

1. If $\vec{r}(t) = r \cos(\omega t) \vec{i} + r \sin(\omega t) \vec{j}$ is the position vector of a point at time t , $\vec{v}(t)$ is the velocity vector of $\vec{r}(t)$ and $\vec{a}(t)$ is the acceleration vector of $\vec{r}(t)$, show that

(a) $\vec{r} \cdot \vec{v} = 0$,

(b) $\vec{r} \times \vec{v} = \text{constant vector}$,

(c) $\vec{a} = \omega^2 \vec{r}$.

Solution:

(a)

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d[r \cos(\omega t) \vec{i} + r \sin(\omega t) \vec{j}]}{dt} = \frac{d(r \cos \omega t)}{dt} \vec{i} + \frac{d(r \sin \omega t)}{dt} \vec{j} = -\omega r \sin(\omega t) \vec{i} + r \omega \cos(\omega t) \vec{j}.$$

$$\vec{r} \cdot \vec{v} = [r \cos(\omega t) \vec{i} + r \sin(\omega t) \vec{j}] \cdot [-\omega r \sin(\omega t) \vec{i} + r \omega \cos(\omega t) \vec{j}] = 0$$

(The velocity is in a tangential direction.)

(b)

$$\vec{r} \times \vec{v} = [r \cos(\omega t) \vec{i} + r \sin(\omega t) \vec{j}] \times [-\omega r \sin(\omega t) \vec{i} + r \omega \cos(\omega t) \vec{j}] = r^2 \omega \cos^2(\omega t) \vec{k} + r^2 \omega \sin^2(\omega t) \vec{k} = r^2 \omega \vec{k}$$

(c)

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d[-\omega r \sin(\omega t) \vec{i} + r \omega \cos(\omega t) \vec{j}]}{dt} = -\omega^2 r \cos(\omega t) \vec{i} - r \omega^2 \sin(\omega t) \vec{j} = -\omega^2 \vec{r}$$

(The acceleration is directed towards the origin.)

2.

(a) Compute the divergence and curl of the vector functions:

(i) $\vec{v} = e^x \cos y \vec{i} + xy^2 \vec{j} + yz^3 \vec{k}$

Solution:

$$\text{div} \vec{v} = \nabla \cdot \vec{v} = \frac{\partial}{\partial x}(e^x \cos y) + \frac{\partial}{\partial y}(xy^2) + \frac{\partial}{\partial z}(yz^3) = e^x \cos y + 2xy + 3yz^2$$

$$\begin{aligned} \text{curl} \vec{v} &= \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \cos y & xy^2 & yz^3 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(yz^3) - \frac{\partial}{\partial z}(xy^2) \right) \vec{i} - \left(\frac{\partial}{\partial x}(yz^3) - \frac{\partial}{\partial z}(e^x \cos y) \right) \vec{j} + \left(\frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(e^x \cos y) \right) \vec{k} \\ &= z^3 \vec{i} + (y^2 + e^x \sin y) \vec{k} \end{aligned}$$

(ii) $\vec{v} = yz \vec{i} + 3zx \vec{j} + z \vec{k}$

Solution:

$$\text{div} \vec{v} = \nabla \cdot \vec{v} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(3zx) + \frac{\partial}{\partial z}(z) = 1$$

$$\begin{aligned} \text{curl } \vec{v} &= \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(3zx) \right) \vec{i} - \left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(yz) \right) \vec{j} + \left(\frac{\partial}{\partial x}(3zx) - \frac{\partial}{\partial y}(yz) \right) \vec{k} = -3x\vec{i} + y\vec{j} + 2z\vec{k} \end{aligned}$$

(b) (i) Find $\text{div}(\text{grad } f)$, for $f(x, y, z) = 1 - x^2 - 4y^2 + 2z^2$

Solution:

$$\text{grad } f = \frac{\partial f_x}{\partial x} \vec{i} + \frac{\partial f_y}{\partial y} \vec{j} + \frac{\partial f_z}{\partial z} \vec{k} = -2x\vec{i} - 8y\vec{j} + 4z\vec{k}$$

$$\text{div}(\text{grad } f) = \frac{\partial}{\partial x}(-2x) + \frac{\partial}{\partial y}(-8y) + \frac{\partial}{\partial z}(4z) = -6$$

