

MA2001 Quiz II

Name: _____

ID: _____

Question (20 marks) Let $\vec{F}(x, y, z) = by \vec{i} + (x + z \cos y) \vec{j} + \sin y \vec{k}$, where b is a real constant.

(i) Suppose $b = 4$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is the path moving from a point $(0,0,0)$ to $(1, \pi/2, \pi/2)$ along a straight line.

(ii) Is $\vec{F}$ conservative with $b = 4$? If not, find a value of b such that $\vec{F}$ is conservative.

(iii) With the value of b in part (ii) such that $\vec{F}$ is conservative, determine a potential function $\varphi(x, y, z)$ such that $\nabla\varphi = \vec{F}$.

Solution: (i) $C: \begin{cases} x=t \\ y=\frac{\pi}{2}t, \quad 0 \leq t \leq 1 \\ z=\frac{\pi}{2}t \end{cases} \Rightarrow \vec{r} = t\vec{i} + (t + \frac{\pi}{2}t \cos \frac{\pi}{2}t)\vec{j} + \sin(\frac{\pi}{2}t)\vec{k}$ and $\frac{d\vec{r}}{dt} = \begin{pmatrix} 1 \\ \frac{\pi}{2} \\ \frac{\pi}{2} \end{pmatrix}$

$$\begin{aligned} \Rightarrow \int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \left[\vec{F} \cdot \frac{d\vec{r}}{dt} \right] dt = \int_0^1 \begin{pmatrix} 4\pi t \\ t + \frac{\pi}{2}t \cos \frac{\pi}{2}t \\ \sin \frac{\pi}{2}t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \frac{\pi}{2} \\ \frac{\pi}{2} \end{pmatrix} dt \\ &= \int_0^1 \left(4\pi t + \frac{\pi}{2}t + \frac{\pi^2}{4}t \cos \frac{\pi}{2}t + \frac{\pi}{2} \sin \frac{\pi}{2}t \right) dt \\ &= \left[\pi t^2 + \frac{\pi}{4}t^2 - \cos \frac{\pi}{2}t \right]_0^1 + \int_0^1 \frac{\pi^2}{4}t \cos \frac{\pi}{2}t dt \\ &= \left(\pi + \frac{\pi}{4} - \cos \frac{\pi}{2} \right) - (0 + 0 - \cos 0) + \int_0^1 \frac{\pi}{2}t d(\sin \frac{\pi}{2}t) \\ &= \frac{5\pi}{4} + 1 + \left[\frac{\pi}{2}t \sin \frac{\pi}{2}t \right]_0^1 - \int_0^1 \frac{\pi}{2} \sin \frac{\pi}{2}t dt \\ &= \frac{5\pi}{4} + 1 + \frac{\pi}{2} + \cos \frac{\pi}{2}t \Big|_0^1 = \frac{5\pi}{4} + 1 + \frac{\pi}{2} + (\cos \frac{\pi}{2} - \cos 0) \\ &= \frac{5\pi}{4} + \frac{\pi}{2} = \boxed{\frac{7\pi}{4}} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \text{curl } \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ by & x+z \cos y & \sin y \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+z \cos y & \sin y \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ by & \sin y \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ by & x+z \cos y \end{vmatrix} \vec{k} \\ &= (\cos y - \cos y) \vec{i} - (0 - 0) \vec{j} + (1 - b) \vec{k} = (1 - b) \vec{k} \end{aligned}$$

Hence, when $b=4$, $\vec{F}$ is not conservative.

$\text{curl } \vec{F} = 0 \Rightarrow b=1$, $\Rightarrow$ when $\boxed{b=1}$, $\vec{F}$ is conservative

(iii) when $b=1$, $\vec{F} = y \vec{i} + (x + z \cos y) \vec{j} + \sin y \vec{k}$

$$\nabla\varphi = \vec{F} \Rightarrow \begin{cases} \varphi_x = y \\ \varphi_y = x + z \cos y \\ \varphi_z = \sin y \end{cases} \Rightarrow \boxed{\varphi(x, y, z) = xy + z \sin y + C}$$

detail:

$$\varphi_x = y \Rightarrow \varphi = xy + g(y, z)$$

$$\Rightarrow \varphi_y = x + \frac{\partial}{\partial y} g(y, z) = x + z \cos y$$

$$\Rightarrow \frac{\partial}{\partial y} g(y, z) = z \cos y$$

$$\Rightarrow g(y, z) = z \sin y + h(z)$$

$$\Rightarrow \varphi = xy + z \sin y + h(z)$$

$$\Rightarrow \varphi_z = \sin y + h'(z) = \sin y$$

$$\Rightarrow h'(z) = 0 \Rightarrow h(z) = C, \text{ Hence } \varphi = xy + z \sin y + C.$$