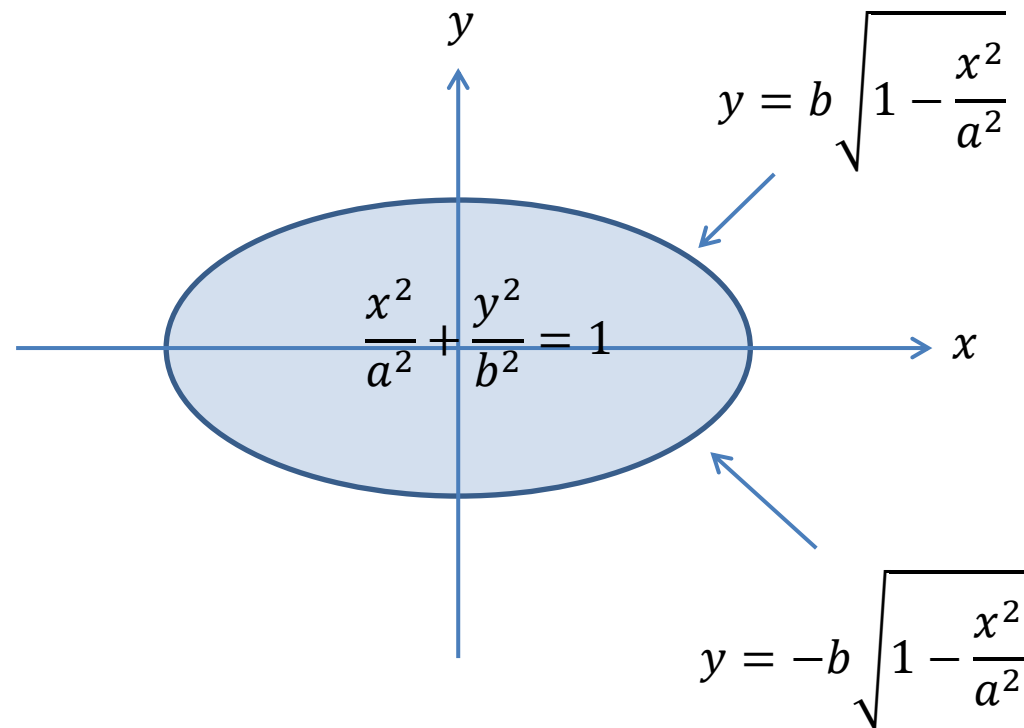
$$\int_{-1}^2 [(4 - x^2) - (x^2 - 2x)] dx = \int_{-1}^2 (-2x^2 + 2x + 4) dx$$

$$= -2 \int_{-1}^2 x^2 dx + 2 \int_{-1}^2 x dx + 4 \int_{-1}^2 dx$$

$$= -2 \left[\frac{x^3}{3} \right]_{-1}^2 + 2 \left[\frac{x^2}{2} \right]_{-1}^2 + 4[x]_{-1}^2 = 9 \text{ (square units)}$$

Example 6

Find the area of the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a, b > 0$.



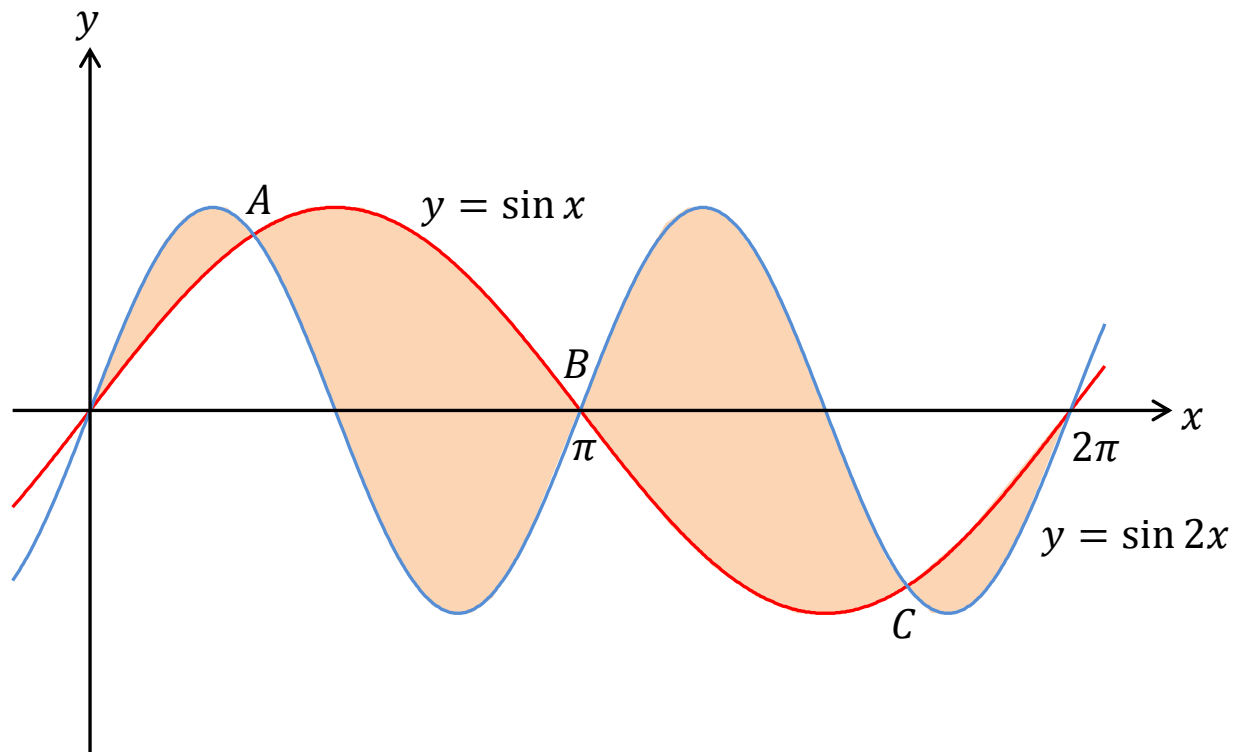
☺Solution:

By symmetry the area of the ellipse is then given by

$$\begin{aligned} 4 \int_0^a b \sqrt{1 - \frac{x^2}{a^2}} dx &= \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} dx \\ x = a \sin \theta, \quad \frac{dx}{d\theta} &= a \cos \theta \\ &\cong \frac{4b}{a} \int_0^{\frac{\pi}{2}} \sqrt{a^2 - (a \sin \theta)^2} (a \cos \theta d\theta) = 4ab \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta \\ &= 4ab \int_0^{\frac{\pi}{2}} \frac{(\cos 2\theta + 1)}{2} d\theta = 2ab \left[\frac{\sin 2\theta}{2} + \theta \right]_0^{\frac{\pi}{2}} = \pi ab. \end{aligned}$$

Example 7

Find the area of the region bounded by the curves $y = \sin 2x$ and $y = \sin x$ for $0 \leq x \leq 2\pi$.



☺Solution

We need to compute the area part by part. First of all, we need to find the intersection points A, B, C between these two curves. We solve the equation

$$\sin 2x = \sin x \Rightarrow \sin 2x - \sin x = 0$$

compound angle
formula

$$\Rightarrow 2 \sin x \cos x - \sin x = 0$$

$$\Rightarrow \sin x (2 \cos x - 1) = 0 \Rightarrow \sin x = 0 \text{ or } \cos x = \frac{1}{2}$$

$$\Rightarrow x = 0, \pi, 2\pi \text{ or } \frac{\pi}{3}, \frac{5\pi}{3}.$$

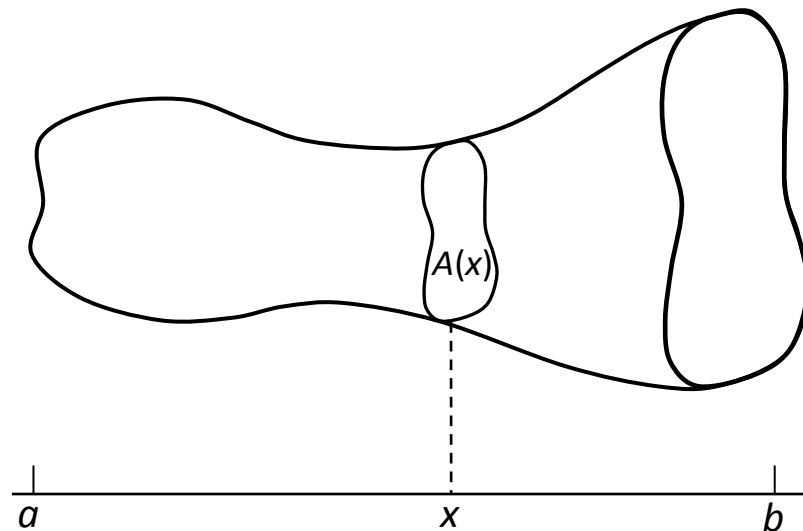
Hence, the coordinates of A, B and C are given by $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right), (\pi, 0), \left(\frac{5\pi}{3}, -\frac{\sqrt{3}}{2}\right)$, respectively.

Then, the area of the entire region is given by

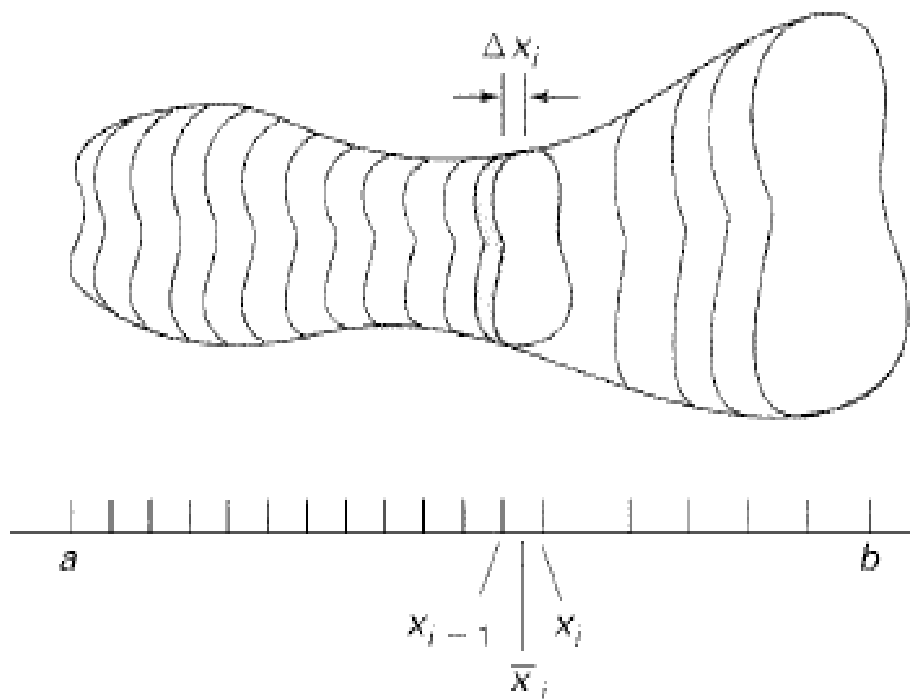
$$\begin{aligned} A &= \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x) dx + \int_{\frac{\pi}{3}}^{\pi} (\sin x - \sin 2x) dx + \int_{\pi}^{\frac{5\pi}{3}} (\sin 2x - \sin x) dx \\ &\quad + \int_{\frac{5\pi}{3}}^{2\pi} (\sin x - \sin 2x) dx \\ &= \left[-\frac{1}{2} \cos 2x + \cos x \right]_0^{\frac{\pi}{3}} + \left[-\cos x + \frac{1}{2} \cos 2x \right]_{\frac{\pi}{3}}^{\pi} + \left[-\frac{1}{2} \cos 2x + \cos x \right]_{\pi}^{\frac{5\pi}{3}} \\ &\quad + \left[-\cos x + \frac{1}{2} \cos 2x \right]_{\frac{5\pi}{3}}^{2\pi} \\ &= \frac{1}{4} + \frac{9}{4} + \frac{9}{4} + \frac{1}{4} = 5. \end{aligned}$$

Calculation of volumes of some objects

- Consider a solid which the cross sections are perpendicular to a given line (say x -axis).
- As an example, suppose we would like to find the volume of the solid shown below



Suppose the cross section area $A(x)$ is known, one can find the volume of the solid using integration. To do this, we first try to cut the solid into n small pieces.