(ii) Find $\nabla \times \nabla(\nabla \cdot \vec{v})$, for $\vec{v}(x, y, z) = e^x \vec{i} + e^y \vec{j} + e^z \vec{k}$

Solution:

$$\nabla \cdot \vec{v} = \text{div } \vec{v} = \frac{\partial e^x}{\partial x} + \frac{\partial e^y}{\partial y} + \frac{\partial e^z}{\partial z} = e^x + e^y + e^z$$

$$\nabla(\nabla \cdot \vec{v}) = \text{grad}(\text{div } \vec{v}) = e^x \vec{i} + e^y \vec{j} + e^z \vec{k}$$

$$\nabla \times \nabla(\nabla \cdot \vec{v}) = \nabla \times (e^x \vec{i} + e^y \vec{j} + e^z \vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & e^y & e^z \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) - \vec{k}(0-0) = \vec{0}$$

Remark: $\nabla \times \nabla f = \vec{0}$

(c) Verify the formula $\text{div}(f\vec{v}) = f \text{div } \vec{v} + \vec{v} \cdot \text{grad } f$ for $f = e^{xyz}$ and $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$.

Solution:

$$\begin{aligned} \text{LHS} &= \text{div}(f\vec{v}) = \text{div}(e^{xyz} x\vec{i} + e^{xyz} y\vec{j} + e^{xyz} z\vec{k}) \\ &= \frac{\partial e^{xyz} x}{\partial x} + \frac{\partial e^{xyz} y}{\partial y} + \frac{\partial e^{xyz} z}{\partial z} = xye^{xyz} + e^{xyz} + xye^{xyz} + e^{xyz} + xye^{xyz} + e^{xyz} = 3e^{xyz}(xyz + 1) \end{aligned}$$

$$\begin{aligned} \text{RHS} &= f \text{div } \vec{v} + \vec{v} \cdot \text{grad } f = e^{xyz} \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) + (x\vec{i} + y\vec{j} + z\vec{k}) \cdot \left(\frac{\partial e^{xyz}}{\partial x} \vec{i} + \frac{\partial e^{xyz}}{\partial y} \vec{j} + \frac{\partial e^{xyz}}{\partial z} \vec{k} \right) \\ &= 3e^{xyz} + (xye^{xyz} + xye^{xyz} + xye^{xyz}) = 3e^{xyz}(xyz + 1) = \text{LHS} \end{aligned}$$

(d) Prove that:

(i) $\text{curl}(\vec{v} + \vec{w}) = \text{curl } \vec{v} + \text{curl } \vec{w}$ for any vector fields $\vec{v}$ and $\vec{w}$ on $\mathbf{R}^3$.

Solution:

$$\text{Let } \vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}, \vec{w} = w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{k}$$

$$\text{LHS} = \text{curl}(\vec{v} + \vec{w}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 + w_1 & v_2 + w_2 & v_3 + w_3 \end{vmatrix}$$

$$\begin{aligned}
&= \left(\frac{\partial(v_3 + w_3)}{\partial y} - \frac{\partial(v_2 + w_2)}{\partial z} \right) \vec{i} - \left(\frac{\partial(v_3 + w_3)}{\partial x} - \frac{\partial(v_1 + w_1)}{\partial z} \right) \vec{j} + \left(\frac{\partial(v_2 + w_2)}{\partial x} - \frac{\partial(v_1 + w_1)}{\partial y} \right) \vec{k} \\
&= \left(\frac{\partial v_3}{\partial y} + \frac{\partial w_3}{\partial y} - \frac{\partial v_2}{\partial z} - \frac{\partial w_2}{\partial z} \right) \vec{i} - \left(\frac{\partial v_3}{\partial x} + \frac{\partial w_3}{\partial x} - \frac{\partial v_1}{\partial z} - \frac{\partial w_1}{\partial z} \right) \vec{j} + \left(\frac{\partial v_2}{\partial x} + \frac{\partial w_2}{\partial x} - \frac{\partial v_1}{\partial y} - \frac{\partial w_1}{\partial y} \right) \vec{k} \\
&= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \vec{i} - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z} \right) \vec{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \vec{k} + \left(\frac{\partial w_3}{\partial y} - \frac{\partial w_2}{\partial z} \right) \vec{i} - \left(\frac{\partial w_3}{\partial x} - \frac{\partial w_1}{\partial z} \right) \vec{j} + \left(\frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial y} \right) \vec{k} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_1 & w_2 & w_3 \end{vmatrix} \\
&= \text{curl } \vec{v} + \text{curl } \vec{w} = \text{RHS}
\end{aligned}$$

