

MA 1201 Calc & Basic Linear Algebra II

Semester B 2014/15, Guo Luo

Supplementary Notes

Scalar Multiplication of a Vector (page 7 & 17, Chapter 4)

- Given a vector $\vec{a}$ and a scalar number c , the vector $c\vec{a}$ has the magnitude

$$|c\vec{a}| = |c||\vec{a}| = \begin{cases} c|\vec{a}|, & c \geq 0 \\ -c|\vec{a}|, & c < 0 \end{cases}.$$

- In component form, if $\vec{a} = \langle a_1, a_2, a_3 \rangle$, then

$$\begin{aligned} |c\vec{a}| &= |c\langle a_1, a_2, a_3 \rangle| = |\langle ca_1, ca_2, ca_3 \rangle| \\ &= \sqrt{(ca_1)^2 + (ca_2)^2 + (ca_3)^2} = \sqrt{c^2a_1^2 + c^2a_2^2 + c^2a_3^2} \\ &= \sqrt{c^2(a_1^2 + a_2^2 + a_3^2)} = \sqrt{c^2} \sqrt{a_1^2 + a_2^2 + a_3^2} = |c||\vec{a}|, \end{aligned}$$

as claimed.

Addition of Vectors I (page 7 & 16, Chapter 4)

- Geometrically (see Figure 1):
 - (a) two vectors $\vec{a}$ and $\vec{b}$ are added by placing the initial point of the second vector $\vec{b}$ at the terminal point of the first vector $\vec{a}$, and then connecting the initial point of $\vec{a}$ to the terminal point of $\vec{b}$;
 - (b) alternatively, the sum $\vec{a} + \vec{b}$ can be viewed as the diagonal of the parallelogram formed by $\vec{a}$ and $\vec{b}$; this is the **parallelogram law** of vector addition; note that $\vec{a}$, $\vec{b}$, $\vec{a} + \vec{b}$ all share the same initial point.
- In component form, if $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\vec{a} + \vec{b} = \langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle;$$

this is in agreement with the geometric definition of $\vec{a} + \vec{b}$ given in Figure 1(a).

Addition of Vectors II (page 7 & 16, Chapter 4)

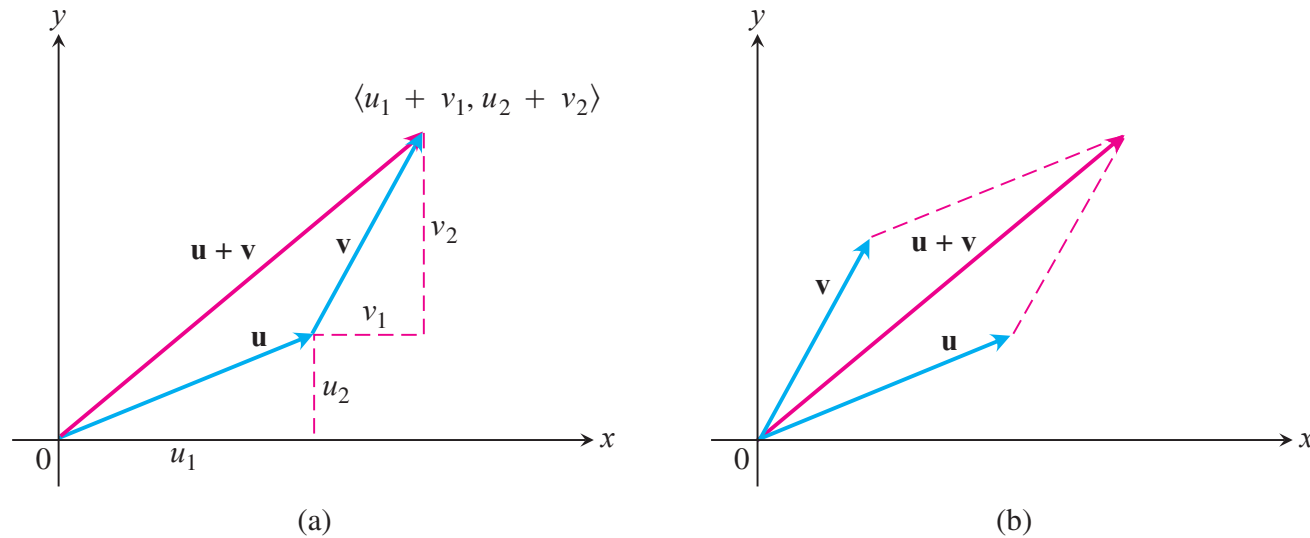


Figure 1: (a) Geometric interpretation of the vector sum, and (b) the parallelogram law of vector addition.

Subtraction of Vectors I (page 8 & 16, Chapter 4)

- Geometrically (see Figure 2):
 - (a) two vectors $\vec{a}$ and $\vec{b}$ are subtracted by placing the initial point of both vectors at the same point, and then connecting the terminal point of the second vector $\vec{b}$ to the terminal point of the first vector $\vec{a}$; this ensures that $\vec{b}$ and $\vec{a} - \vec{b}$ add up to $\vec{a}$;
 - (b) alternatively, the difference $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$ can be viewed as the sum of the two vectors $\vec{a}$ and $-\vec{b}$.
- In component form, if $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\vec{a} - \vec{b} = \langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle.$$

Subtraction of Vectors II (page 8 & 16, Chapter 4)

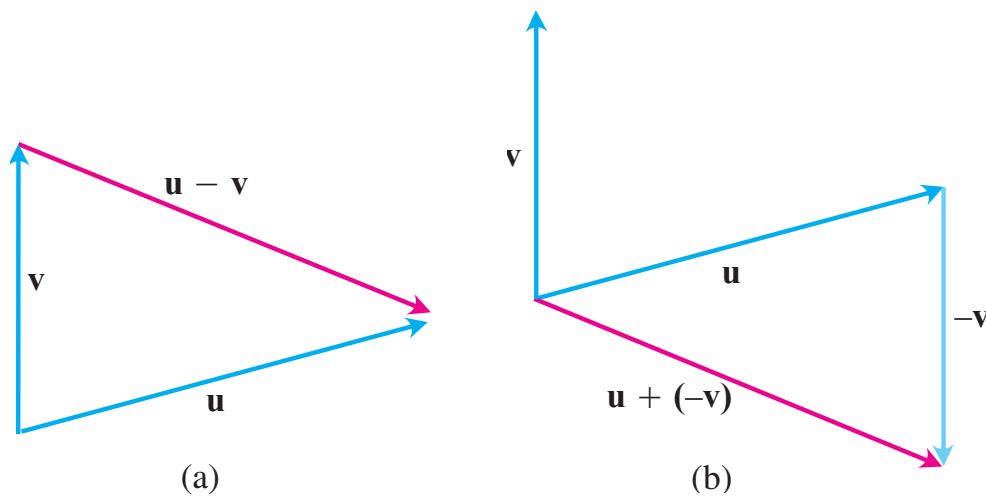


Figure 2: (a) The vector $\vec{u} - \vec{v}$, when added to $\vec{v}$, gives $\vec{u}$, and (b) $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$.

Properties of Scalar Product (page 22, Chapter 4)

- The various properties of scalar product can be readily verified using the definition; for example, if $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$, and $\vec{c} = \langle c_1, c_2, c_3 \rangle$, then

$$\begin{aligned}
 (k\vec{a} + l\vec{b}) \cdot \vec{c} &= (k\langle a_1, a_2, a_3 \rangle + l\langle b_1, b_2, b_3 \rangle) \cdot \langle c_1, c_2, c_3 \rangle \\
 &= (\langle ka_1, ka_2, ka_3 \rangle + \langle lb_1, lb_2, lb_3 \rangle) \cdot \langle c_1, c_2, c_3 \rangle \\
 &= \langle ka_1 + lb_1, ka_2 + lb_2, ka_3 + lb_3 \rangle \cdot \langle c_1, c_2, c_3 \rangle \\
 &= (ka_1 + lb_1)c_1 + (ka_2 + lb_2)c_2 + (ka_3 + lb_3)c_3 \\
 &= (ka_1c_1 + lb_1c_1) + (ka_2c_2 + lb_2c_2) + (ka_3c_3 + lb_3c_3) \\
 &= (ka_1c_1 + ka_2c_2 + ka_3c_3) + (lb_1c_1 + lb_2c_2 + lb_3c_3) \\
 &= k(a_1c_1 + a_2c_2 + a_3c_3) + l(b_1c_1 + b_2c_2 + b_3c_3) \\
 &= k(\vec{a} \cdot \vec{c}) + l(\vec{b} \cdot \vec{c}),
 \end{aligned}$$

which is property 3.

