
EE3121

Differential Equations for Engineers

Tutorial 4: Existence, Uniqueness, and Qualitative Analysis



Instructor: Prof. Jie Chen

DEPARTMENT OF ELECTRICAL ENGINEERING

Summary

- In a defined rectangle $R : |t - t_0| \leq a, |y - y_0| \leq b$ around the point (t_0, y_0) , the function $f(t, y)$ is sufficiently **smooth** (is continuous and $|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|$ for some K), then a unique solution exists to

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,$$

in a certain interval around t_0 : $|t - t_0| \leq \min(a, b/M)$.

- The existence and uniqueness theorem provides only a **lower bound** on the length of time in which a solution exists, leaving the possibility that solutions may only be **local**.

Summary

- Successive approximations: The sequence $y(t_1), y(t_2), \dots, y(t_n)$

$$\frac{dy_1}{dt} = f(t, y_0), \quad y_1(t_0) = y_0$$

$$\frac{dy_2}{dt} = f(t, y_1), \quad y_2(t_0) = y_0$$

$\vdots$

$$\frac{dy_n}{dt} = f(t, y_{n-1}), \quad y_n(t_0) = y_0$$

- Integral calculus:

$$y_n(t) = y_0 + \int_{t_0}^t f(\tau, y_{n-1}(\tau)) d\tau$$

Summary

- A direction field and direction line can provide useful insight into the behavior of the solution to an ODE when we cannot solve it.
- Direction field: a graph of vectors at discrete points in the (y, t) plane with the tangent of the angle of the vector drawn proportional to the value of $f(t, y)$.
- Direction line is created by drawing arrows at discrete points along the y axis, the direction of the arrow up if $f(y) > 0$, and down if $f(y) < 0$.
- The direction line immediately reveals the ODE's equilibrium points and identifies whether each equilibrium point is stable or unstable.

Summary: Solved Problems

- **SP4.1** Compute the first four iterations $(y_0(t), y_1(t), y_2(t), y_3(t))$ in successive approximation to solution of

$$\frac{dy}{dt} - 4y = 1, \quad y(0) = 2.$$

- *Solution:*

- The equation for successive approximation:

$$\frac{dy_{n+1}}{dt} = 1 + 4y_n, \quad y_{n+1}(0) = 2.$$

- The zeroth approximation:

$$y_0(t) = y(0) = 2.$$

Summary: Solved Problems

■ *Solution:*

■ The first approximation:

$$\frac{dy_1}{dt} = 1 + 4y_0 = 1 + 4 \cdot 2 = 9$$

$$y_1(t) = y_1(0) + \int_{y_0(t)}^{y_1(t)} \frac{dy_1}{d\tau} d\tau = y_1(0) + \int_0^t 9 d\tau = 2 + 9t.$$

■ The second approximation:

$$\frac{dy_2}{dt} = 1 + 4y_1 = 1 + 4 \cdot (2 + 9t) = 9 + 36t$$

$$\begin{aligned} y_2(t) &= y_2(0) + \int_{y_0(t)}^{y_2(t)} \frac{dy_2}{d\tau} d\tau = y_2(0) + \int_0^t (9 + 36\tau) d\tau \\ &= 2 + 9t + 18t^2. \end{aligned}$$

Summary: Solved Problems

■ *Solution:*

■ The third approximation:

$$\frac{dy_3}{dt} = 1 + 4y_2 = 1 + 4 \cdot (2 + 9t + 18t^2) = 9 + 36t + 72t^2$$

$$\begin{aligned} y_3(t) &= y_3(0) + \int_{y_0(t)}^{y_3(t)} \frac{dy_3}{d\tau} d\tau = y_3(0) + \int_0^t (9 + 36\tau + 72\tau^2) d\tau \\ &= 2 + 9t + 18t^2 + 24t^3. \end{aligned}$$

Summary: Solved Problems

- **SP4.2** Compute the first four iterations $(y_0(t), y_1(t), y_2(t), y_3(t))$ in successive approximation to solution of

$$\frac{dy}{dt} + 3y = 2t, \quad y(0) = 4.$$

- *Solution:*

- The equation for successive approximation:

$$\frac{dy_{n+1}}{dt} = 2t - 3y_n, \quad y_{n+1}(0) = 4.$$

- The zeroth approximation:

$$y_0(t) = y(0) = 4.$$

Summary: Solved Problems

■ *Solution:*

■ The first approximation:

$$\frac{dy_1}{dt} = 2t - 3y_0 = 2t - 3 \cdot 4 = -12 + 2t$$

$$y_1(t) = y_1(0) + \int_{y_0(t)}^{y_1(t)} \frac{dy_1}{d\tau} d\tau = y_1(0) + \int_0^t (-12 + 2\tau) d\tau = 4 - 12t + t^2.$$