Each small piece “looks like” a cylinder.

The volume of each small piece is approximated by

$$\Delta V_i \approx A(\bar{x}_i)\Delta x_i$$

where $\bar{x}_i$ is some point between x_{i-1} and x_i .

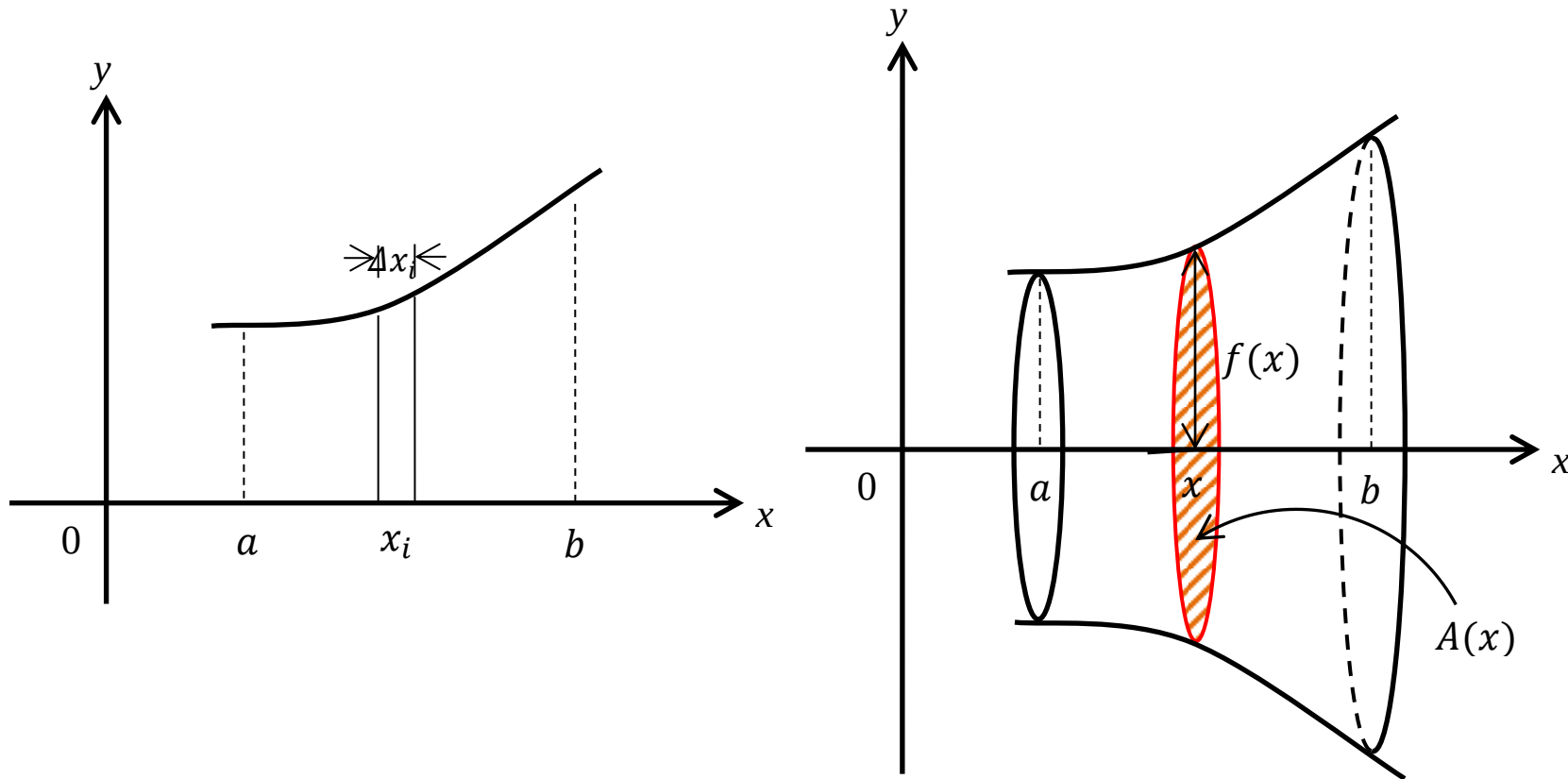
Therefore, the volume of the solid is approximated by

$$V \approx \sum_{i=1}^n A(\bar{x}_i)\Delta x_i.$$

If we cut the solid into more pieces ($n \rightarrow \infty$ and $\Delta x_i \rightarrow 0$), then

$$V = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(\bar{x}_i)\Delta x_i = \int_a^b A(x)dx.$$

Suppose we would like to find the volume of the solid formed by rotating a continuous curve $y = f(x)$ about the x -axis.



- Each cross-section is a circle. Therefore

$$A(x) = \pi[f(x)]^2.$$

- Therefore, the volume of the solid is given by

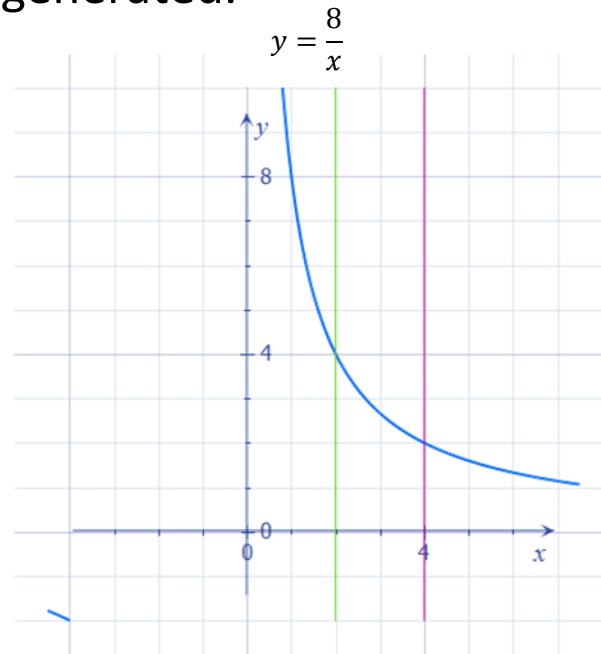
$$V_x = \int_a^b A(x)dx = \int_a^b \pi[f(x)]^2 dx.$$

Remark:

- Different from finding the area bounded by curves, we need not to care whether the curve is above the x -axis or below the x -axis.

Example 8

The portion of the curve $xy = 8$ from $x = 2$ to $x = 4$ is rotated about the x -axis. Find the volume of the solid generated.



☺Solution: The volume

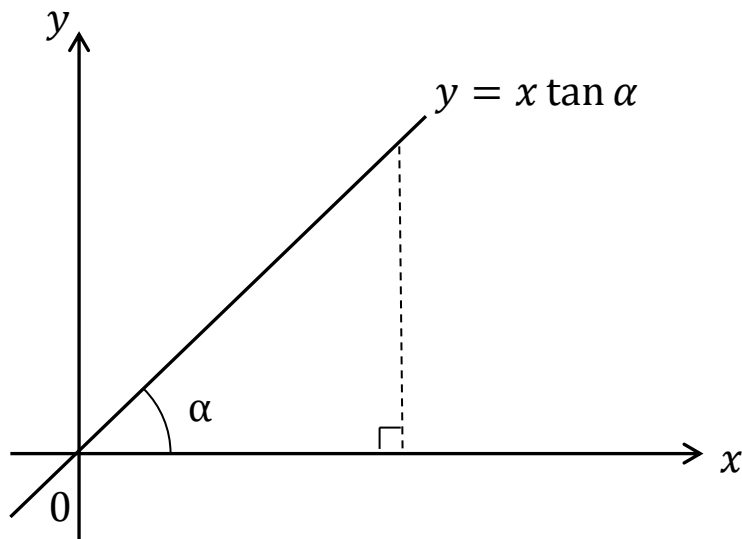
$$V_x = \int_2^4 \pi y^2 dx = \int_2^4 \pi \left(\frac{8}{x}\right)^2 dx = 64\pi \int_2^4 \frac{1}{x^2} dx = 64\pi \left[-\frac{1}{x}\right]_2^4 = 16\pi.$$

Example 9

Find the volume of a right circular cone of height h and semi-vertical angle α .

☺Solution:

The required circular cone can be generated by rotating the straight line $y = x \tan \alpha$ from $x = 0$ to $x = h$ about the x -axis.

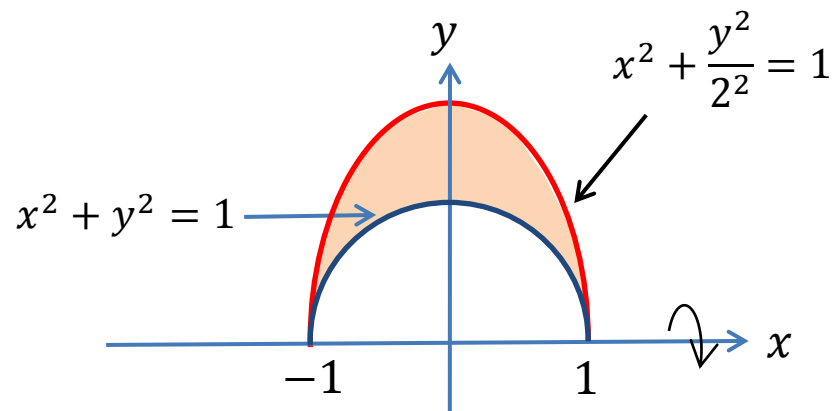


The volume is then given by

$$\begin{aligned}
 V_x &= \int_0^h \pi y^2 dx = \int_0^h \pi (x \tan \alpha)^2 dx = \pi \tan^2 \alpha \int_0^h x^2 dx \\
 &= \pi \tan^2 \alpha \left[\frac{x^3}{3} \right]_0^h = \frac{1}{3} \pi \tan^2 \alpha h^3.
 \end{aligned}$$

Example 10

Find the volume of the solid formed by rotating the region bounded by the upper half of the ellipse $x^2 + \frac{y^2}{2^2} = 1$ and the upper half of the circle $x^2 + y^2 = 1$ about the x -axis.