Finding the Vector Projection I (page 32, Chapter 4)

- Given two vectors $\vec{a}$ and $\vec{b}$, the unit direction vector $\hat{b} = \vec{b}/|\vec{b}|$ of $\vec{b}$, and the angle θ between $\vec{a}$ and $\vec{b}$, the **vector projection** of $\vec{a}$ onto $\vec{b}$ is given by (recall that $|\hat{b}| = 1$)

$$\begin{aligned}\text{proj}_{\vec{b}}\vec{a} &= \text{magnitude} \times \text{direction} \\ &= (|\vec{a}| \cos \theta) \hat{b} = (|\vec{a}| |\hat{b}| \cos \theta) \hat{b} \\ &= (\vec{a} \cdot \hat{b}) \hat{b} = \left(\vec{a} \cdot \frac{\vec{b}}{|\vec{b}|} \right) \frac{\vec{b}}{|\vec{b}|} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \right) \vec{b}.\end{aligned}$$

- The **scalar projection** of $\vec{a}$ onto $\vec{b}$, on the other hand, is the scalar component of the vector projection $\text{proj}_{\vec{b}}\vec{a}$, given by

$$\text{comp}_{\vec{b}}\vec{a} = \vec{a} \cdot \hat{b} = \vec{a} \cdot \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}.$$

Finding the Vector Projection II (page 33, Chapter 4)

- **Example.**^a If $\vec{a} = \langle 3, 0, 1 \rangle$ and $\vec{b} = \langle 0, 2, 4 \rangle$, then the vector projection of $\vec{a}$ onto $\vec{b}$ is

$$\begin{aligned}
 \text{proj}_{\vec{b}} \vec{a} &= \left(\vec{a} \cdot \frac{\vec{b}}{|\vec{b}|} \right) \frac{\vec{b}}{|\vec{b}|} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \right) \vec{b} \\
 &= \left(\frac{\langle 3, 0, 1 \rangle \cdot \langle 0, 2, 4 \rangle}{|\langle 0, 2, 4 \rangle|^2} \right) \langle 0, 2, 4 \rangle \\
 &= \left(\frac{(3)(0) + (0)(2) + (1)(4)}{(0)^2 + (2)^2 + (4)^2} \right) \langle 0, 2, 4 \rangle \\
 &= \frac{4}{20} \langle 0, 2, 4 \rangle = \left\langle 0, \frac{2}{5}, \frac{4}{5} \right\rangle.
 \end{aligned}$$

^aSee Example 8 on page 33.

Direction of Vector Product (page 40, Chapter 4)

- The direction of a vector product $\vec{a} \times \vec{b}$ is the unit normal vector that points the way your right thumb points when your fingers curl through the angle θ from $\vec{a}$ to $\vec{b}$; this is the **right-hand rule**.

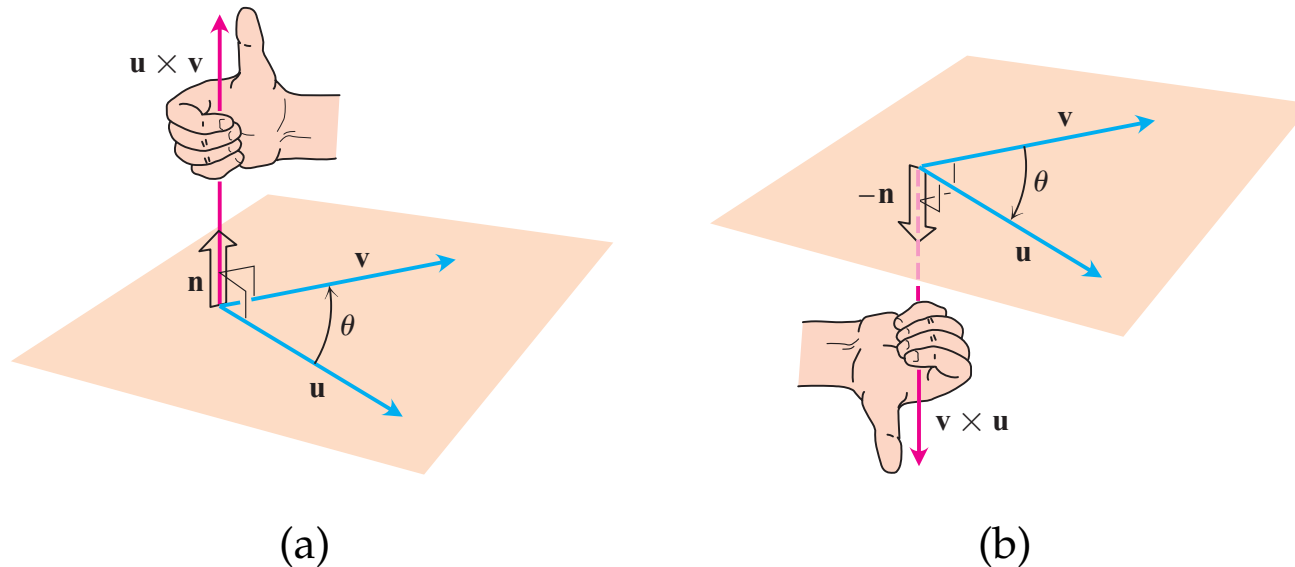


Figure 3: The construction of (a) $\vec{u} \times \vec{v}$ and (b) $\vec{v} \times \vec{u}$.

Vector Products of Fundamental Vectors (page 41, Chapter 4)

- The pairwise vector products of the fundamental vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ can be conveniently remembered using the following permutation diagram.

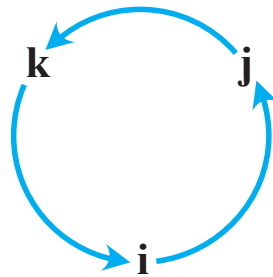


Figure 4: The permutation diagram for recalling pairwise vector products of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

Determinant Formula for Vector Product I (page 42, Chapter 4)

- If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, then

$$\begin{aligned}
 \vec{a} \times \vec{b} &= (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) \\
 &= (a_1b_1)(\vec{i} \times \vec{i}) + (a_1b_2)(\vec{i} \times \vec{j}) + (a_1b_3)(\vec{i} \times \vec{k}) \\
 &\quad + (a_2b_1)(\vec{j} \times \vec{i}) + (a_2b_2)(\vec{j} \times \vec{j}) + (a_2b_3)(\vec{j} \times \vec{k}) \\
 &\quad + (a_3b_1)(\vec{k} \times \vec{i}) + (a_3b_2)(\vec{k} \times \vec{j}) + (a_3b_3)(\vec{k} \times \vec{k}) \\
 &= (a_1b_1)(\vec{0}) + (a_1b_2)(\vec{k}) + (a_1b_3)(-\vec{j}) \\
 &\quad + (a_2b_1)(-\vec{k}) + (a_2b_2)(\vec{0}) + (a_2b_3)(\vec{i}) \\
 &\quad + (a_3b_1)(\vec{j}) + (a_3b_2)(-\vec{i}) + (a_3b_3)(\vec{0}) \\
 &= (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}.
 \end{aligned}$$

Determinant Formula for Vector Product II (page 42, Chapter 4)

- Defining 2×2 and 3×3 determinants by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix},$$

the vector product $\vec{a} \times \vec{b}$ can be represented as a determinant:

$$\begin{aligned} \vec{a} \times \vec{b} &= (a_2b_3 - a_3b_2)\vec{i} - (a_1b_3 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k} \\ &= \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \end{aligned}$$

Determinant Formula for Vector Product III (page 42, Chapter 4)

- **Example.**^a If $\vec{a} = \langle 2, 1, 0 \rangle$ and $\vec{b} = \langle 0, -1, 2 \rangle$, then

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 0 \\ 0 & -1 & 2 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} \\ &= [(1)(2) - (0)(-1)]\vec{i} - [(2)(2) - (0)(0)]\vec{j} + [(2)(-1) - (1)(0)]\vec{k} \\ &= \langle 2, -4, -2 \rangle.\end{aligned}$$

^aSee Example 11 on page 42.

Finding the Distance between Two Lines I (page 54, Chapter 4)

- **Example.**^a If L_1 is the line passing through the points

$$A = (0, 1, 4), \quad B = (3, 4, 1),$$

and L_2 is the line passing through the points

$$C = (-1, 4, 11), \quad D = (1, 5, 9),$$

then the distance d between L_1 and L_2 is found as follows:

- (a) find the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ that lie on the lines L_1 and L_2 :

$$\overrightarrow{AB} = (3, 4, 1) - (0, 1, 4) = \langle 3, 3, -3 \rangle,$$

$$\overrightarrow{CD} = (1, 5, 9) - (-1, 4, 11) = \langle 2, 1, -2 \rangle;$$

^aSee Example 16 on page 54.