■ The second approximation:

$$\frac{dy_2}{dt} = 2t - 3y_1 = 2t - 3 \cdot (4 - 12t + t^2) = -12 + 38t - 3t^2$$

$$\begin{aligned} y_2(t) &= y_2(0) + \int_{y_0(t)}^{y_2(t)} \frac{dy_2}{d\tau} d\tau = y_2(0) + \int_0^t (-12 + 38\tau - 3\tau^2) d\tau \\ &= 4 - 12t + 19t^2 - t^3. \end{aligned}$$

Summary: Solved Problems

■ *Solution:*

■ The third approximation:

$$\frac{dy_3}{dt} = 2t - 3y_2 = 2t - 3 \cdot (4 - 12t + 19t^2 - t^3) = -12 + 38t - 57t^2 + 3t^3$$

$$\begin{aligned} y_3(t) &= y_3(0) + \int_{y_0(t)}^{y_3(t)} \frac{dy_3}{d\tau} d\tau = y_3(0) + \int_0^t (-12 + 38\tau - 57\tau^2 + 3\tau^3) d\tau \\ &= 4 - 12t + 19t^2 - 19t^3 + (3/4)t^4. \end{aligned}$$

Summary: Solved Problems

- **SP4.3** Without solving the equation, determine the equilibrium points for

$$\frac{dy}{dt} = 2y^3 - 24y^2 + 88y - 96$$

Use a direction line to determine which equilibrium points are stable and which are unstable.

- *Solution:*

- Simplify the equation:

$$\frac{dy}{dt} = 2(y^3 - 12y^2 + 44y - 48) = 2(y - 2)(y - 4)(y - 6).$$

- Three equilibrium points for this system $y = 2, 4, 6$.

Summary: Solved Problems

■ *Solution:*

■ Three equilibrium points for this system $y = 2, 4, 6$.

■ Then,

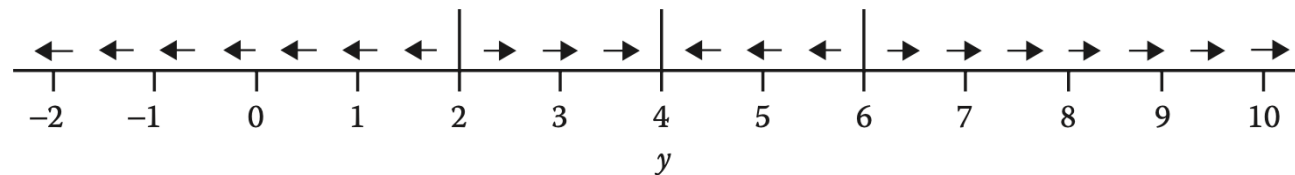
$$\frac{dy}{dt} < 0 \text{ for } y < 2$$

$$\frac{dy}{dt} > 0 \text{ for } 2 < y < 4$$

$$\frac{dy}{dt} < 0 \text{ for } 4 < y < 6$$

$$\frac{dy}{dt} > 0 \text{ for } y > 6$$

■ Create the direction line



■ Unstable equilibrium points: $y = 2$ and $y = 6$; Stable equilibrium point $y = 4$.

Summary: Solved Problems

■ SP4.4 Solve

$$\frac{dy}{dt} = (y - 1)(y - 2), \quad y(0) = y_0$$

and examine the behavior of the solution as it depends on y_0 .

■ *Solution:*

■ Separating variables:

$$\frac{dy}{(y - 1)(y - 2)} = dt.$$

■ Expanding in partial fractions:

$$\frac{1}{(y - 1)(y - 2)} = \frac{c_1}{y - 1} + \frac{c_2}{y - 2} = \frac{(c_1 + c_2)y - 2c_1 - c_2}{(y - 1)(y - 2)}.$$

■ Equating coefficients of similar powers of y in the numerator:

$$c_1 + c_2 = 0, \quad -2c_1 - c_2 = 1 \implies c_1 = -1, \quad c_2 = 1.$$

Summary: Solved Problems

■ *Solution:*

■ Then

$$\frac{dy}{(y-1)(y-2)} = \frac{-dy}{y-1} + \frac{dy}{y-2} = dt.$$

$$\int_{y_0}^y \left(\frac{-dy}{y-1} + \frac{dy}{y-2} \right) = t.$$

$$-\ln(y-1) + \ln(y_0-1) + \ln(y-2) - \ln(y_0-2) = t.$$

$$\ln \left[\left(\frac{y_0-1}{y-1} \right) \left(\frac{y-2}{y_0-2} \right) \right] = t.$$

$$\left(\frac{y_0-1}{y-1} \right) \left(\frac{y-2}{y_0-2} \right) = e^t.$$

Summary: Solved Problems

■ *Solution:*



$$\left(\frac{y_0 - 1}{y - 1}\right) \left(\frac{y - 2}{y_0 - 2}\right) = e^t \Rightarrow (y - 2)(y_0 - 1) = (y - 1)(y_0 - 2)e^t \Rightarrow$$

$$y(t) = \frac{e^t - 2\frac{y_0 - 1}{y_0 - 2}}{e^t - \frac{y_0 - 1}{y_0 - 2}}.$$

■ Let $y_0 = 0$. Then

$$y(t) = \frac{e^t - 1}{e^t - 1/2},$$

from which we see that a solution exists for all $t > 0$ and furthermore that $\lim_{t \rightarrow \infty} y(t) = 1$.