(ii) $\text{div}(\text{curl } \vec{v}) = 0$

Solution:

Let $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$

$$\begin{aligned}
\text{LHS} &= \text{div}(\text{curl } \vec{v}) = \text{div} \left(\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \right) \\
&= \text{div} \left(\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \vec{i} - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z} \right) \vec{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \vec{k} \right) \\
&= \frac{\partial}{\partial x} \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) - \frac{\partial}{\partial y} \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \\
&= \frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} - \frac{\partial^2 v_3}{\partial y \partial x} + \frac{\partial^2 v_1}{\partial y \partial z} + \frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y} = 0
\end{aligned}$$

3. It is given that $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{p} = a\vec{i} + b\vec{j} + c\vec{k}$ is a constant vector and $\vec{u} = (\vec{p} \cdot \vec{r})\vec{r}$.

(a) Evaluate $\vec{u} = (\vec{p} \cdot \vec{r})\vec{r}$.

(b) Show that

(i) $\nabla \cdot \vec{u} = 4\vec{p} \cdot \vec{r}$,

(ii) $\nabla \times \vec{u} = \vec{p} \times \vec{r}$,

(iii) $\nabla \times (\vec{p} \times \vec{r}) = 2\vec{p}$.

Solution:

(a)

$$\vec{u} = (\vec{p} \cdot \vec{r})\vec{r} = \left[(a\vec{i} + b\vec{j} + c\vec{k}) \cdot (x\vec{i} + y\vec{j} + z\vec{k}) \right] (x\vec{i} + y\vec{j} + z\vec{k}) = (ax + by + cz)(x\vec{i} + y\vec{j} + z\vec{k})$$

(b)(i)

$$\begin{aligned}
\nabla \cdot \vec{u} &= \nabla \cdot (\vec{p} \cdot \vec{r}) \vec{r} = \nabla \cdot [(ax + by + cz)(xi + yj + zk)] \\
&= \frac{d[(ax + by + cz)x]}{dx} + \frac{d[(ax + by + cz)y]}{dy} + \frac{d[(ax + by + cz)z]}{dz} \\
&= (ax + by + cz) + ax + (ax + by + cz) + by + (ax + by + cz) + cz \\
&= 4(ax + by + cz) = 4\vec{p} \cdot \vec{r}
\end{aligned}$$

(b)(ii)

$$\begin{aligned}
\nabla \times \vec{u} &= \nabla \times [(\vec{p} \cdot \vec{r}) \vec{r}] = \nabla \times [(ax + by + cz)x\vec{i} + (ax + by + cz)y\vec{j} + (ax + by + cz)z\vec{k}] \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (ax + by + cz)x & (ax + by + cz)y & (ax + by + cz)z \end{vmatrix} = (bz - cy)\vec{i} + (cx - az)\vec{j} + (ay - bx)\vec{k} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ x & y & z \end{vmatrix} = \vec{p} \times \vec{r}
\end{aligned}$$

(b)(iii)

$$\nabla \times (\vec{p} \times \vec{r}) = \nabla \times [(bz - cy)\vec{i} + (cx - az)\vec{j} + (ay - bx)\vec{k}] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix} = 2a\vec{i} + 2b\vec{j} + 2c\vec{k} = 2\vec{p}$$

4. Let $\vec{F}(x, y, z) = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ be a vector field on $\mathbf{R}^3$, where a, b and c are real constants.

(a) Find the values of a, b and c such that $\vec{F}$ is irrotational.

(b) With the values of a, b and c obtained in (a), determine a potential function φ on $\mathbf{R}^3$ for which $\nabla\varphi = \vec{F}$.