Finding the Distance between Two Lines II (page 54, Chapter 4)

- (b) find the normal vector $\vec{n}$ (not necessarily unit) perpendicular to both lines L_1 and L_2 :

$$\begin{aligned}\vec{n} = \overrightarrow{AB} \times \overrightarrow{CD} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 3 & -3 \\ 2 & 1 & -2 \end{vmatrix} = \vec{i} \begin{vmatrix} 3 & -3 \\ 2 & -2 \end{vmatrix} - \vec{j} \begin{vmatrix} 3 & -3 \\ 2 & -2 \end{vmatrix} + \vec{k} \begin{vmatrix} 3 & 3 \\ 2 & 1 \end{vmatrix} \\ &= [(3)(-2) - (-3)(1)]\vec{i} - [(3)(-2) - (-3)(2)]\vec{j} \\ &\quad + [(3)(1) - (3)(2)]\vec{k} = \langle -3, 0, -3 \rangle;\end{aligned}$$

- (c) find a vector connecting two arbitrary points on L_1 and L_2 , say:

$$\overrightarrow{AD} = (1, 5, 9) - (0, 1, 4) = \langle 1, 4, 5 \rangle;$$

any other vector $\overrightarrow{AC}$, $\overrightarrow{BC}$, or $\overrightarrow{BD}$ will do;

Finding the Distance between Two Lines III (page 54, Chapter 4)

(d) find the **scalar projection** of $\overrightarrow{AD}$ onto $\vec{n}$:

$$\begin{aligned}\text{comp}_{\vec{n}}\overrightarrow{AD} &= \overrightarrow{AD} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{\overrightarrow{AD} \cdot \vec{n}}{|\vec{n}|} \\ &= \frac{\langle 1, 4, 5 \rangle \cdot \langle -3, 0, -3 \rangle}{|\langle -3, 0, -3 \rangle|} \\ &= \frac{(1)(-3) + (4)(0) + (5)(-3)}{\sqrt{(-3)^2 + (0)^2 + (-3)^2}} \\ &= -\frac{18}{\sqrt{18}} = -\sqrt{18} = -3\sqrt{2};\end{aligned}$$

(e) find the distance d as the absolute value of $\text{comp}_{\vec{n}}\overrightarrow{AD}$:

$$d = |\text{proj}_{\vec{n}}\overrightarrow{AD}| = |\text{comp}_{\vec{n}}\overrightarrow{AD}| = 3\sqrt{2}.$$

Antiderivatives and Indefinite Integrals I (page 2, Chapter 1)

- **Definition 1.** Let $f(x)$ be a given function. A function $F(x)$ is called an *antiderivative* of $f(x)$ on an interval $I = (a, b)$ if

$$\frac{d}{dx}F(x) = F'(x) = f(x)$$

for all x in I . The set of all antiderivatives of $f(x)$ is called the *indefinite integral* of $f(x)$ (with respect to x), and is denoted by

$$\int f(x) dx.$$

The symbol $\int$ is called the *integral sign*, the function $f(x)$ is called the *integrand* of the integral, and x is called the *variable of integration*.

Antiderivatives and Indefinite Integrals II (page 2, Chapter 1)

- **Lemma 2.** *Let $f(x)$ be a continuous function and $F_1(x)$, $F_2(x)$ be two antiderivatives of $f(x)$. Then there exists a constant C such that*

$$F_1(x) - F_2(x) = C.$$

- **Proof.** Since both $F_1(x)$ and $F_2(x)$ are antiderivatives of $f(x)$, we have, by definition,

$$[F_1(x) - F_2(x)]' = F_1'(x) - F_2'(x) = f(x) - f(x) = 0.$$

This shows that the function $G(x)$ defined by $G(x) = F_1(x) - F_2(x)$ satisfies $G'(x) = 0$, so there must exist a constant C such that $G(x) = C$. This implies that $F_1(x) - F_2(x) = C$, as claimed. $\square$

Antiderivatives and Indefinite Integrals III (page 2, Chapter 1)

- **Corollary 3.** *If $F(x)$ is an antiderivative of $f(x)$, then*

$$\int f(x) dx = F(x) + C,$$

where C is an arbitrary constant.

- **Proof.** Let $F_1(x)$ be any other antiderivative of $f(x)$, then by Lemma 2, there exists a constant C_1 such that

$$F_1(x) - F(x) = C_1,$$

or equivalently,

$$F_1(x) = F(x) + C_1.$$

Antiderivatives and Indefinite Integrals IV (page 2, Chapter 1)

Thus the indefinite integral of $f(x)$, being the set of all antiderivatives of $f(x)$, must have the form

$$\int f(x) dx = F(x) + C_1,$$

where C_1 is **some** constant. Since $F(x) + C$ is clearly an antiderivative of $f(x)$ for any value of C :

$$[F(x) + C]' = F'(x) = f(x),$$

it follows that

$$\int f(x) dx = F(x) + C,$$

where C is an **arbitrary** constant. □

The Indefinite Integral of x^{-1} I (page 6, Chapter 1)

- The indefinite integral of x^{-1} can be “derived” from the indefinite integrals of x^a , $a \neq -1$ as follows:

(a) recall from example 2 on page 5 that

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C =: F(x; C),$$

where C is an arbitrary constant;

(b) for each $a \neq -1$, set $C = -1/(a+1)$ in the above formula to obtain

$$F(x, -1/(a+1)) = \frac{x^{a+1}}{a+1} - \frac{1}{a+1} = \frac{x^{a+1} - 1}{a+1} = \frac{x^{a-(-1)} - x^0}{a - (-1)};$$

The Indefinite Integral of x^{-1} II (page 6, Chapter 1)

- (c) the above expression, with a viewed as the variable, is reminiscent of the definition of a derivative; assuming $x > 0$ is fixed and letting $a \rightarrow -1$, we deduce

$$\begin{aligned} \lim_{a \rightarrow -1} F(x, -1/(a+1)) &= \lim_{a \rightarrow -1} \frac{x^{a-(-1)} - x^0}{a - (-1)} \\ &= \left. \frac{d}{da} x^{a-(-1)} \right|_{a=-1} = x^{a+1} \log x \Big|_{a=-1} = \log x; \end{aligned}$$

this can be used as a motivation for the integral formula

$$\int x^{-1} dx = \int \frac{1}{x} dx = \log|x| + C.$$

Properties of Indefinite Integrals I (page 16, Chapter 1)

- Let $f(x)$ and $g(x)$ be two continuous functions with antiderivatives $F(x)$ and $G(x)$, then:

(a) $cF(x)$ is an antiderivative of $cf(x)$, since

$$[cF(x)]' = cF'(x) = cf(x);$$

(b) $F(x) \pm G(x)$ is an antiderivative of $f(x) \pm g(x)$, since

$$[F(x) \pm G(x)]' = F'(x) \pm G'(x) = f(x) \pm g(x);$$

(c) $a^{-1}F(ax + b)$ is an antiderivative of $f(ax + b)$, since

$$[a^{-1}F(ax + b)]' = \frac{1}{a} [F(ax + b)]' = \frac{1}{a} [aF'(ax + b)] = f(ax + b).$$

Properties of Indefinite Integrals II (page 16, Chapter 1)

- Thus by Corollary 3, we deduce (note C, C_1, C_2 are **arbitrary**)

(a)

$$\int cf(x) dx = cF(x) + C = c[F(x) + C_1] = c \int f(x) dx;$$

(b)

$$\begin{aligned} \int [f(x) \pm g(x)] dx &= [F(x) \pm G(x)] + C \\ &= [F(x) + C_1] \pm [G(x) + C_2] = \int f(x) dx \pm \int g(x) dx; \end{aligned}$$

(c)

$$\int f(ax + b) dx = \frac{1}{a} F(ax + b) + C.$$

Approximating Area with Finite Sums I (page 29, Chapter 1)

- The method of **integration** is a tool for calculating quantities such as areas and volumes by breaking them into small pieces, and then summing the contributions from each small part.
- **Example.** As an example, consider the area A of the shaded region R that lies above the x -axis, below the graph of $y = 1 - x^2$, and between the vertical lines $x = 0$ and $x = 1$ (see Figure 5):
 - (a) there is no simple formula for calculating the area A , but it can be approximated by summing the areas of a collection of rectangles;

Approximating Area with Finite Sums II (page 29, Chapter 1)