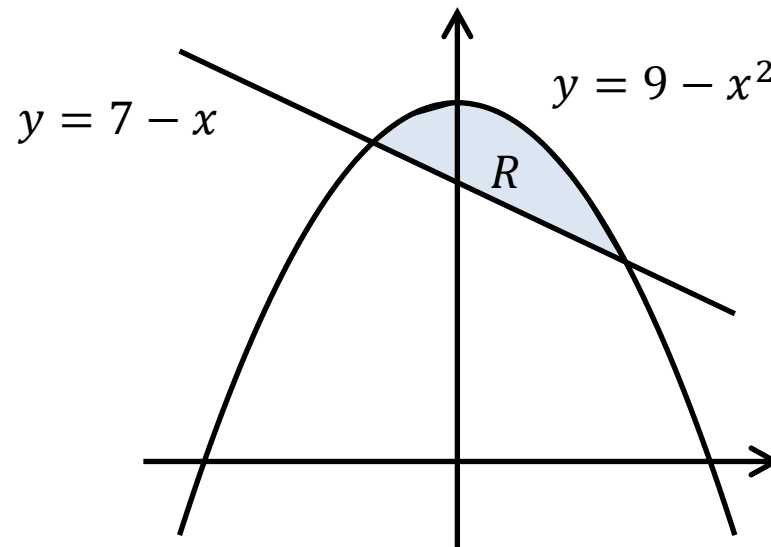
☺Solution:

Using the graph above, the volume of the solid is then given by

$$\begin{aligned} V_x &= \underbrace{\int_{-1}^1 \pi \left(2\sqrt{1-x^2} \right)^2 dx}_{\text{volume of ellipsoid}} - \underbrace{\int_{-1}^1 \pi \left(\sqrt{1-x^2} \right)^2 dx}_{\text{volume of sphere}} \\ &= 3\pi \int_{-1}^1 (1-x^2) dx = 6\pi \int_0^1 (1-x^2) dx \quad \text{as the integrand is an even function} \\ &= 6\pi \left[x - \frac{x^3}{3} \right]_0^1 \\ &= 4\pi. \end{aligned}$$

Example 11

Let R be the region bounded by $y = 9 - x^2$ and $y = 7 - x$. Find the volume of the solid generated by rotating the region R about the x -axis.



☺Solution:

The intersection points between two given curves are found to be

$$\begin{cases} y = 9 - x^2 \\ y = 7 - x \end{cases} \Rightarrow 9 - x^2 = 7 - x \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0$$

$$\Rightarrow x = 2 (y = 5), \quad x = -1 (y = 8)$$

Using similar technique as in Example 10, the required volume is given by

$$V_x = \pi \int_{-1}^2 (9 - x^2)^2 dx - \pi \int_{-1}^2 (7 - x)^2 dx$$

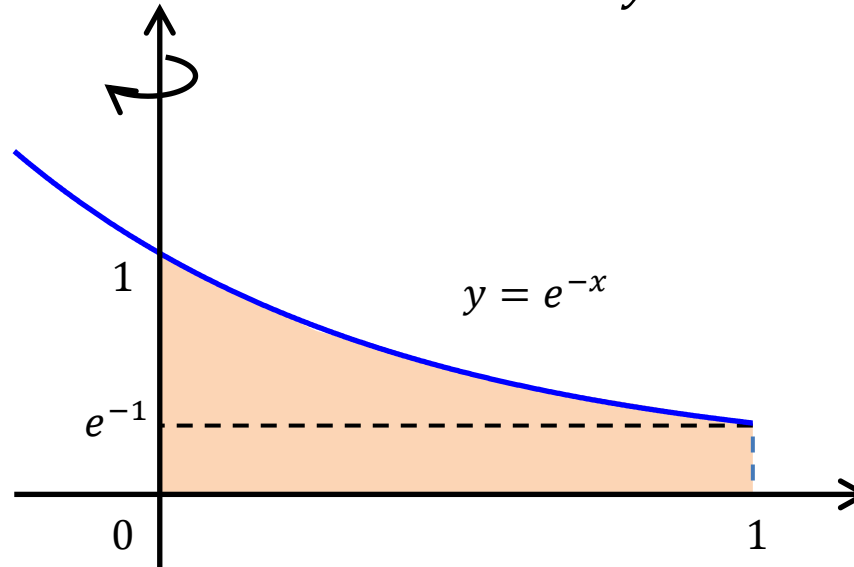
$$= \pi \int_{-1}^2 (x^4 - 19x^2 + 14x + 32) dx$$

$$= \pi \left[\frac{x^5}{5} - \frac{19x^3}{3} + 7x^2 + 32x \right]_{-1}^2 = \frac{333\pi}{5}.$$

The rotation axis is the y -axis: Volume $V_y = \int_c^d \pi[g(y)]^2 dy$ for revolving the region bounded by the curve $x = g(y)$ over $[c, d]$ about the y -axis

Example 12

Find the volume of the solid generated by revolving the area bounded by the curve $y = e^{-x}$, x -axis, y -axis and $x = 1$ about the y -axis.



☺Solution:

Although the rotation axis is not the x -axis, one can use the formula to obtain the volume by *interchanging the role of x and y* .

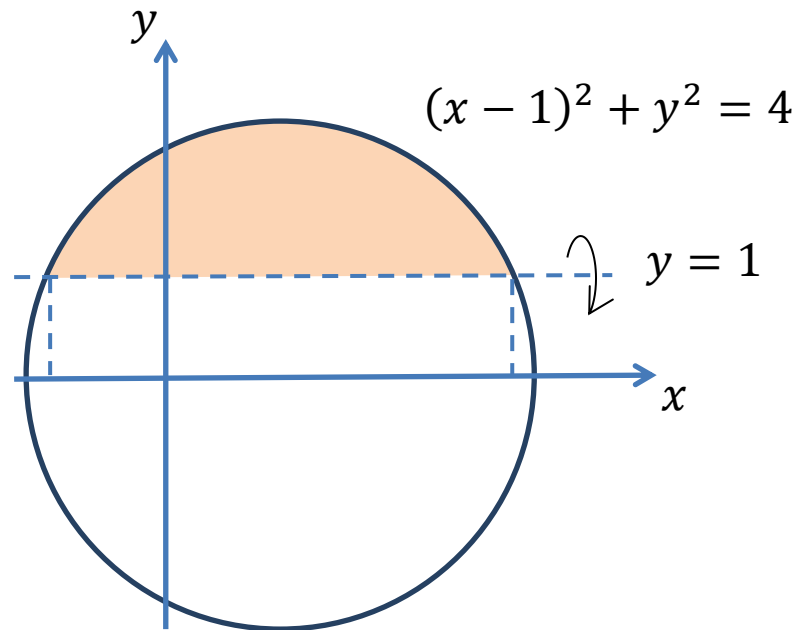
The equation of the graph $y = e^{-x}$ can be rewritten as $x = -\ln y$. The volume is then given by

$$\begin{aligned} V_y &= \int_0^{e^{-1}} \pi(1)^2 dy + \int_{e^{-1}}^1 \pi(-\ln y)^2 dy \\ &= \pi \int_0^{e^{-1}} 1 dy + \pi \int_{e^{-1}}^1 \underbrace{(\ln y)^2}_u \underbrace{dy}_{dv} \\ &= \pi e^{-1} + \pi \left[y(\ln y)^2 \Big|_{e^{-1}}^1 - \int_{e^{-1}}^1 y d(\ln y)^2 \right] \text{ by integration by parts} \\ &= \pi e^{-1} - \pi e^{-1} - \pi \int_{e^{-1}}^1 y \left(\frac{2 \ln y}{y} \right) dy \end{aligned}$$

$$\begin{aligned}
&= -2\pi \int_{e^{-1}}^1 \underbrace{\ln y}_u \underbrace{dy}_{dv} \\
&= -2\pi \left[y \ln y \Big|_{e^{-1}}^1 - \int_{e^{-1}}^1 y d(\ln y) \right] \text{ by integration by parts} \\
&= -2\pi e^{-1} + 2\pi \int_{e^{-1}}^1 1 dy \\
&= -2\pi e^{-1} + 2\pi(1 - e^{-1}) = 2\pi(1 - 2e^{-1}).
\end{aligned}$$

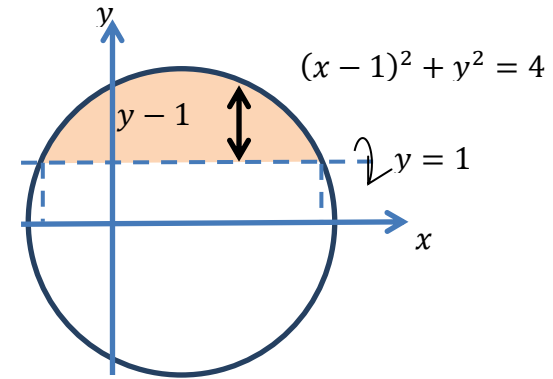
Example 13

Let R be the region bounded by upper half of a circle $(x - 1)^2 + y^2 = 4$ and the line $y = 1$. Find the volume of the solid generated by revolving the region R about the line $y = 1$ (not the x -axis or the y -axis).



☺Solution:

Using the graph on the R.H.S, the volume is given by



$$V = \int_{1-\sqrt{3}}^{1+\sqrt{3}} \pi \left(\sqrt{4 - (x-1)^2} - \color{red}{1} \right)^2 dx$$

$$= \pi \int_{1-\sqrt{3}}^{1+\sqrt{3}} \left(5 - (x-1)^2 - 2\sqrt{4 - (x-1)^2} \right) dx$$

$$\stackrel{(x-1)=2\sin\theta}{\cong} \pi \int_{1-\sqrt{3}}^{1+\sqrt{3}} (4 + 2x - x^2) dx - 2\pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sqrt{4 - 4\sin^2\theta} (2\cos\theta d\theta)$$

$$= \pi \left[4x + x^2 - \frac{x^3}{3} \right]_{1-\sqrt{3}}^{1+\sqrt{3}} - 8\pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos^2\theta d\theta$$

$$= 8\sqrt{3}\pi - 8\pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\cos 2\theta + 1}{2} d\theta = \dots = 6\sqrt{3}\pi - \frac{8\pi^2}{3}.$$

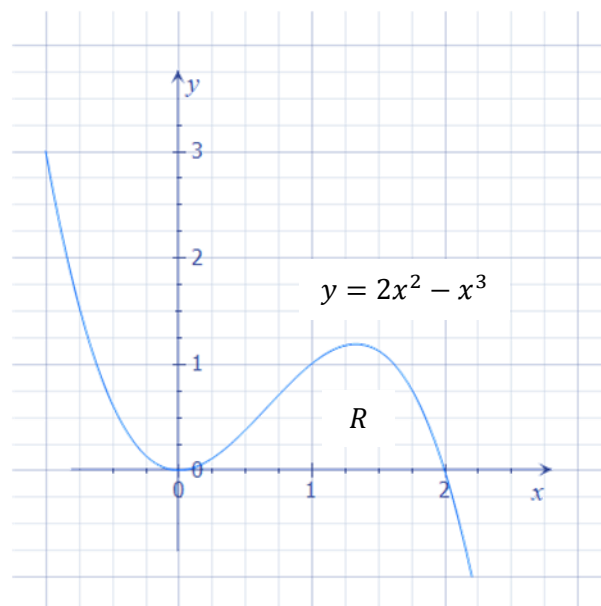
Volume by cylindrical shells: Shell method

The volume of the solid generated by revolving the region bounded by the curve $y = f(x)$ over $[a, b]$ one complete revolution about the y -axis:

$$V_y = \lim_{n \rightarrow \infty} \sum_{i=1}^n \underbrace{2\pi x_i f(x_i)}_{\substack{\text{area of the shell} \\ \text{with radius } r_i = x_i \\ \text{and height } h_i = f(x_i)}} \underbrace{\Delta x_i}_{\substack{\text{shell} \\ \text{width}}} = \int_a^b 2\pi x f(x) dx$$

Example 14

Find the volume of the solid generated by revolving the region R bounded by the curve $y = 2x^2 - x^3$ and the x -axis about the y -axis.