Summary: Solved Problems

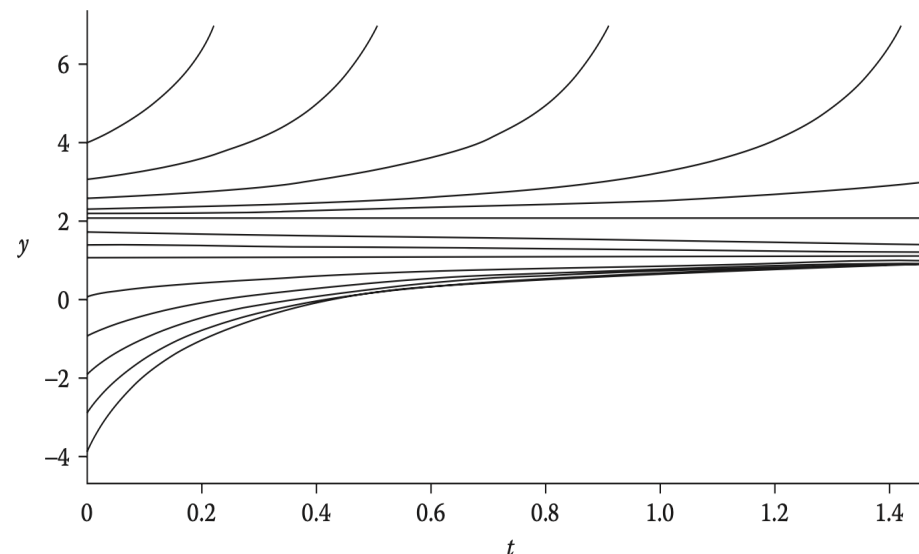
■ *Solution:*

- Let $y_0 = 3$. Then

$$y(t) = \frac{e^t - 4}{e^t - 1},$$

from which we see that a solution exists for all $t > 0$ and furthermore that $\lim_{t \rightarrow \infty} y(t) = 1$. When $t = \ln 2 = 0.693$, the denominator goes to zero. Hence a solution exists only for a finite time.

- The solution for various initial conditions.



Summary: Solved Problems

■ SP4.5 Solve

$$\frac{dy}{dt} = \frac{1}{4-y}, \quad y(0) = 0.$$

and determine the maximum time a solution exists.

■ *Solution:*

■ Separating variables:

$$(4-y)dy = dt$$

■ Integrating:

$$\int_{y(0)}^y (4-y)dy = t.$$

$$4y - \frac{y^2}{2} = t.$$

$$y^2 - 8y + 2t = 0.$$

$$y(t) = 4 \pm \sqrt{16 - 2t}$$

Summary: Solved Problems

- *Solution:*

- Integrating:

$$y(t) = 4 \pm \sqrt{16 - 2t}$$

- Since $y(0) = 0$,

$$y(t) = 4 - \sqrt{16 - 2t}$$

- In Lecture 4 (page 24), via the existence and uniqueness theorem, a solution must exist for at least $t \leq t_E = 4$. Using the true solution above, the ODE has a valid solution for $t \leq 8$.
- The solution is local.

Summary: Solved Problems

- **SP4.6** Without solving the equation, determine the highest estimate of the time, according to the terms and conditions of the existence and uniqueness theorem, in which a solution to the following equation is certain to exist:

$$\frac{dy}{dt} = (1 + y)^2, \quad y(0) = 0.$$

Then solve the equation above and determine the maximum time a solution actually exists.

- *Solution:*

- Denote $f(t, y) = (1 + y)^2$. To apply the theorem, we set a bound on y such that $y \leq b$. Then, $(1 + y)^2$ is bounded by $M = (1 + b)^2$. Also,

$$\frac{\partial f}{\partial y} = 2(1 + y) \leq 2(1 + b).$$

$f(t, y)$ satisfies a Lipschitz condition and the conditions

Summary: Solved Problems

■ *Solution:*

- $f(t, y)$ is independent of t . Set an arbitrarily large value on the bound a .
- A solution is certain to exist for a time $t_E = \min(a, b/M) = \min(a, b/(1+b)^2)$. Set $a > b/(1+b)^2$, then $t_E = b/(1+b)^2$.
- $\max t_E = t_E(b^*) = \frac{1}{4}$ with $b^* = 1$.
- The theorem's highest bound on the time in which a solution is certain to exist is $\frac{1}{4}$.
- Solving the equation actually:
- Separate variables:

$$\frac{dy}{(1+y)^2} = dt.$$

- Integrating:

$$\int_{y(0)}^y \frac{dy}{(1+y)^2} = t.$$
$$1 - \frac{1}{1+y} = t.$$

Summary: Solved Problems

- *Solution:*

- Integrating:

$$y(t) = \frac{t}{1-t}.$$

- The solution is valid for $t < 1$.
 - The actual value is considerably greater than the lower bound $1 > \frac{1}{4}$.