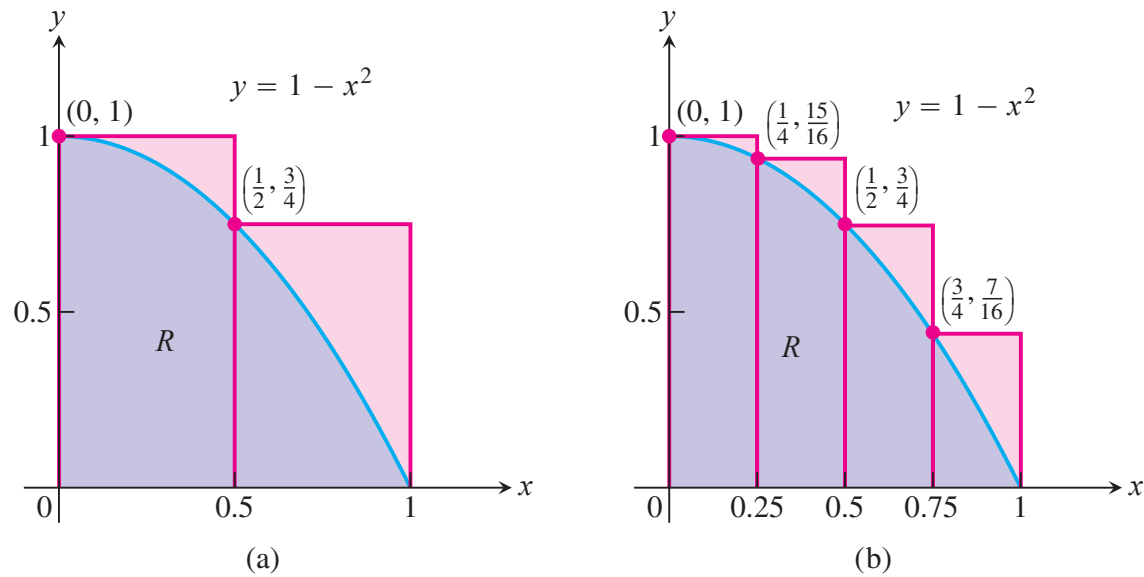


Figure 5: Approximating the area of R by using (a) two rectangles and (b) four rectangles.

Approximating Area with Finite Sums III (page 29, Chapter 1)

- (b) Figure 5(a) shows two rectangles that together contain the region R ; the total area of the two rectangles approximates the area A of the region R :

$$A \approx 1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{7}{8} = 0.875;$$

- (c) Figure 5(b) shows four thinner rectangles that together contain the region R ; these four rectangles give the approximation:

$$A \approx 1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = \frac{25}{32} = 0.78125;$$

Approximating Area with Finite Sums IV (page 29, Chapter 1)

- (d) in each of these computed sums, the interval $[a, b]$ over which the function $f(x)$ is defined is subdivided into n subintervals of equal width $\Delta x = (b - a)/n$, and $f(x)$ is evaluated at a point in each subinterval: c_1 in the first subinterval, c_2 in the second subinterval, and so on; the finite sums then all take the form

$$f(c_1) \cdot \Delta x + f(c_2) \cdot \Delta x + \cdots + f(c_n) \cdot \Delta x;$$

Approximating Area with Finite Sums V (page 29, Chapter 1)

- (e) by taking more and more rectangles, with each rectangle thinner than before, it appears that these finite sums give better and better approximations to the true area A of the region R :

n	Finite sum
2	0.875
4	0.78125
16	0.697265625
50	0.6766
100	0.67165
1000	0.6671665

with the techniques developed below, we will be able to show that the area of R is exactly $2/3$.

Finite Sums and Sigma Notation I (page 29, Chapter 1)

- **Sigma notation** enables us to write a sum with many terms in the compact form

$$\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n,$$

where the Greek letter Σ stands for “sum”, and the **index of summation** i tells us where the sum begins and where it ends.

- For example,

$$\sum_{i=1}^3 (-1)^i i = (-1)^1(1) + (-1)^2(2) + (-1)^3(3) = -1 + 2 - 3 = -2,$$

$$\sum_{i=4}^5 \frac{i^2}{i-1} = \frac{4^2}{4-1} + \frac{5^2}{5-1} = \frac{16}{3} + \frac{25}{4} = \frac{139}{12}.$$

Finite Sums and Sigma Notation II (page 29, Chapter 1)

- It is not difficult to check that finite sums, when written in sigma notation, satisfy the following algebraic rules:

$$\begin{aligned}\sum_{i=1}^n (a_i + b_i) &= \sum_{i=1}^n a_i + \sum_{i=1}^n b_i, & \sum_{i=1}^n c a_i &= c \sum_{i=1}^n a_i, \\ \sum_{i=1}^n (a_i - b_i) &= \sum_{i=1}^n a_i - \sum_{i=1}^n b_i, & \sum_{i=1}^n c &= n \cdot c.\end{aligned}$$

- For example, to verify the sum rule, we only need to compute

$$\begin{aligned}\sum_{i=1}^n (a_i + b_i) &= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) \\ &= (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i;\end{aligned}$$

the other rules can be verified in a similar way.

Finite Sums and Sigma Notation III (page 29, Chapter 1)

- **Example.** As an example, let's use the above rules to compute

$$\sum_{i=1}^n (3i - i^2) = \sum_{i=1}^n 3i - \sum_{i=1}^n i^2 = 3 \sum_{i=1}^n i - \sum_{i=1}^n i^2,$$

$$\sum_{i=1}^n (-a_i) = \sum_{i=1}^n (-1) \cdot a_i = (-1) \sum_{i=1}^n a_i = - \sum_{i=1}^n a_i,$$

$$\sum_{i=1}^3 (i + 4) = \sum_{i=1}^3 i + \sum_{i=1}^3 4 = (1 + 2 + 3) + 3 \cdot 4 = 6 + 12 = 18,$$

$$\sum_{i=1}^n \frac{1}{n} = n \cdot \frac{1}{n} = 1.$$

Finite Sums and Sigma Notation IV (page 29, Chapter 1)

- It is also useful to record the following formulas for future use:

the sum of the first n integers : $\sum_{i=1}^n i = \frac{1}{2} n(n+1),$

the sum of the first n squares : $\sum_{i=1}^n i^2 = \frac{1}{6} n(n+1)(2n+1),$

the sum of the first n cubes : $\sum_{i=1}^n i^3 = \frac{1}{4} n^2(n+1)^2.$

- Although the first formula can be proved directly, the second and third ones need to be proved using mathematical induction or complex variable methods; we omit the details here.

Limits of Finite Sums I (page 30, Chapter 1)

- With the aid of sigma notation, we now find the limiting value of finite sum approximations to the area of the region R shown in Figure 5 as the number of rectangles approaches infinity:

(a) divide $[0, 1]$ into n equal subintervals

$$[0, 1/n], [1/n, 2/n], \dots, [(n-1)/n, 1],$$

each of which has width $1/n$;

(b) on the i -th subinterval $[(i-1)/n, i/n]$, construct a rectangle with width $1/n$ and height $f(i/n)$; this leads to the finite sum^a

$$f\left(\frac{1}{n}\right) \cdot \frac{1}{n} + f\left(\frac{2}{n}\right) \cdot \frac{1}{n} + \cdots + f\left(\frac{n}{n}\right) \cdot \frac{1}{n};$$

^aThis is called a **right Riemann sum** (see below); the finite sums constructed on page 27 correspond to a **left Riemann sum**.

Limits of Finite Sums II (page 30, Chapter 1)

- (c) writing this sum in sigma notation and simplifying, we obtain (recall $f(x) = 1 - x^2$)

$$\begin{aligned} \sum_{i=1}^n f\left(\frac{i}{n}\right) \cdot \frac{1}{n} &= \sum_{i=1}^n \left(1 - \left(\frac{i}{n}\right)^2\right) \cdot \frac{1}{n} \\ &= \sum_{i=1}^n \left(\frac{1}{n} - \frac{i^2}{n^3}\right) = \sum_{i=1}^n \frac{1}{n} - \sum_{i=1}^n \frac{i^2}{n^3} = n \cdot \frac{1}{n} - \frac{1}{n^3} \sum_{i=1}^n i^2 \\ &= 1 - \frac{1}{n^3} \cdot \frac{1}{6} n(n+1)(2n+1) = 1 - \frac{2n^3 + 3n^2 + n}{6n^3}; \end{aligned}$$

- (d) as $n \rightarrow \infty$, the finite sums converge to

$$\lim_{n \rightarrow \infty} \left(1 - \frac{2n^3 + 3n^2 + n}{6n^3}\right) = 1 - \frac{2}{6} = \frac{2}{3};$$

- (e) it can be shown that **any** finite sum approximation with equal sub-interval widths converges to the same value $2/3$.