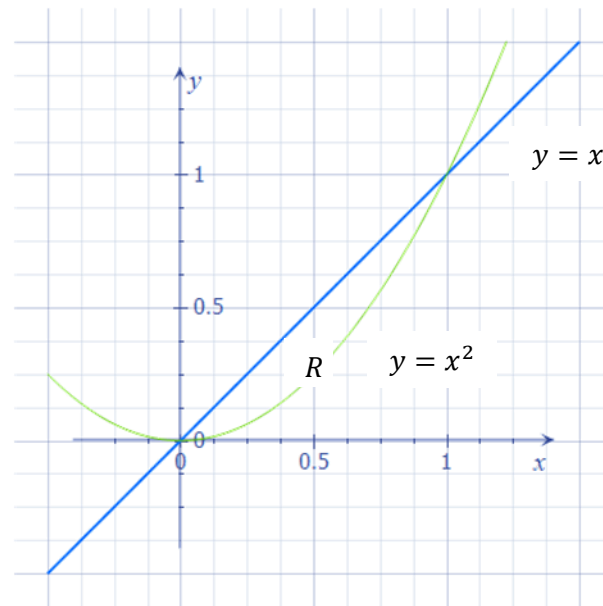
😊Solution:

Lower and upper limits: $0 = 2x^2 - x^3 = x^2(2 - x) \Rightarrow x = 0$ or 2 .

$$V_y = 2\pi \int_0^2 xy dx = 2\pi \int_0^2 x(2x^2 - x^3) dx = 2\pi \int_0^2 (2x^3 - x^4) dx = \frac{16\pi}{5}.$$

Example 15

Find the volume of the solid generated by revolving the region R bounded by the curve $y = x$ and $y = x^2$ about the y -axis.



☺Solution:

Lower and upper limits: $x = x^2 \Rightarrow 0 = x^2 - x = x(x - 1) \Rightarrow x = 0, 1$.

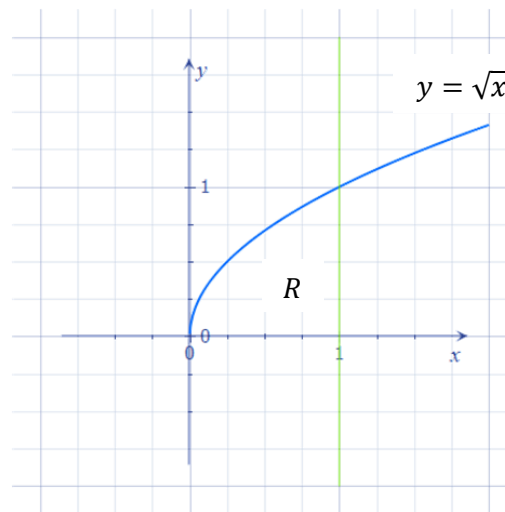
Height: $h = x - x^2$

$$V_y = 2\pi \int_0^1 x h dx = 2\pi \int_0^1 x(x - x^2) dx = 2\pi \int_0^1 (x^2 - x^3) dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = 2\pi \left[\frac{1}{3} - \frac{1}{4} \right] = \frac{2\pi}{12} = \frac{\pi}{6}.$$

Example 16

Find the volume of the solid generated by revolving the region R bounded by the curve $y = \sqrt{x}$, the x -axis and $x = 1$ about the x -axis.



☺Solution:

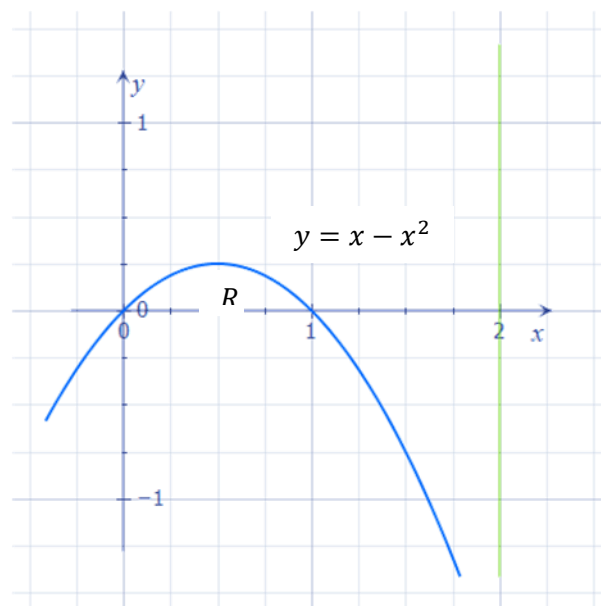
Lower and upper limits on y : $x = 0 \Rightarrow y = \sqrt{0} = 0$; $x = 1 \Rightarrow y = \sqrt{1} = 1$.

Horizontal height: $h = 1 - y^2$

$$\begin{aligned} V_x &= 2\pi \int_0^1 y h dy = 2\pi \int_0^1 y(1 - y^2) dy = 2\pi \int_0^1 (y - y^3) dy \\ &= 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 = 2\pi \left[\frac{1}{2} - \frac{1}{4} \right] = \frac{2\pi}{4} = \frac{\pi}{2}. \end{aligned}$$

Example 17

Find the volume of the solid generated by revolving the region R bounded by the curve $y = x - x^2$ and the x -axis about the line $x = 2$.



☺Solution: Lower and upper limits: $0 = x - x^2 = x(1 - x) \Rightarrow x = 0, 1$.

Radius: $r = 2 - x$; Height: y

$$V = 2\pi \int_0^1 r h dx = 2\pi \int_0^1 (2 - x)(x - x^2) dx = 2\pi \int_0^1 (2x - 3x^2 + x^3) dx$$

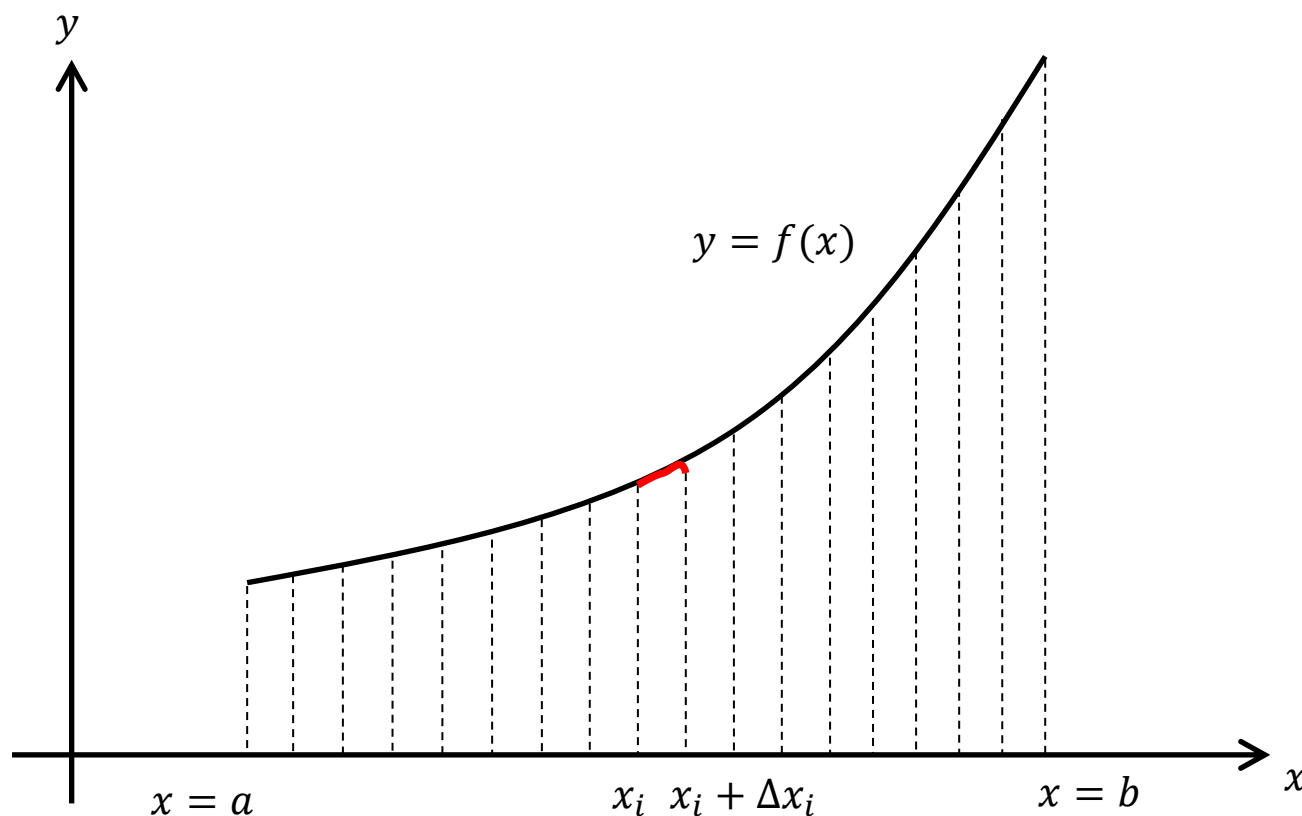
$$= 2\pi \left[x^2 - x^3 + \frac{x^4}{4} \right]_0^1 = 2\pi \left[1 - 1 + \frac{1}{4} \right] = \frac{2\pi}{4} = \frac{\pi}{2}.$$