Solution:

(a) Since

$$\begin{aligned}
\nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} \\
&= \left(\frac{\partial(4x + cy + 2z)}{\partial y} - \frac{\partial(bx - 3y - z)}{\partial z} \right) \vec{i} - \left(\frac{\partial(4x + cy + 2z)}{\partial x} - \frac{\partial(x + 2y + az)}{\partial z} \right) \vec{j} \\
&\quad + \left(\frac{\partial(bx - 3y - z)}{\partial x} - \frac{\partial(x + 2y + az)}{\partial y} \right) \vec{k} \\
&= (c + 1)\vec{i} - (4 - a)\vec{j} + (b - 2)\vec{k}
\end{aligned}$$

The vector field $\vec{F}$ is irrotational provided $\nabla \times \vec{F} = \vec{0}$

$$\therefore c = -1, a = 4, b = 2$$

(b) Let ϕ be a scalar field on $\mathbf{R}^3$ such that $\nabla \phi = \vec{F}$, i.e.

$$\frac{\partial \phi}{\partial x} = x + 2y + 4z, \frac{\partial \phi}{\partial y} = 2x - 3y - z, \frac{\partial \phi}{\partial z} = 4x - y + 2z$$

From the first equality,

$$\phi(x, y, z) = \int (x + 2y + 4z) dx = \frac{x^2}{2} + 2xy + 4xz + f(y, z),$$

where f is a function to be determined. Then $\frac{\partial \phi}{\partial y} = 2x + \frac{\partial f(y, z)}{\partial y}$.

Equating this with the equality $\frac{\partial \phi}{\partial y} = 2x - 3y - z$ gives

$$\frac{\partial f(y, z)}{\partial y} = -3y - z$$

It follows that $f(y, z) = \int (-3y - z) dy = -\frac{3y^2}{2} - yz + g(z)$ for a function g .

$$\text{Thus, } \phi(x, y, z) = \frac{x^2}{2} + 2xy + 4xz - \frac{3y^2}{2} - yz + g(z)$$

$$\text{and } \frac{\partial \phi}{\partial z} = 4x - y + g'(z)$$

Equating with $\frac{\partial \phi}{\partial z} = 4x - y + 2z$, we have

$$g'(z) = 2z, \text{ so that } g(z) = \int 2z dz = z^2 \quad (\text{we set the constant of integration to be 0.})$$

$$\therefore \text{ A potential function } \phi \text{ is given by } \phi(x, y, z) = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4xz$$

5. Let $\vec{G}(x, y, z) = 3yz\vec{i} + x^2\vec{j} + x \cos y\vec{k}$ be a vector field on $\mathbf{R}^3$.

(a) Show that $\vec{G}$ is solenoidal.

(b) Find a vector field $\vec{F}(x, y, z) = f_1(x, y, z)\vec{i} + f_2(x, y, z)\vec{j}$ on $\mathbf{R}^3$ such that $\nabla \times \vec{F} = \vec{G}$.

Solution:

(a) The vector field $\vec{G}$ is solenoidal, since

$$\text{div} \vec{G} = \frac{\partial 3yz}{\partial x} + \frac{\partial x^2}{\partial y} + \frac{\partial x \cos y}{\partial z} = 0$$

(b) Let $\vec{F}(x, y, z) = f_1(x, y, z)\vec{i} + f_2(x, y, z)\vec{j}$ be a vector field on $\mathbf{R}^3$. Then

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & 0 \end{vmatrix} = \left(-\frac{\partial f_2}{\partial z} \right) \vec{i} - \left(-\frac{\partial f_1}{\partial z} \right) \vec{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \vec{k}$$

To find f_1 and f_2 such that

$$\frac{\partial f_1}{\partial z} = x^2, \frac{\partial f_2}{\partial z} = -3yz \text{ and } \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = x \cos y$$

From the first two equalities,

$$f_1(x, y, z) = \int x^2 dz = x^2 z + \varphi(x, y)$$

and
$$f_2(x, y, z) = -\int 3yz dz = -\frac{3yz^2}{2} + \psi(x, y)$$

for two functions φ and ψ . Then
$$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = \frac{\partial \psi}{\partial x} - \frac{\partial \varphi}{\partial y}$$

Equating this with the equality $\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = x \cos y$ gives

$$\frac{\partial \psi}{\partial x} - \frac{\partial \varphi}{\partial y} = x \cos y$$

By setting $\varphi = 0$, we have

$$\frac{\partial \psi}{\partial x} = x \cos y, \text{ so that } \psi(x, y) = \int x \cos y dx = \frac{x^2 \cos y}{2} + h(y)$$

By choosing $h(y) = 0$, we have $\psi(x, y) = \frac{x^2 \cos y}{2}$.