Riemann Sums and Definite Integrals I (page 30, Chapter 1)

- The theory of limits of finite approximations was made precise by the notion of a **Riemann sum**, which underlies the theory of definite integrals; to define a Riemann sum:
 - (a) consider an arbitrary function $f(x)$ on a closed interval $[a, b]$;
 - (b) divide the interval $[a, b]$ into n subintervals, not necessarily of equal widths, by choosing points $\{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$ satisfying

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b;$$

the set $P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$ is called a **partition** of $[a, b]$;

- (c) denote the width of the i -th subinterval $[x_{i-1}, x_i]$ by $\Delta x_i = x_i - x_{i-1}$, and choose a point c_i in $[x_{i-1}, x_i]$; the area under the graph of $f(x)$ between x_{i-1} and x_i is approximated by the area $f(c_i) \cdot \Delta x_i$ of the rectangle with width Δx_i and height $f(c_i)$ (see Figure 6);

Riemann Sums and Definite Integrals II (page 30, Chapter 1)

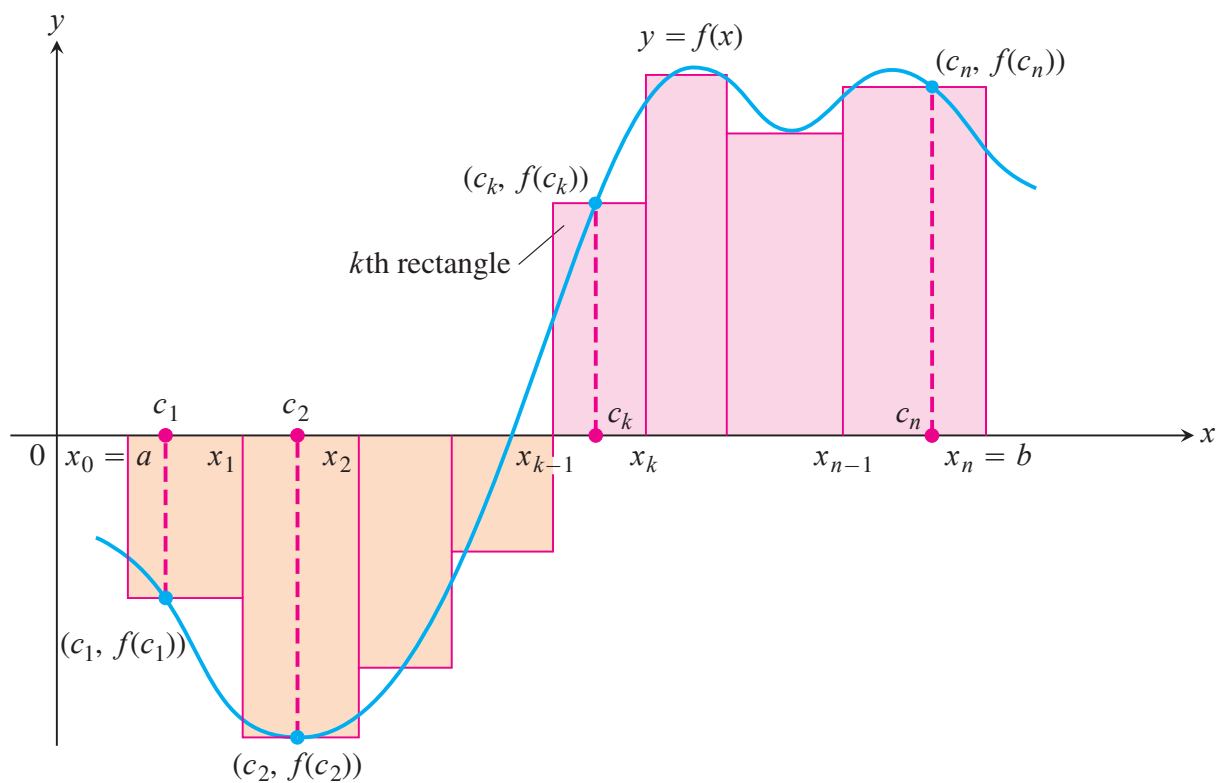


Figure 6: Constructing a Riemann sum.

Riemann Sums and Definite Integrals III (page 30, Chapter 1)

- (d) summing the areas of all these rectangles yields a **Riemann sum** for $f(x)$ on the interval $[a, b]$:

$$\begin{aligned} S_P &= f(c_1) \cdot \Delta x_1 + f(c_2) \cdot \Delta x_2 + \cdots \\ &\quad + f(c_n) \cdot \Delta x_n = \sum_{i=1}^n f(c_i) \cdot \Delta x_i; \end{aligned}$$

clearly, the value of S_P depends both on the partition P and on the choices of the points c_i in the subintervals.

- The **norm** of a partition P , denoted by $\|P\|$, is defined to be the largest of all the subinterval widths:

$$\|P\| = \max_i \Delta x_i = \max \left\{ \Delta x_1, \Delta x_2, \dots, \Delta x_n \right\}.$$

Riemann Sums and Definite Integrals IV (page 30, Chapter 1)

- **Definition 4.** Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that $f(x)$ is **(Riemann) integrable** over $[a, b]$ if the Riemann sums $\sum_{i=1}^n f(c_i)\Delta x_i$ converge to a number I in the following sense: given any number $\epsilon > 0$, there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_i in $[x_{i-1}, x_i]$, we have

$$\left| \sum_{i=1}^n f(c_i)\Delta x_i - I \right| < \epsilon.$$

The number I is called the **definite integral** of $f(x)$ over $[a, b]$, and it is denoted by

$$I = \int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(c_i)\Delta x_i.$$

Riemann Sums and Definite Integrals V (page 30, Chapter 1)

- The symbol $\int_a^b f(x) dx$ used to denote a definite integral can be viewed as a “continuum limit” of the symbol $\sum_{i=1}^n f(c_i) \Delta x_i$ used to denote a Riemann sum, where in the limit of infinitely refined partition ($\|P\| \rightarrow 0$):
 - (a) the sum $\sum_{i=1}^n$ of areas of finitely many rectangles becomes the sum $\int_a^b$ of areas of infinitely many (infinitesimally thin) rectangles;
 - (b) the heights $f(c_i)$ of the finitely many rectangles become the heights $f(x)$ of the infinitely many (infinitesimally thin) rectangles;
 - (c) the widths Δx_i of the finitely many rectangles become the widths dx of the infinitely many (infinitesimally thin) rectangles.

Existence of Definite Integrals I (page 30, Chapter 1)

- **Theorem 5** (Riemann integrability of continuous functions). *Let $f(x)$ be a continuous function defined on a closed interval $[a, b]$. Then $f(x)$ is Riemann integrable over $[a, b]$.*
- ***Proof.**^a Recall first that a continuous function defined on a closed interval $[a, b]$ is uniformly continuous on $[a, b]$. Thus for any given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{b - a},$$

whenever $|x - y| < \delta$. Consider any partition P of $[a, b]$ with $\|P\| < \delta$. For any choices c_i, d_i of the points in $[x_{i-1}, x_i]$, there clearly holds

$$|c_i - d_i| \leq x_i - x_{i-1} = \Delta x_i \leq \|P\| < \delta.$$

^aOptional reading; skip if you are not familiar with the ϵ - δ language.

Existence of Definite Integrals II (page 30, Chapter 1)

Thus by uniform continuity, there must hold

$$|f(c_i) - f(d_i)| < \frac{\epsilon}{b-a}, \quad i = 1, 2, \dots, n.$$

This implies that

$$\begin{aligned} & \left| \sum_{i=1}^n f(c_i) \Delta x_i - \sum_{i=1}^n f(d_i) \Delta x_i \right| \\ &= \left| \sum_{i=1}^n [f(c_i) - f(d_i)] \Delta x_i \right| \leq \sum_{i=1}^n |f(c_i) - f(d_i)| \Delta x_i \\ &< \sum_{i=1}^n \frac{\epsilon}{b-a} \Delta x_i = \frac{\epsilon}{b-a} \sum_{i=1}^n \Delta x_i = \frac{\epsilon}{b-a} (b-a) = \epsilon, \end{aligned}$$

which shows that the sequence of Riemann sums $\sum_{i=1}^n f(c_i) \cdot \Delta x_i$ is Cauchy. The existence of the definite integral of $f(x)$ over $[a, b]$ as the limit of the Riemann sums then follows from the completeness of the real numbers. $\square$

Existence of Definite Integrals III (page 30, Chapter 1)

- Theorem 5 guarantees the integrability of continuous functions over a closed interval $[a, b]$; functions that are not continuous, on the other hand, may or may not be integrable:
 - (a) discontinuous functions that are integrable include those that are **monotone** on $[a, b]$ and those that are **piecewise continuous** on $[a, b]$ (continuous except at a finite number of points);
 - (b) an example of a nonintegrable function is given by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}, \quad x \in [0, 1];$$

Existence of Definite Integrals IV (page 30, Chapter 1)

indeed, for any partition P of $[0, 1]$, let c_i be a rational number contained in $[x_{i-1}, x_i]$ and d_i be an irrational number contained in $[x_{i-1}, x_i]$ (note that such c_i and d_i always exist); then

$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n (1) \Delta x_i = 1,$$

while

$$\sum_{i=1}^n f(d_i) \Delta x_i = \sum_{i=1}^n (0) \Delta x_i = 0;$$

this shows that the limit of the Riemann sums $\sum_{i=1}^n f(c_i) \Delta x_i$ cannot exist, since it depends on the choices of the points c_i : if the c_i 's are always rational, then the limit is 1; if the c_i 's are always irrational, then the limit is 0.