Remark: If we apply the shell method for Example 12,

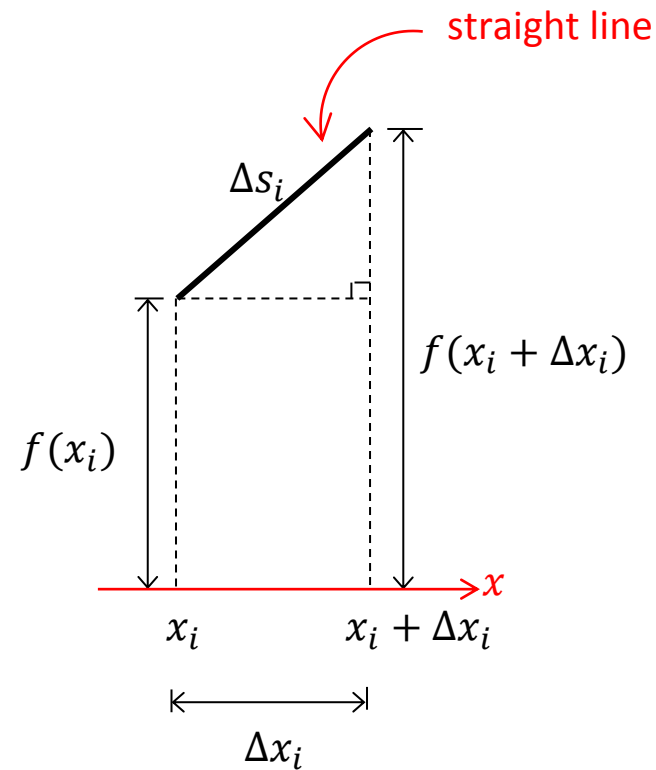
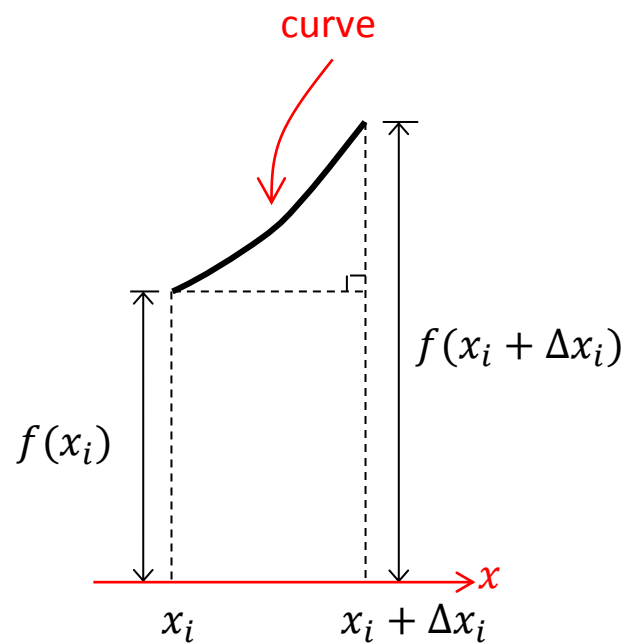
$$\begin{aligned} V_y &= 2\pi \int_0^1 xy dx = 2\pi \int_0^1 xe^{-x} dx = 2\pi \int_0^1 x d(-e^{-x}) \\ &= 2\pi \{[-xe^{-x}]_0^1 + \int_0^1 e^{-x} dx\} \\ &= 2\pi \{[-e^{-1} + 0] + [-e^{-x}]_0^1\} = 2\pi \{-e^{-1} + [-e^{-1} + 1]\} \\ &= 2\pi(1 - e^{-1}). \end{aligned}$$

Arc length of a curve

- We adopt the similar procedure as we did when finding the volume or surface area of the solid. We first divide the curve into n parts:



- Each part of curve is approximated by a straight line joining the endpoints of the curve.



- The length of straight line Δs_i is found to be

$$\begin{aligned}\Delta s_i &= \sqrt{(\Delta x_i)^2 + [f(x_i + \Delta x_i) - f(x_i)]^2} \\ &= \Delta x_i \sqrt{1 + \left[\frac{f(x_i + \Delta x_i) - f(x_i)}{\Delta x_i} \right]^2} \approx \Delta x_i \sqrt{1 + (f'(x_i))^2}.\end{aligned}$$

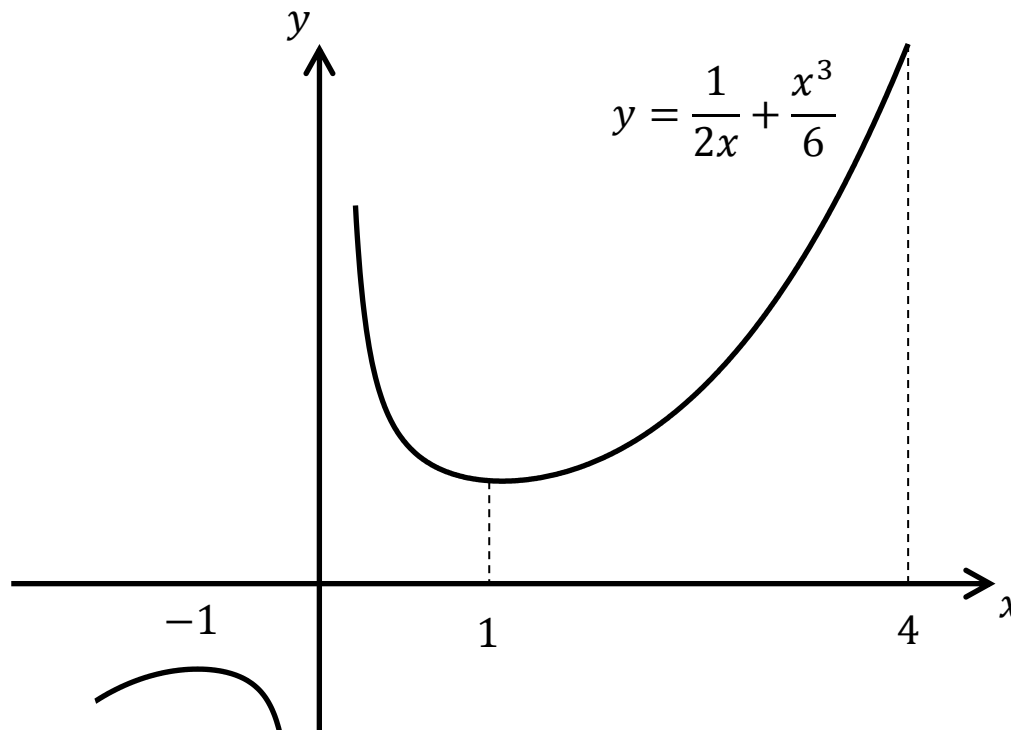
Therefore, the total length of the curve (arc length)

$$s = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{1 + (f'(x_i))^2} \Delta x_i$$

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Example 18

Find the arc length of the curve $6xy = 3 + x^4$ (or $y = \frac{3+x^4}{6x}$) between the points whose abscissa (x -coordinate) are 1 and 4.



☺Solution:

Take $f(x) = \frac{3+x^4}{6x}$, note that

$$f'(x) = \frac{1}{6} \frac{x(4x^3) - (3+x^4)}{x^2} = \frac{1}{6} \frac{3x^4 - 3}{x^2} = \frac{x^4 - 1}{2x^2}.$$

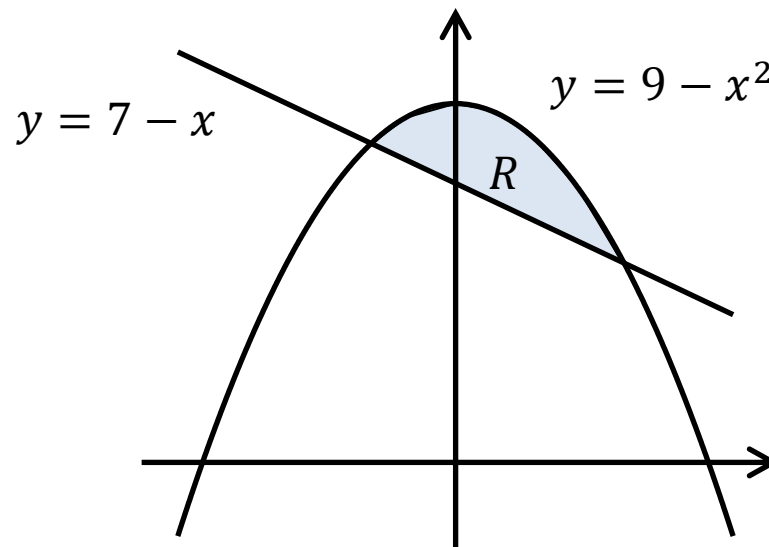
So the arc length is seen to be

$$\begin{aligned} s &= \int_1^4 \sqrt{1 + [f'(x)]^2} dx = \int_1^4 \sqrt{1 + \left(\frac{x^4 - 1}{2x^2}\right)^2} dx \\ &= \int_1^4 \sqrt{\frac{(2x^2)^2 + (x^4 - 1)^2}{(2x^2)^2}} dx \\ &= \int_1^4 \sqrt{\frac{4x^4 + x^8 - 2x^4 + 1}{4x^4}} dx \end{aligned}$$

$$\begin{aligned}
&= \int_1^4 \sqrt{\frac{x^8 + 2x^4 + 1}{4x^4}} dx \\
&= \int_1^4 \sqrt{\frac{(x^4 + 1)^2}{4x^4}} dx \\
&= \int_1^4 \frac{x^4 + 1}{2x^2} dx = \frac{1}{2} \int_1^4 \left(\frac{x^4}{x^2} + \frac{1}{x^2} \right) dx \\
&= \frac{1}{2} \int_1^4 \left(x^2 + \frac{1}{x^2} \right) dx = \frac{1}{2} \left[\frac{x^3}{3} - \frac{1}{x} \right]_1^4 = \frac{35}{4} \text{ (units)}.
\end{aligned}$$

Example 19

Let R be the region bounded by $y = 9 - x^2$ and $y = 7 - x$. Find the arc length of the boundary curve of the region R .



☺Solution

Note that the boundary curves consists of two different parts: (1) the curve $y = 9 - x^2$ from $x = -1$ to $x = 2$ and (2) the straight line $y = 7 - x$.

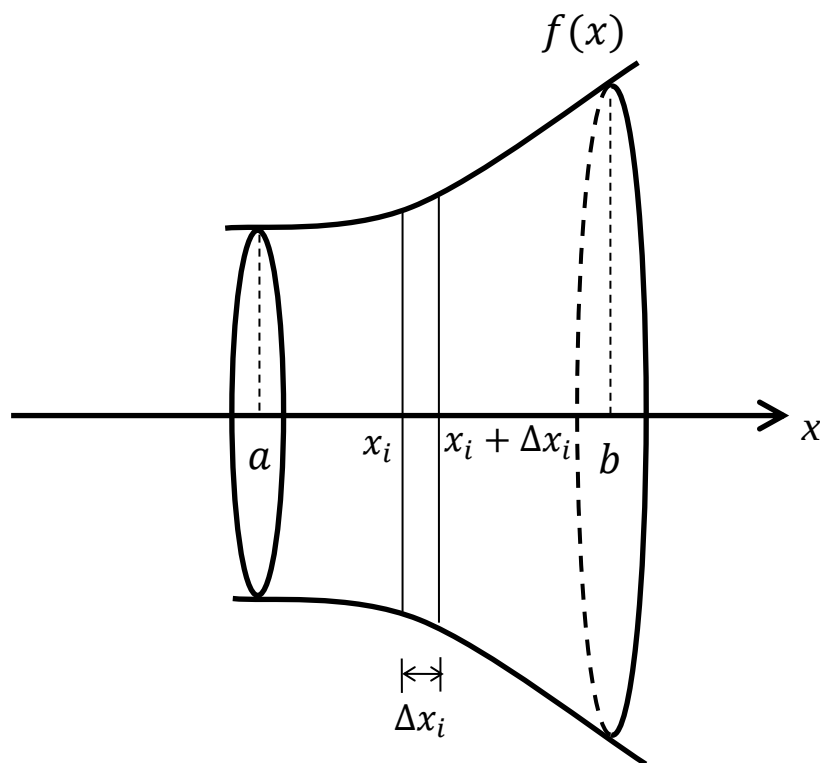
Hence the total arc length is given by

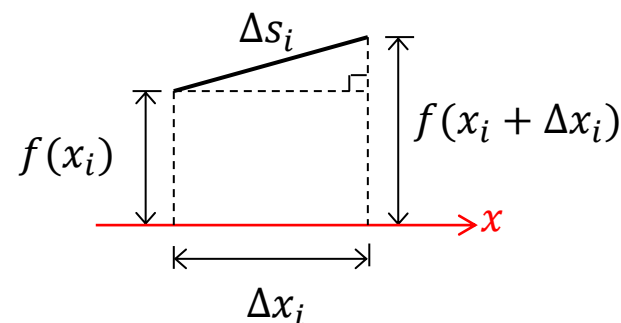
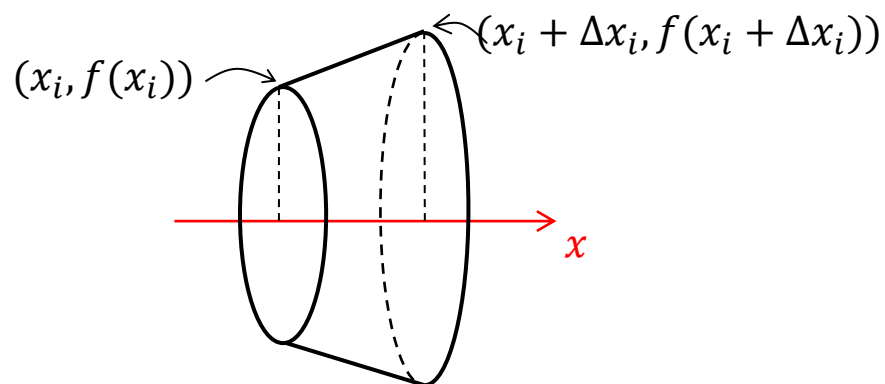
$$\underbrace{\int_{-1}^2 \sqrt{1 + \left(\frac{d}{dx}(9 - x^2)\right)^2} dx}_{\text{arc length of upper curve}} + \underbrace{\int_{-1}^2 \sqrt{1 + \left(\frac{d}{dx}(7 - x)\right)^2} dx}_{\text{arc length of lower curve}} = \int_{-1}^2 \sqrt{1 + 4x^2} dx + \int_{-1}^2 \sqrt{2} dx$$