Hence, one of the vector field $\vec{F}$ is

$$\vec{F}(x, y, z) = x^2 z \vec{i} + \frac{x^2 \cos y - 3yz^2}{2} \vec{j}$$

Remark: The vector field $\vec{F}$ is not unique.

6. (a) A vector field $\vec{F}$ is said to be solenoidal if $\nabla \cdot \vec{F} = 0$. Let $\vec{F} = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$. Show that $\vec{F}$ is solenoidal.
- (b) As a consequence of $\vec{F}$ being solenoidal, there exists a vector field $\vec{H}$ such that $\vec{F} = \nabla \times \vec{H}$. Find a vector field $\vec{H} = h_1(x, y, z)\vec{i} + h_2(x, y, z)\vec{j} + h_3(x, y, z)\vec{k}$ with $h_2(x, y, z) \equiv 0$ such that $\vec{F} = \nabla \times \vec{H}$.
- (c) Observe that if φ is a scalar field and $\vec{H}, \vec{F}$ are vector fields such that $\vec{F} = \nabla \times \vec{H}$, then we have $\nabla \times (\vec{H} + \nabla \varphi) = \nabla \times \vec{H} + \nabla \times \nabla \varphi = \nabla \times \vec{H} = \vec{F} \dots \dots (I)$. Using (b) and observation (I), find a vector field $\vec{G} = g_1(x, y, z)\vec{i} + g_2(x, y, z)\vec{j} + g_3(x, y, z)\vec{k}$ such that $\vec{F} = \nabla \times \vec{G}$ and $g_2(x, y, z) = 2y$.

Solution:

(a)
$$\nabla \cdot \vec{F} = \frac{\partial (y+z)}{\partial x} + \frac{\partial (x+z)}{\partial y} + \frac{\partial (x+y)}{\partial z} = 0.$$

(b)
$$\nabla \times \vec{H} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ h_1 & h_2 & h_3 \end{pmatrix} = \left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z} \right) \vec{i} + \left(\frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x} \right) \vec{j} + \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y} \right) \vec{k}$$

$$\vec{F} = \nabla \times \vec{H} \text{ and } h_2 \equiv 0 \Rightarrow \vec{F} = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k} = \frac{\partial h_3}{\partial y} \vec{i} + \left(\frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x} \right) \vec{j} - \frac{\partial h_1}{\partial y} \vec{k}.$$

That is,
$$\begin{cases} y+z = \frac{\partial h_3}{\partial y} \\ x+z = \frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x} \\ x+y = -\frac{\partial h_1}{\partial y} \end{cases}.$$

$$x+y = -\frac{\partial h_1}{\partial y} \Rightarrow h_1 = -y - \frac{y^2}{2}, \quad y+z = \frac{\partial h_3}{\partial y} \Rightarrow h_3 = \frac{y^2}{2} + zy + C(x, z).$$

$$x+z = \frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x} = -\frac{\partial C}{\partial x} \Rightarrow C(y, z) = -\frac{x^2}{2} - zx. \text{ Thus, } h_3 = \frac{y^2}{2} + zy - \frac{x^2}{2} - zx = \frac{y^2 - x^2}{2} + z(y - x).$$

$$\text{So, } \vec{H} = \left(-xy - \frac{y^2}{2}\right)\vec{i} + \left[\frac{y^2 - x^2}{2} + z(y - x)\right]\vec{k}.$$

(c)

$$\text{Let } \varphi(x, y, z) = y^2 \Rightarrow \nabla \varphi = 2y\vec{j}.$$

$$\vec{G} = \vec{H} + \nabla \varphi = \left(-xy - \frac{y^2}{2}\right)\vec{i} + 2y\vec{j} + \left[\frac{y^2 - x^2}{2} + z(y - x)\right]\vec{k}.$$

-End-