Fundamental Theorem of Calculus I (page 34, Chapter 1)

- **Theorem 6** (The fundamental theorem of calculus, part I). *Let $f(x)$ be a continuous function defined on a closed interval $[a, b]$. Then the function*

$$F(x) = \int_a^x f(t) dt$$

is continuous on $[a, b]$ and differentiable on (a, b) , with

$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x).$$

- ***Proof.**^a We only prove $F'(x) = f(x)$ for each $x \in (a, b)$, from which the differentiability of $F(x)$ on (a, b) and the continuity of $F(x)$ on (a, b) follows. The continuity of $F(x)$ at $x = a^+$ and $x = b^-$ can be verified in a similar way.

^aOptional reading; skip if you are not familiar with the ϵ - δ language.

Fundamental Theorem of Calculus II (page 34, Chapter 1)

For any $x \in (a, b)$, observe that

$$\begin{aligned} \frac{1}{h} [F(x+h) - F(x)] \\ = \frac{1}{h} \left(\int_a^{x+h} f(t) dt - \int_a^x f(t) dt \right) = \frac{1}{h} \int_x^{x+h} f(t) dt. \end{aligned}$$

On the other hand, it is not difficult to see that

$$f(x) = \frac{1}{h} [f(x)h] = \frac{1}{h} \left(f(x) \int_x^{x+h} 1 dt \right) = \frac{1}{h} \int_x^{x+h} f(x) dt.$$

Thus

$$\begin{aligned} \frac{1}{h} [F(x+h) - F(x)] - f(x) \\ = \frac{1}{h} \int_x^{x+h} f(t) dt - \frac{1}{h} \int_x^{x+h} f(x) dt = \frac{1}{h} \int_x^{x+h} [f(t) - f(x)] dt. \end{aligned}$$

Fundamental Theorem of Calculus III (page 34, Chapter 1)

Since $f(x)$ is continuous, for any given $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(t) - f(x)| < \epsilon$ whenever $|t - x| < \delta$. Thus for all $h < \delta$, there must hold

$$\begin{aligned} & \left| \frac{1}{h} [F(x+h) - F(x)] - f(x) \right| \\ &= \left| \frac{1}{h} \int_x^{x+h} [f(t) - f(x)] dt \right| \leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt \\ &< \frac{1}{h} \int_x^{x+h} \epsilon dt = \frac{\epsilon}{h} \int_x^{x+h} 1 dt = \frac{\epsilon}{h} h = \epsilon. \end{aligned}$$

This shows that

$$\lim_{h \rightarrow 0} \frac{1}{h} [F(x+h) - F(x)] = f(x),$$

and by definition, that $F'(x)$ exists and equals $f(x)$. □

Fundamental Theorem of Calculus IV (page 34, Chapter 1)

- Geometrically, the area under the graph of $f(x)$ between x and $x + h$, represented by the difference

$$F(x + h) - F(x) = \int_a^{x+h} f(t) dt - \int_a^x f(t) dt = \int_x^{x+h} f(t) dt,$$

can be well approximated by the area $f(x) \cdot h$ of the rectangle with width h and height $f(x)$, provided that h is sufficiently small (see Figure 7); hence

$$F'(x) \approx \frac{1}{h} [F(x + h) - F(x)] \approx \frac{1}{h} [f(x)h] = f(x),$$

which is essentially what Theorem 6 claims.

Fundamental Theorem of Calculus V (page 34, Chapter 1)

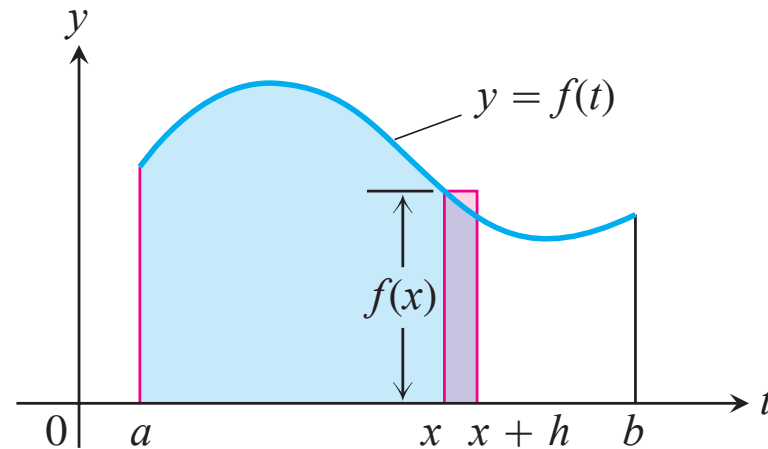


Figure 7: Approximating the difference quotient $[F(x + h) - F(x)]/h$ by the height $f(x)$ of the rectangle.

Fundamental Theorem of Calculus VI (page 34, Chapter 1)

- **Example.** As an example, let's use Theorem 6 to compute

$$\frac{d}{dx} \int_a^x \cos t \, dt = \cos x,$$

$$\frac{d}{dx} \int_0^x \frac{1}{1+t^2} \, dt = \frac{1}{1+x^2},$$

$$\frac{d}{dx} \int_x^5 3t \sin t \, dt = \frac{d}{dx} \left(- \int_5^x 3t \sin t \, dt \right) = -3x \sin x.$$

- More generally, one can show that

$$\frac{d}{dx} \left(\int_{a(x)}^{b(x)} f(t) \, dt \right) = f(b(x)) \cdot b'(x) - f(a(x)) \cdot a'(x);$$

a proof of this fact can be found in the review sheet.

Fundamental Theorem of Calculus VII (page 47, Chapter 1)

- **Example.**^a Using the above formula, we may easily compute

$$\begin{aligned}\frac{d}{dx} \int_1^{2x} e^{3y^2-1} dy &= e^{3(2x)^2-1} \cdot (2x)' - e^{3(1)^2-1} \cdot (1)' \\ &= e^{3(4x^2)-1} \cdot (2) - e^{3(1)-1} \cdot (0) = 2e^{12x^2-1},\end{aligned}$$

$$\begin{aligned}\frac{d}{dx} \int_1^{x^2} \cos t dt &= \cos(x^2) \cdot (x^2)' - \cos(1) \cdot (1)' \\ &= \cos(x^2) \cdot (2x) - \cos(1) \cdot (0) = 2x \cos x^2.\end{aligned}$$

^aSee Example 22 on page 47.

Fundamental Theorem of Calculus VIII (page 34, Chapter 1)

- **Theorem 7** (The fundamental theorem of calculus, part II). *Let $f(x)$ be a continuous function defined on a closed interval $[a, b]$, and let $F(x)$ be any antiderivative of $f(x)$ on $[a, b]$. Then*

$$\int_a^b f(t) dt = F(b) - F(a) =: F(x) \Big|_a^b.$$

- Theorem 7 says that to compute the definite integral of $f(x)$ over $[a, b]$, all we need to do is:
 - (a) find an antiderivative $F(x)$ of $f(x)$, and
 - (b) calculate the number $F(b) - F(a)$.

Fundamental Theorem of Calculus IX (page 34, Chapter 1)

- **Proof.** By Theorem 6, the function $G(x)$ defined by

$$G(x) = \int_a^x f(t) dt$$

is **an** antiderivative of $f(x)$ (since $G'(x) = f(x)$). According to Lemma 2, however, the two antiderivatives $F(x)$ and $G(x)$ of $f(x)$ must differ only by a constant, namely,

$$F(x) - G(x) = C, \quad \text{or} \quad F(x) = G(x) + C,$$

for some constant C . Thus

$$\begin{aligned} \int_a^b f(t) dt &= \int_a^b f(t) dt - \int_a^a f(t) dt \\ &= G(b) - G(a) = [G(b) + C] - [G(a) + C] = F(b) - F(a), \end{aligned}$$

as is to be shown. □

Fundamental Theorem of Calculus X (page 34, Chapter 1)

- The conclusions of the Fundamental Theorem 6–7 reveal that:
 - (a) the integration of a function $f(x)$ followed by the differentiation of the result gives back the function $f(x)$, namely (Theorem 6):

$$\frac{d}{dx} \int_a^x f(t) dt = f(x);$$

- (b) the differentiation of a function $F(x)$ followed by the integration of the result gives back the function $F(x)$, adjusted by an integration constant (Theorem 7; note that $F(x)$ is an antiderivative of $F'(x)$):

$$\int_a^x \frac{d}{dt} F(t) dt = F(x) - F(a);$$

- (c) in other words, the processes of integration and differentiation are “inverses” of each other;

Fundamental Theorem of Calculus XI (page 34, Chapter 1)

- (d) the indefinite integral of a function $f(x)$, being the set of all antiderivatives of $f(x)$, satisfies

$$\int f(x) dx = \int_a^x f(t) dt + C,$$

where C is an arbitrary constant (Corollary 3; note that $\int_a^x f(t) dt$ is an antiderivative of $f(x)$ by Theorem 6); this explains the notation $\int f(x) dx$ used for the indefinite integral of $f(x)$.