$$\begin{aligned} & x = \frac{1}{2} \tan \theta \\ \Rightarrow \frac{dx}{d\theta} &= \frac{1}{2} \sec^2 \theta \\ & \cong \int_{\tan^{-1}(-2)}^{\tan^{-1} 4} \sqrt{1 + 4\left(\frac{1}{4} \tan^2 \theta\right)} \left(\frac{1}{2} \sec^2 \theta d\theta\right) + [\sqrt{2}x]_{-1}^2 \\ &= \frac{1}{2} \int_{\tan^{-1}(-2)}^{\tan^{-1} 4} \sec^3 \theta d\theta + 3\sqrt{2} \\ &= \frac{1}{2} \left[\frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \right]_{\tan^{-1}(-2)}^{\tan^{-1} 4} + 3\sqrt{2} = \dots \\ &= \sqrt{17} + \frac{\sqrt{5}}{2} + \frac{1}{4} \ln |\sqrt{17} + 4| - \frac{1}{4} \ln |\sqrt{5} - 2| + 3\sqrt{2} \\ &= \sqrt{17} + \frac{\sqrt{5}}{2} + 3\sqrt{2} + \frac{1}{4} \ln |\sqrt{17} + 4| + \frac{1}{4} \ln |\sqrt{5} + 2|. \end{aligned}$$

Area of surface of revolution

Suppose we would like to find the area of surface generated by rotating a continuous curve $y = f(x)$ about the x -axis.





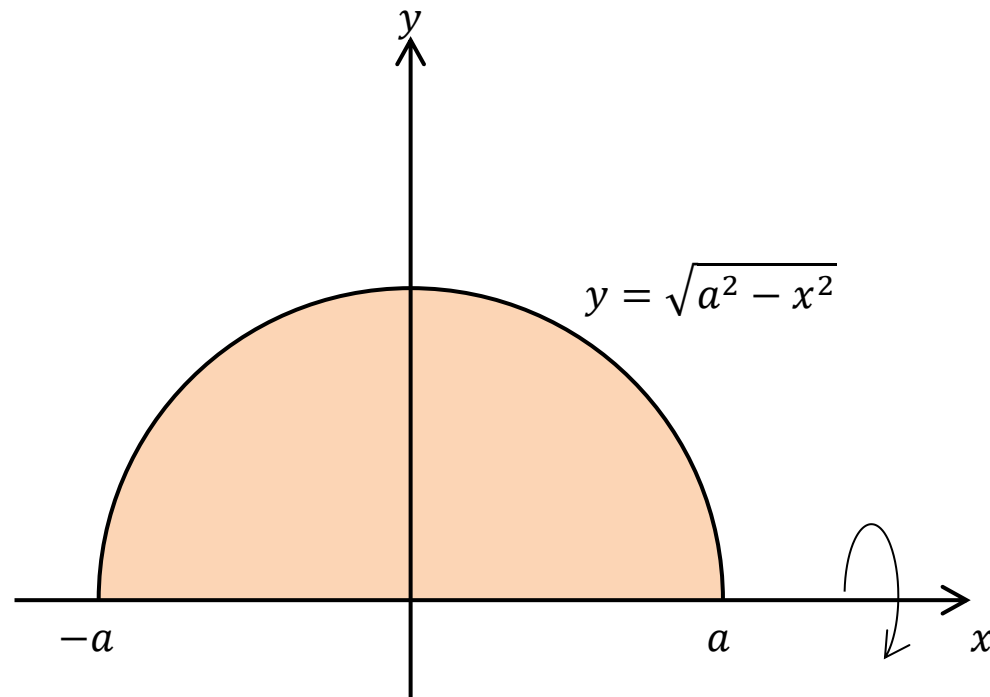
Surface area of the solid generated by revolving the curve $y = f(x)$ over $[a, b]$ about the x -axis:

$$S_x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \underbrace{2\pi f(x_i)}_{\substack{\text{perimeter} \\ \text{with radius} \\ r_i = f(x_i)}} \underbrace{\sqrt{1 + (f'(x_i))^2} \Delta x_i}_{\text{width } \Delta s_i}$$

$$S_x = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx.$$

Example 20

The sphere with radius a ($a > 0$) can be formed by rotating the upper half of the curve $y = \sqrt{a^2 - x^2}$ (or $x^2 + y^2 = a^2$) from $x = -a$ to $x = a$ about the x -axis. Find the surface area of the sphere.



☺Solution:

$$\text{Let } f(x) = \sqrt{a^2 - x^2}.$$

Step 1: Compute $f'(x)$

$$\begin{aligned} f'(x) &= \frac{d}{dx} \sqrt{a^2 - x^2} = \frac{d}{dx} (a^2 - x^2)^{\frac{1}{2}} \\ &= \frac{d\left((a^2 - x^2)^{\frac{1}{2}}\right)}{d(a^2 - x^2)} \frac{d(a^2 - x^2)}{dx} && \text{(Chain Rule)} \\ &= -\frac{x}{\sqrt{a^2 - x^2}}. \end{aligned}$$

Step 2: Calculate the surface area

$$A = 4\pi \int_0^a f(x) \sqrt{1 + [f'(x)]^2} dx \quad \text{by symmetry}$$

$$= 4\pi \int_0^a \sqrt{a^2 - x^2} \sqrt{1 + \left(-\frac{x}{\sqrt{a^2 - x^2}}\right)^2} dx$$

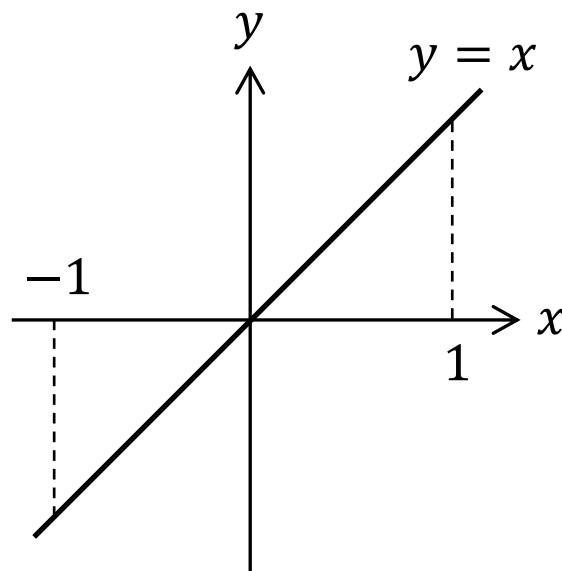
$$= 4\pi \int_0^a \sqrt{a^2 - x^2} \sqrt{1 + \frac{x^2}{a^2 - x^2}} dx$$

$$= 4\pi \int_0^a \sqrt{a^2} dx = 4\pi \int_0^a a dx = 4\pi a \int_0^a dx$$

$$= 4\pi a [x]_0^a = 4\pi a(a) = 4\pi a^2.$$

Example 21

Find the area A of the surface generated by rotating the line $y = x$ from $x = -1$ to $x = 1$ about the x -axis.



☹️ Wrong Solution

Take $f(x) = x$, we have $f'(x) = 1$.

The required surface area

$$\begin{aligned} A &= 2\pi \int_{-1}^1 f(x) \sqrt{1 + [f'(x)]^2} dx \\ &= 2\pi \int_{-1}^1 x \sqrt{1 + 1} dx \\ &= 2\sqrt{2}\pi \int_{-1}^1 x dx \\ &= 2\sqrt{2}\pi \left[\frac{x^2}{2} \right]_{-1}^1 = 0 < \text{--- (Oops)} \end{aligned}$$

Remark about the surface area formula

- The formula works well when the entire curve $y = f(x)$ lies above the x -axis (i.e., $f(x)$ is positive).

- When the curve $y = f(x)$ lies below the x -axis, the integral

$$2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx$$

becomes a negative number.

- If such case happens, split the surface into two main parts (lies above the x -axis/ below the x -axis) and compute the surface area separately.

☺Correct Solution of Example 21

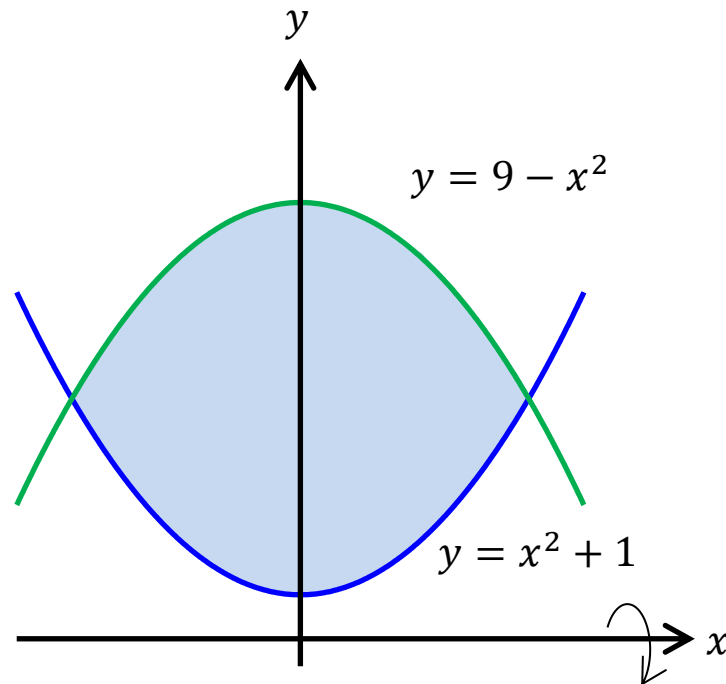
Take $f(x) = x$, we have $f'(x) = 1$.

