

Homework #7

Solutions:

Course: EE3121 *Differential Equations for Electrical Engineering*
Date:

Problem 1

Solution. Consider the trial solution $T(x, t) = \Phi(x)\Theta(t)$, which gives

$$\frac{\partial^2 T}{\partial x^2} = \frac{d^2 \Phi(x)}{dx^2} \Theta(t) = \frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\Phi(x)}{\alpha} \frac{d\Theta(t)}{dt}.$$

It follows that

$$\frac{1}{\Phi(x)} \frac{d^2 \Phi(x)}{dx^2} = \frac{1}{\alpha \Theta(t)} \frac{d\Theta(t)}{dt} = -\omega^2,$$

which leads to two ODEs

$$\frac{d^2 \Phi(x)}{dx^2} + \omega^2 \Phi(x) = 0, \quad \frac{d\Theta(t)}{dt} + \alpha \omega^2 \Theta(t) = 0.$$

These ODEs admit the solutions

$$\Phi(x) = c_1 \cos \omega x + c_2 \sin \omega x, \quad \Theta(t) = c_3 \exp(-\alpha \omega^2 t).$$

From the boundary conditions $T(0, t) = 0$, $T(L, t) = 0$, we find that

$$\Phi(0) = c_2 = 0, \quad \Phi(L) = c_1 \cos \omega L + c_2 \sin \omega L = 0.$$

Clearly, c_2 cannot be zero. As such, we have $\sin \omega L = 0$, which in turn means that

$$\omega = \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots$$

We try a sum over all n

$$T(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) \exp\left(-\alpha\left(\frac{n\pi}{L}\right)^2 t\right).$$

The initial condition $T(x, 0) = \frac{4T_0}{L^2}x(L-x)$ requires that

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) = \frac{4T_0}{L^2}x(L-x).$$

It follows that

$$c_n = \frac{2}{L} \int_0^L \frac{4T_0}{L^2} x(L-x) \sin \frac{n\pi x}{L} dx = \frac{8T_0}{L^3} \int_0^L x(L-x) \sin \frac{n\pi x}{L} dx.$$

Then we use the integration by parts to compute the integral

$$\begin{aligned}
 \int_0^L x(L-x) \sin \frac{n\pi x}{L} dx &= -\frac{L}{n\pi} x(L-x) \cos \frac{n\pi x}{L} \Big|_0^L + \frac{L}{n\pi} \int_0^L (-2x+L) \cos \frac{n\pi x}{L} dx \\
 &= \left(\frac{L}{n\pi}\right)^2 \left((-2x+L) \sin \frac{n\pi x}{L} \Big|_0^L + 2 \int_0^L \sin \frac{n\pi x}{L} dx \right) \\
 &= -2 \left(\frac{L}{n\pi}\right)^3 \cos \frac{n\pi x}{L} \Big|_0^L \\
 &= 2 \left(\frac{L}{n\pi}\right)^3 (1 - (-1)^n)
 \end{aligned}$$

Therefore,

$$c_n = \begin{cases} \frac{32T_0}{\pi^3 n^3} & n = 1, 3, 5, \dots, \\ 0 & n = 2, 4, 6, \dots. \end{cases}$$

Substituting c_n into the series expression of $T(x, t)$, we find a solution. Finally, we check the result at the point $x = L/2, t = 0$

$$T\left(\frac{L}{2}, 0\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{2}\right) = \frac{32T_0}{\pi^3} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3},$$

which, along with the identity that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32},$$

verifies the solution at $x = L/2, t = 0$.