Properties of Definite Integrals I (page 40, Chapter 1)

- The various properties of definite integrals can be proved using the definition of the definite integral as a limit of Riemann sums.
- For example, if $f(x)$ and $g(x)$ are integrable over $[a, b]$, then
 - (a) for every partition P of $[a, b]$ and for every choice of the points d_i ,

$$\begin{aligned}\sum_{i=1}^n [c_1 f(d_i) + c_2 g(d_i)] \Delta x_i &= \sum_{i=1}^n c_1 f(d_i) \Delta x_i + \sum_{i=1}^n c_2 g(d_i) \Delta x_i \\ &= c_1 \sum_{i=1}^n f(d_i) \Delta x_i + c_2 \sum_{i=1}^n g(d_i) \Delta x_i;\end{aligned}$$

- (b) since $f(x)$ and $g(x)$ are integrable, the limits of the Riemann sums on the right-hand side of the above equation exist as $\|P\| \rightarrow 0$;

Properties of Definite Integrals II (page 40, Chapter 1)

- (c) consequently, the limit of the Riemann sum on the left-hand side of the above equation also exists as $\|P\| \rightarrow 0$, and equals

$$\begin{aligned}
 \int_a^b [c_1 f(x) + c_2 g(x)] dx &= \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n [c_1 f(d_i) + c_2 g(d_i)] \Delta x_i \\
 &= \lim_{\|P\| \rightarrow 0} \left(c_1 \sum_{i=1}^n f(d_i) \Delta x_i + c_2 \sum_{i=1}^n g(d_i) \Delta x_i \right) \\
 &= \lim_{\|P\| \rightarrow 0} c_1 \sum_{i=1}^n f(d_i) \Delta x_i + \lim_{\|P\| \rightarrow 0} c_2 \sum_{i=1}^n g(d_i) \Delta x_i \\
 &= c_1 \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(d_i) \Delta x_i + c_2 \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n g(d_i) \Delta x_i \\
 &= c_1 \int_a^b f(x) dx + c_2 \int_a^b g(x) dx;
 \end{aligned}$$

this establishes property 4.

Substitution for Indefinite Integrals I (page 3, Chapter 2)

- **Theorem 8** (Substitution rule for indefinite integrals). *Let $u = g(x)$ be a differentiable function whose range is an interval I and $f(u)$ be a continuous function on I . Then*

$$\int f(g(x))g'(x) dx = \int f(u) du.$$

- Theorem 8 provides the following method to evaluate the integral $\int f(g(x))g'(x) dx$ when f and g' are continuous functions:
 - (a) substitute $u = g(x)$, $du = g'(x) dx$ to obtain the integral $\int f(u) du$;
 - (b) integrate with respect to u ;
 - (c) replace u by $g(x)$ in the result.

Substitution for Indefinite Integrals II (page 3, Chapter 2)

- **Proof.** Let $F(u)$ be any antiderivative of $f(u)$ and let $G(x)$ be the function defined by $G(x) = F(g(x))$. By the chain rule, $G(x)$ is an antiderivative of $f(g(x)) \cdot g'(x)$:

$$G'(x) = \frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x).$$

So by Corollary 3 (applied to $G(x)$),

$$\int f(g(x))g'(x) dx = \int G'(x) dx = G(x) + C = F(g(x)) + C.$$

With the substitution $u = g(x)$, another application of Corollary 3 shows that

$$F(g(x)) + C = F(u) + C = \int F'(u) du = \int f(u) du.$$

Hence the theorem follows. □

Substitution for Definite Integrals I (page 9, Chapter 2)

- To evaluate a definite integral using substitution, we may:
 - (a) find an antiderivative (indefinite integral) using Theorem 8, and then evaluate the definite integral by applying Theorem 7; or
 - (b) extend the process of substitution directly to definite integrals.
- **Theorem 9** (Substitution rule for definite integrals). *Let $g'(x)$ be continuous on the interval $[a, b]$ and $f(u)$ be continuous on the range of $g(x)$. Then*

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

Substitution for Definite Integrals II (page 9, Chapter 2)

- **Proof.** Let $F(u)$ and $G(x)$ be as defined in the proof of Theorem 8, and observe that $G(x)$ is an antiderivative of $f(g(x)) \cdot g'(x)$. Then by Theorem 7,

$$\int_a^b f(g(x))g'(x) dx = G(x) \Big|_{x=a}^{x=b} = F(g(x)) \Big|_{x=a}^{x=b} = F(g(b)) - F(g(a)).$$

With the substitution $u = g(x)$, another application of Theorem 7 shows that

$$F(g(b)) - F(g(a)) = F(u) \Big|_{u=g(a)}^{u=g(b)} = \int_{g(a)}^{g(b)} F'(u) du = \int_{g(a)}^{g(b)} f(u) du.$$

Hence the theorem follows. □

Review of Differentials I (page 3, Chapter 2)

- The successful application of substitution rules (Theorem 8–9) requires a thorough understanding of the symbol dx , the so-called **differentials**, hidden in the definition of definite integrals:
 - (a) the differential dx in a definite integral $\int_a^b f(x) \, dx$ is an infinitesimal analog of the subinterval width Δx introduced in a Riemann sum $\sum_{i=1}^n f(c_i) \Delta x_i$; it represents **infinitesimal changes** in the variable x ;
 - (b) likewise, if $y = f(x)$ is a differentiable function of x , then the differential dy represents the infinitesimal change in the variable y as an infinitesimal change of dx occurs in the variable x :

$$dy = f(x + dx) - f(x);$$

Review of Differentials II (page 3, Chapter 2)

- (c) to express dy in terms of dx , consider first a small but **finite** change Δx in the variable x , and the associated finite change

$$\Delta y = f(x + \Delta x) - f(x)$$

incurred in the variable y ;

- (d) since $f(x)$ is differentiable, the Mean Value Theorem implies that

$$\Delta y = f'(\xi) [(x + \Delta x) - x] = f'(\xi) \Delta x,$$

where ξ is some point lying between x and $x + \Delta x$;

Review of Differentials III (page 3, Chapter 2)

- (e) as the size of the finite change Δx reduces, the point ξ gets closer and closer to x ; when Δx becomes “infinitely small” (infinitesimal), ξ coincides with x and the above equation becomes

$$dy = f'(x) dx;$$

- (f) using the Leibniz notation dy/dx for the derivative of y with respect to x , the above equation can be written in the more revealing form as

$$dy = f'(x) dx = \frac{dy}{dx} dx;$$

this says that the derivative of $y = f(x)$ can indeed be interpreted as the quotient of the differentials dy and dx .

Review of Differentials IV (page 3, Chapter 2)

- **Example.** As an example, let's find the differential dy for

$$y = x^5 + 37x : dy = (x^5 + 37x)' dx = (5x^4 + 37) dx,$$

$$y = \tan 2x : dy = (\tan 2x)' dx = \sec^2(2x) \cdot (2x)' dx = 2 \sec^2(2x) dx.$$

In the last equation, the differential dy can also be expressed in terms of the intermediate variable $u = 2x$, which gives:

$$y = \tan u : dy = \frac{dy}{du} du = \sec^2 u du = \sec^2(2x) d(2x) = 2 \sec^2(2x) dx.$$

It is important to work this equation backward to see how an expression of the form $g(x) dx$ can be recognized as a differential dy :

$$\begin{aligned} 2 \sec^2(2x) dx &= \sec^2(2x) \cdot 2 dx = \sec^2(2x) d(2x) \\ &= \sec^2 u du = d(\tan u) = d(\tan 2x). \end{aligned}$$

Review of Differentials V (page 3, Chapter 2)

- Every differentiation formula has a corresponding differential form; for example, if $u(x)$, $v(x)$ are differentiable functions and c is a constant, then

$$d(cu) = c \, du,$$

$$d(u \pm v) = du \pm dv,$$

$$d(uv) = v \, du + u \, dv,$$

$$d\left(\frac{u}{v}\right) = \frac{v \, du - u \, dv}{v^2}.$$

- These formulas can either be derived from the definition, e.g.

$$d(uv) = (uv)' \, dx = (u'v + uv') \, dx = vu' \, dx + uv' \, dx = v \, du + u \, dv,$$

or, more simply, by multiplying the corresponding differentiation formula by dx , e.g.

$$\left(\frac{d(uv)}{dx}\right) dx = \left(v \frac{du}{dx}\right) dx + \left(u \frac{dv}{dx}\right) dx.$$

Linear Substitution Revisited (page 16, Chapter 1)

- The integral formula

$$\int f(ax + b) dx = \frac{1}{a} F(ax + b) + C,$$

stated on page 16 of Chapter 1 can be viewed as a special case of Theorem 8; indeed, if $u = ax + b$ and $F(u)$ is an antiderivative of $f(u)$, then

$$\begin{aligned} \int f(ax + b) dx &= \frac{1}{a} \int f(\textcolor{red}{ax} + \textcolor{red}{b}) a dx = \frac{1}{a} \int f(\underbrace{\textcolor{red}{ax} + \textcolor{red}{b}}_{\textcolor{red}{u}}) \underbrace{a dx}_{u' dx} \\ &= \frac{1}{a} \int f(u) du = \frac{1}{a} [F(u) + C_1] = \frac{1}{a} F(ax + b) + C, \end{aligned}$$

where $C = C_1/a$ is again an arbitrary constant.