The required surface area

$$\begin{aligned} A &= 2\pi \left[\int_0^1 f(x) \sqrt{1 + [f'(x)]^2} dx + \left(- \int_{-1}^0 f(x) \sqrt{1 + [f'(x)]^2} dx \right) \right] \\ &= 2\pi \left[\int_0^1 x \sqrt{1 + 1} dx - \int_{-1}^0 x \sqrt{1 + 1} dx \right] \\ &= 2\pi \left\{ \sqrt{2} \left[\frac{x^2}{2} \right]_0^1 - \sqrt{2} \left[\frac{x^2}{2} \right]_{-1}^0 \right\} \\ &= 2\sqrt{2}\pi. \end{aligned}$$

Example 22

Let R be the region bounded by the curves $y = x^2 + 1$ and $y = 9 - x^2$. Find the area A of the surface generated by rotating the region R about the x -axis.



☺Solution

The surface area of the entire object

= surface area of upper half of the solid + surface area of lower half of the solid

$$= 2\pi \int_{-2}^2 (9 - x^2) \sqrt{1 + [-2x]^2} dx + 2\pi \int_{-2}^2 (1 + x^2) \sqrt{1 + [2x]^2} dx$$

$$= 20\pi \int_{-2}^2 \sqrt{1 + 4x^2} dx = 40\pi \int_0^2 \sqrt{1 + 4x^2} dx \quad \text{by symmetry}$$

$$\begin{aligned} & \stackrel{x=\frac{\tan \theta}{2}}{\cong} 40\pi \int_0^{\tan^{-1} 4} \sqrt{1 + \tan^2 \theta} \left(\frac{\sec^2 \theta}{2} d\theta \right) = 20\pi \int_0^{\tan^{-1} 4} \sec^3 \theta d\theta \\ & = 10\pi [\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|]_0^{\tan^{-1} 4} \end{aligned}$$

$$= 40\sqrt{17}\pi + 10\pi \ln |\sqrt{17} + 4|.$$

Parametric equations of curve

- Alternatively, the curve in 2-D plane can be expressed in the following form:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}, \quad a \leq t \leq b$$

where t is another variable called **parameter**.

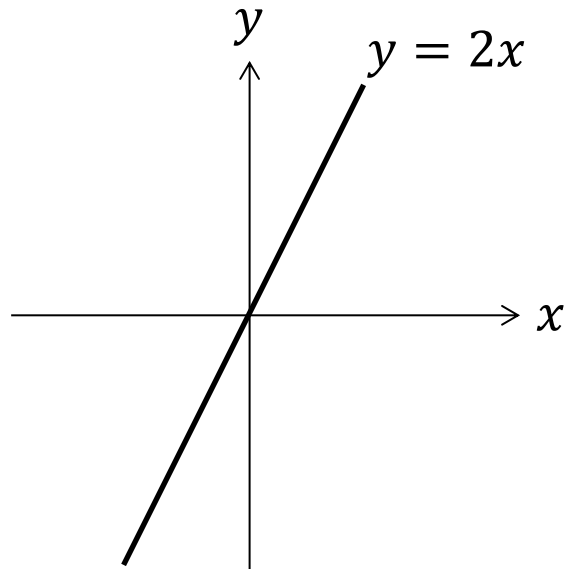
- This notion is commonly used in Physics. If we interpret t as time, then the above pair of equations describes the position of a moving particle at different time t .

Example 23a

Suppose (x, y) is governed by the following parametric equations:

$$\begin{cases} x = t \\ y = 2t \end{cases} \quad \text{where } t \geq 0.$$

- Then the particle is moving along the straight line $y = 2x$ (starting from $(0,0)$).

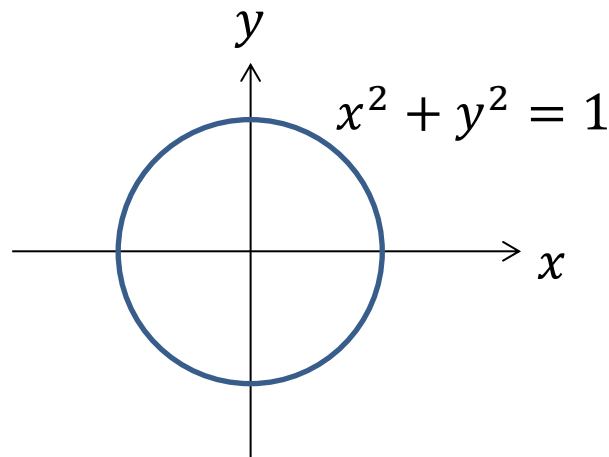


Example 23b

Suppose (x, y) is governed by the following parametric equations:

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad \text{where } 0 \leq t \leq 2\pi.$$

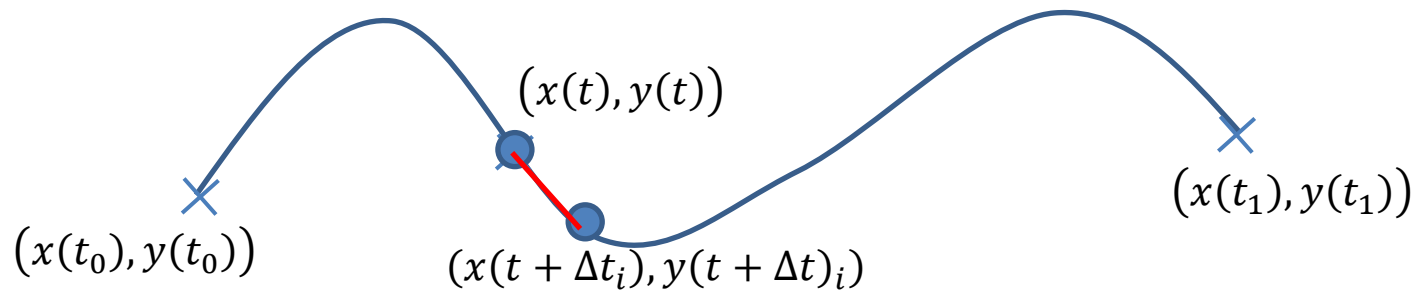
- Then the particle is moving along the circle with equation $x^2 + y^2 = 1$ along anti-clockwise direction once.



How to compute the arc length for this case?

Given a curve with parametrization $x = x(t)$ and $y = y(t)$, we can find the arc length using the following approach:

We first shall divide the whole curve into n small segments.



Since each small segment is close to a straight line, the arc length is given by

$$\begin{aligned}\Delta s_i &= \sqrt{[x(t_i + \Delta t_i) - x(t_i)]^2 + [y(t_i + \Delta t_i) - y(t_i)]^2} \\ &= \sqrt{\left(\frac{x(t_i + \Delta t_i) - x(t_i)}{\Delta t_i}\right)^2 + \left(\frac{y(t_i + \Delta t_i) - y(t_i)}{\Delta t_i}\right)^2} \Delta t_i \\ &\approx \sqrt{[x'(t_i)]^2 + [y'(t_i)]^2} \Delta t_i\end{aligned}$$

Then the entire arc length is given by

$$s = \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta s_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{[x'(t_i)]^2 + [y'(t_i)]^2} \Delta t_i$$

$$s = \int_{t_0}^{t_1} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$$

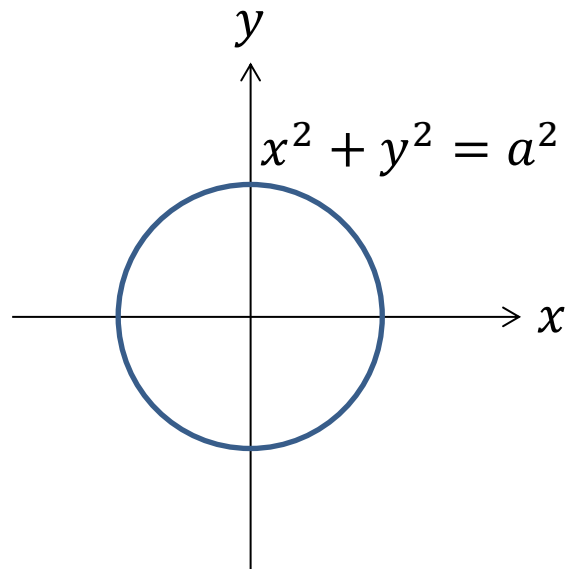
Example 24

Find the arc length of the curve with parametric equations (circle)

$$x(t) = a \cos t, \quad y(t) = a \sin t$$

for $0 \leq t \leq 2\pi$.

(Remark: It traces out a circle with equation $x^2 + y^2 = a^2$)



☺Solution:

Take $x(t) = a \cos t$ and $y(t) = a \sin t$, then

$$x'(t) = -a \sin t, \quad y'(t) = a \cos t.$$

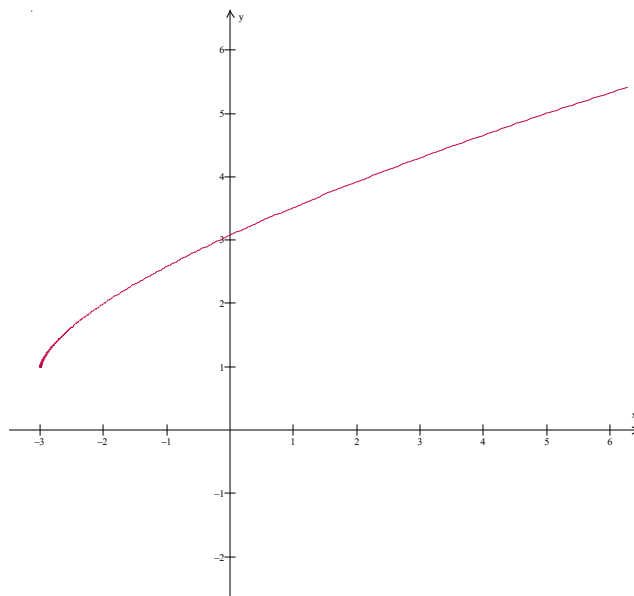
So the arc length is then given by

$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt = \int_0^{2\pi} \sqrt{(-a \sin t)^2 + (a \cos t)^2} dt \\ &= \int_0^{2\pi} \sqrt{a^2(\sin^2 t + \cos^2 t)} dt \\ &= \int_0^{2\pi} a dt = a \int_0^{2\pi} dt = a[t]_0^{2\pi} = 2\pi a. \end{aligned}$$

Example 25

Find the arc length of the curve with parametric equations

$$x = t^3 - 3, \quad y = t^2 + 1 \quad \text{for } 1 \leq t \leq 2.$$



☺Solution:

Let $x(t) = t^3 - 3$ and $y(t) = t^2 + 1$, then

$$x'(t) = 3t^2 \text{ and } y'(t) = 2t.$$

Then the arc length is given by

$$s = \int_1^2 \sqrt{[x'(t)]^2 + [y'(t)]^2} dt = \int_1^2 \sqrt{9t^4 + 4t^2} dt = \int_1^2 t\sqrt{9t^2 + 4} dt$$

Let $u = 9t^2 + 4$, then $\frac{du}{dt} = 18t \Rightarrow dt = \frac{1}{18t} du$.

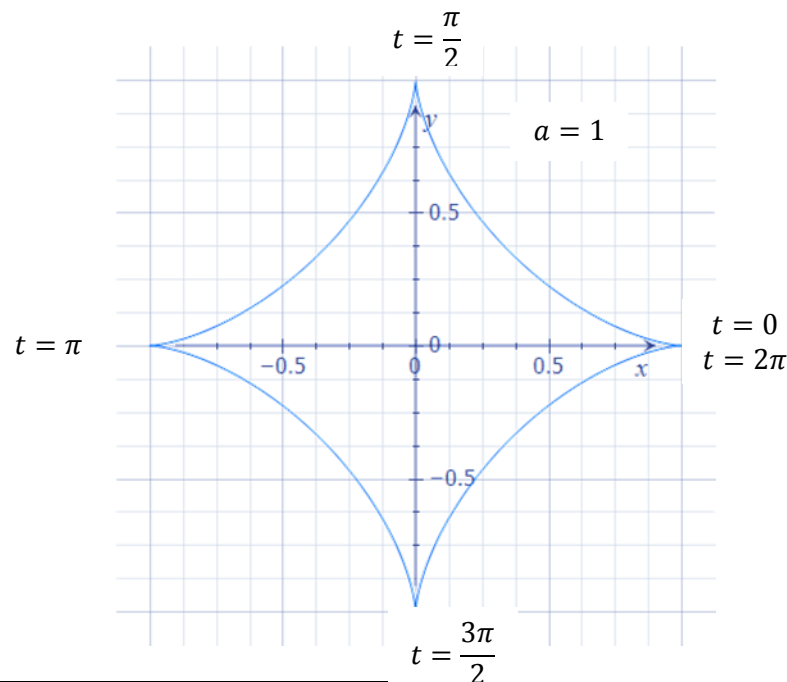
When $t = 2$, $u = 40$; when $t = 1$, $u = 13$.

$$\begin{aligned} \int_1^2 t\sqrt{9t^2 + 4} dt &= \int_{13}^{40} t\sqrt{u} \left(\frac{1}{18t} du \right) = \frac{1}{18} \int_{13}^{40} \sqrt{u} du = \frac{1}{18} \int_{13}^{40} u^{\frac{1}{2}} du = \frac{1}{18} \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_{13}^{40} \\ &= \frac{1}{27} \left(40^{\frac{3}{2}} - 13^{\frac{3}{2}} \right). \end{aligned}$$

Example 26

Find the arc length of the curve with parametric equations (Astroid: $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$):

$$\begin{cases} x(t) = a \cos^3 t \\ y(t) = a \sin^3 t \end{cases}, \quad 0 \leq t \leq 2\pi, \quad a > 0.$$



☺Solution

$$s = \int_0^{2\pi} \sqrt{\left(\frac{d}{dt} a \cos^3 t\right)^2 + \left(\frac{d}{dt} a \sin^3 t\right)^2} dt$$

$$\begin{aligned}
&= \int_0^{2\pi} \sqrt{(-3a \cos^2 t \sin t)^2 + (3a \sin^2 t \cos t)^2} dt \\
&= \int_0^{2\pi} \sqrt{9a^2 \cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t} dt \\
&= \int_0^{2\pi} \sqrt{9a^2 \cos^2 t \sin^2 t \underbrace{(\cos^2 t + \sin^2 t)}_{=1}} dt \\
&= \int_0^{2\pi} |3a \cos t \sin t| dt \quad \overset{\sin t \cos t = \frac{1}{2} \sin 2t}{\cong} \quad \frac{3a}{2} \int_0^{2\pi} |\sin 2t| dt \\
&= \frac{3a}{2} \left[\int_0^{\frac{\pi}{2}} \sin 2t dt + \int_{\frac{\pi}{2}}^{\pi} (-\sin 2t) dt + \int_{\pi}^{\frac{3\pi}{2}} \sin 2t dt + \int_{\frac{3\pi}{2}}^{2\pi} (-\sin 2t) dt \right] \\
&= \frac{3a}{2} \left(\left[-\frac{1}{2} \cos 2t \right]_0^{\frac{\pi}{2}} + \left[\frac{1}{2} \cos 2t \right]_{\frac{\pi}{2}}^{\pi} + \left[-\frac{1}{2} \cos 2t \right]_{\pi}^{\frac{3\pi}{2}} + \left[\frac{1}{2} \cos 2t \right]_{\frac{3\pi}{2}}^{2\pi} \right) \\
&= \frac{3a}{2} (1 + 1 + 1 + 1) = 6a.
\end{aligned}$$

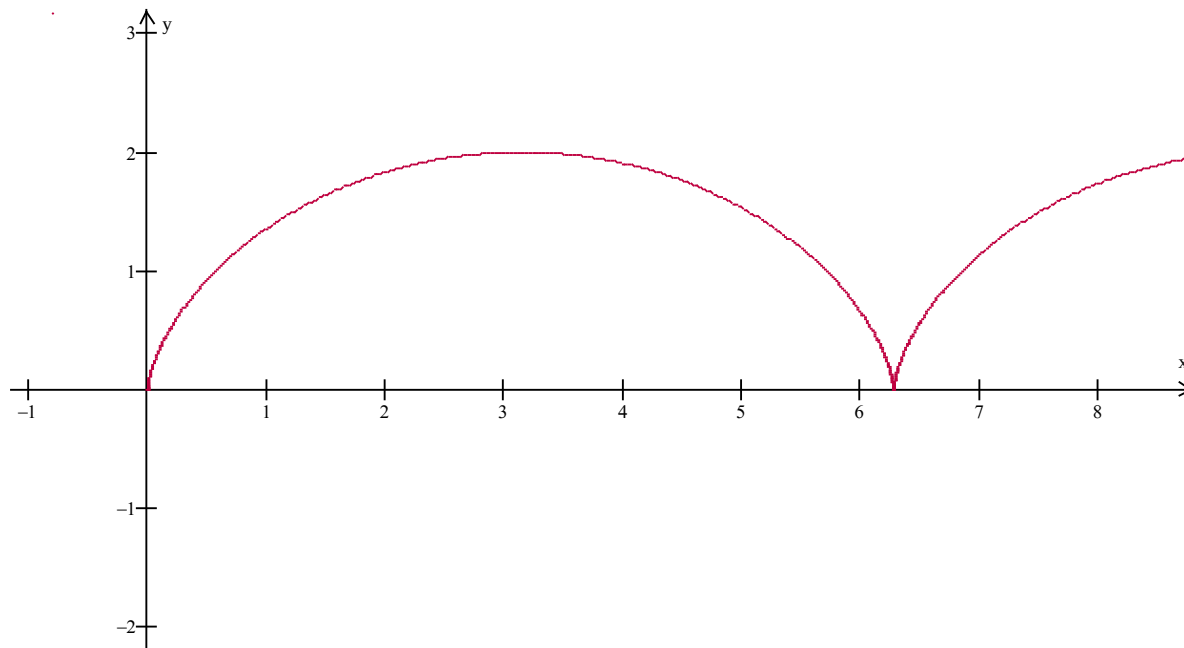
Remark

It is worthwhile to emphasize that the absolute value sign must be included after taking the square root, it is because the function $3 \cos t \sin t = \frac{3}{2} \sin 2t$ can be *negative* for some t within 0 and 2π while the square root of a function is assumed to be *positive* in our context.

Example 27

Find the surface area of the solid generated by revolving the **Cycloid** about the x -axis:

$$x = a(t - \sin t), \quad y = a(1 - \cos t), \quad t: 0 \rightarrow 2\pi, \quad a > 0$$



$$a = 1$$

☺Solution

$$\frac{dx}{dt} = a(1 - \cos t), \quad \frac{dy}{dt} = a \sin t,$$

$$\begin{aligned}\frac{ds}{dt} &= \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{a^2(1 - \cos t)^2 + a^2 \sin^2 t} \\ &= a\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2}a\sqrt{1 - \cos t}\end{aligned}$$

$$\begin{aligned}S_x &= \int_0^{2\pi a} 2\pi y ds = 2\pi \int_0^{2\pi} a(1 - \cos t) \sqrt{2}a\sqrt{1 - \cos t} dt \\ &= 2\sqrt{2}\pi a^2 \int_0^{2\pi} \left(\frac{1 - \cos t}{2 \sin^2 \frac{t}{2}} \right)^{\frac{3}{2}} dt = 8\pi a^2 \int_0^{2\pi} \sin^3 \frac{t}{2} dt \\ &= 16\pi a^2 \int_0^{2\pi} (1 - \cos^2 \frac{t}{2}) d(-\cos \frac{t}{2}) = -16\pi a^2 \left[\cos \frac{t}{2} - \frac{\cos^3 \frac{t}{2}}{3} \right]_0^{2\pi} \\ &= -16\pi a^2 \left[\left(\underbrace{\cos \pi}_{-1} - \underbrace{\cos 0}_1 \right) - \frac{1}{3} \left(\underbrace{\cos^3 \pi}_{-1} - \underbrace{\cos^3 0}_1 \right) \right] \\ &= -16\pi a^2 \left[-2 + \frac{2}{3} \right] = \frac{64}{3} \pi a^2\end{aligned}$$

Summary of formula

	Formula	$x(t_0) = a, \quad x(t_1) = b$ or $y(t_0) = c, \quad y(t_1) = d$
Area bounded by the region R_x : $y_{\text{top}} = f(x)$, $y_{\text{bottom}} = g(x)$, $x = a$ and $x = b$	$\int_a^b [f(x) - g(x)]dx$	$\int_{t_0}^{t_1} [f(x(t)) - g(x(t))]x'(t)dt$
Area bounded by the region R_y : $x_{\text{right}} = \phi(y)$, $x_{\text{left}} = \psi(y)$, $y = c$ and $y = d$	$\int_c^d [\phi(y) - \psi(y)]dy$	$\int_{t_0}^{t_1} [\phi(y(t)) - \psi(y(t))]y'(t)dt$
Volume of solid generated by rotating R_x about the line $y = y_0$ (not cut R_x)	$\pi \int_a^b [f(x) - y_0]^2 - [g(x) - y_0]^2 dx$	$\pi \int_{t_0}^{t_1} [f(x(t)) - y_0]^2 - [g(x(t)) - y_0]^2 x'(t)dt$
Volume of solid generated by rotating R_x about the line $x = x_0$ (not cut R_x)	$2\pi \int_a^b x - x_0 [f(x) - g(x)]dx$	$2\pi \int_{t_0}^{t_1} x(t) - x_0 [f(x(t)) - g(x(t))]x'(t)dt$

Volume of solid generated by rotating R_y about the line $x = x_0$ (not cut R_y)	$\pi \int_c^d [\phi(y) - x_0]^2 - [\psi(y) - x_0]^2 dy$	$\pi \int_{t_0}^{t_1} [\phi(y(t)) - x_0]^2 - [\psi(y(t)) - x_0]^2 y'(t) dt$
Volume of solid generated by rotating R_y about the line $y = y_0$ (not cut R_y)	$2\pi \int_c^d y - y_0 [\phi(y) - \psi(y)] dy$	$2\pi \int_{t_0}^{t_1} y(t) - y_0 [\phi(y(t)) - \psi(y(t))] y'(t) dt$
Arc length of $y = f(x)$ over $[a, b]$	$\int_a^b \sqrt{1 + [f'(x)]^2} dx$	$\int_{t_0}^{t_1} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$
Area of surface generated by rotating $y = f(x)$ over $[a, b]$ about the line $y = y_0$	$2\pi \int_a^b f(x) - y_0 \sqrt{1 + [f'(x)]^2} dx$	$2\pi \int_{t_0}^{t_1} y(t) - y_0 \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$
Area of surface generated by rotating $x = \phi(y)$ over $[c, d]$ about the line $x = x_0$	$2\pi \int_c^d \phi(y) - x_0 \sqrt{1 + [\phi'(y)]^2} dy$	$2\pi \int_{t_0}^{t_1} x(t) - x_0 \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$

Note: x -axis: $y = 0$; y -axis: $x = 0$