Rules of Thumb for Substitution I (page 4–14, Chapter 2)

- If there is a power (fraction) in the integrand, let the base (denominator) be the new variable; for example^a:

$$\begin{aligned}\int \frac{x+1}{x^2+2x+7} dx &= \int \frac{1}{\textcolor{red}{x}^2 + \textcolor{red}{2}x + 7} \cdot (x+1) dx \\ &= \int \frac{1}{\underbrace{\textcolor{red}{x}^2 + \textcolor{red}{2}x + 7}_u} \cdot \underbrace{(x+1) dx}_{\frac{1}{2}u' dx} = \int \frac{1}{u} \cdot \frac{1}{2} du,\end{aligned}$$

$$\begin{aligned}\int x^5 \sqrt{x^2+3} dx &= \int x^4 (\textcolor{red}{x}^2 + \textcolor{red}{3})^{1/2} \cdot x dx \\ &= \int \underbrace{(x^2)_{u-3}}^2 \underbrace{(\textcolor{red}{x}^2 + \textcolor{red}{3})}_u^{1/2} \cdot \underbrace{x dx}_{\frac{1}{2}u' dx} = \int (u-3)^2 u^{1/2} \cdot \frac{1}{2} du.\end{aligned}$$

^aSee Example 6 on page 14 and Example 2 on page 6.

Rules of Thumb for Substitution II (page 4–14, Chapter 2)

- If there is a trigonometric function in the integrand, let its argument be the new variable; for example^a:

$$\begin{aligned}
 \int x^2 \cos(x^3 + 1) dx &= \int \cos(\underbrace{x^3 + 1}_u) \cdot \underbrace{x^2 dx}_{\frac{1}{3}u' dx} \\
 &= \int \cos(\underbrace{x^3 + 1}_u) \cdot \underbrace{x^2 dx}_{\frac{1}{3}u' dx} = \int \cos u \cdot \frac{1}{3} du, \\
 \int_1^4 \frac{\sin(1/x)}{x^2} dx &= \int_1^4 \sin(\underbrace{1/x}_u) \cdot \frac{1}{x^2} dx \\
 &= \int_1^4 \sin(\underbrace{1/x}_u) \cdot \underbrace{\frac{1}{x^2} dx}_{-u' dx} = \int_{1/1}^{1/4} \sin u \cdot (-du).
 \end{aligned}$$

^aSee Example 1 on page 4 and Example 3 on page 9.

Rules of Thumb for Substitution III (page 4–14, Chapter 2)

- If there is an exponential function in the integrand, let its exponent be the new variable; for example^a:

$$\begin{aligned}\int \frac{e^{1+(1/x^2)}}{x^3} dx &= \int \exp[1 + (1/x^2)] \cdot \frac{1}{x^3} dx \\ &= \int \exp[\underbrace{1 + (1/x^2)}_u] \cdot \underbrace{\frac{1}{x^3} dx}_{-\frac{1}{2}u' dx} = \int e^u \cdot \left(-\frac{1}{2} du\right),\end{aligned}$$

$$\begin{aligned}\int_1^2 x e^{x^2-1} dx &= \int_1^2 \exp(x^2 - 1) \cdot x dx \\ &= \int_1^2 \exp(\underbrace{x^2 - 1}_u) \cdot \underbrace{x dx}_{\frac{1}{2}u' dx} = \int_{1^2-1}^{2^2-1} e^u \cdot \frac{1}{2} du.\end{aligned}$$

^aSee Problem 1 (a)(g) in Homework 2.

The Key to Success I (page 11–19, Chapter 2)

- The key for substitution to succeed is that, after identifying the new variable u as part of the integrand, the rest of the integrand must be expressible **as a product** $g(u)u'$ for some function g .
- **Example.**^a As an example, consider the integral

$$I = \int_0^{\log 2} \frac{e^{2x}}{\sqrt{e^x + 1}} dx = \int_0^{\log 2} (e^x + 1)^{-1/2} e^{2x} dx.$$

This integral can be evaluated using substitution in three steps:

- (a) identify the “trouble maker” $(e^x + 1)^{-1/2}$;
- (b) try a substitution $u = e^x + 1$;

^aSee Example 4 on page 11.

The Key to Success II (page 11–19, Chapter 2)

(c) rewrite the rest of the integrand e^{2x} as a product of $u' = e^x$ and some function of $u = e^x + 1$; the result is

$$I = \int_0^{\log 2} \underbrace{(e^x + 1)}_u^{-1/2} \underbrace{e^x}_{u-1} \cdot \underbrace{e^x dx}_{u' dx} = \int_{e^0+1}^{e^{\log 2}+1} u^{-1/2} (u-1) du.$$

Note that in step (b) above, one may also try the substitution $u = (e^x + 1)^{-1/2}$. It works, but the computation is somewhat more involved:

$$u' = -\frac{1}{2} (e^x + 1)^{-3/2} e^x = -\frac{1}{2} u^3 (u^{-2} - 1),$$

$$e^{2x} dx = (e^x)^2 \frac{1}{u'} \cdot u' dx = -2(u^{-5} - u^{-3}) du,$$

$$I = \int_0^{\log 2} \underbrace{(e^x + 1)^{-1/2}}_u e^{2x} dx = -2 \int_{1/\sqrt{2}}^{1/\sqrt{3}} u(u^{-5} - u^{-3}) du.$$

The Key to Success III (page 11–19, Chapter 2)

- **Example.^a** As another example, consider the integral

$$I = \int \frac{2x - 5}{x^2 - 6x + 10} dx.$$

Let's first try evaluating this integral using the same three steps:

- (a) identify the “trouble maker” $1/(x^2 - 6x + 10)$;
- (b) try a substitution $u = x^2 - 6x + 10$;
- (c) rewrite the rest of the integrand $2x - 5$ as a product of $u' = 2x - 6$ and some function of $u = x^2 - 6x + 10$; unfortunately, this does not seem possible.

^aSee Example 7 on page 15.

The Key to Success IV (page 11–19, Chapter 2)

To proceed, observe that while $2x - 5$ cannot be represented as a **product** of $u' = 2x - 6$ and a function of u , it can be written as a **sum** of u' and a constant function:

$$2x - 5 = (2x - 6) + 1 = u' + 1.$$

This motivates the decomposition

$$I = \int \frac{2x - 6}{x^2 - 6x + 10} dx + \int \frac{1}{x^2 - 6x + 10} dx =: I_1 + I_2,$$

where the first integral can now be evaluated by substitution:

$$I_1 = \int \frac{1}{\underbrace{x^2 - 6x + 10}_u} \cdot \underbrace{(2x - 6) dx}_{u' dx} = \int \frac{1}{u} du.$$

The evaluation of the second integral is standard.

General Procedure for Substitution I (page 4–19, Chapter 2)

- Besides the procedure outlined above, there is an alternative, more general way of applying substitution:

(a) suppose a substitution $u = g(x)$ is made for an integral $\int f(x) dx$;

(b) the idea is to solve for x , the old variable, in terms of u , the new variable, from the substitution equation:

$$u = g(x) \implies x = h(u), \quad \text{e.g. } u = \frac{1}{x} \implies x = \frac{1}{u};$$

(c) then compute dx in terms of u and du :

$$x = h(u) \implies dx = h'(u) du, \quad \text{e.g. } x = \frac{1}{u} \implies dx = -\frac{1}{u^2} du;$$

(d) and rewrite the original integral in terms of the new variable:

$$\int f(x) dx \implies \int f(h(u))h'(u) du.$$

General Procedure for Substitution II (page 4–19, Chapter 2)

- **Example.^a** As an example, consider again the integral

$$I = \int_0^{\log 2} \frac{e^{2x}}{\sqrt{e^x + 1}} dx = \int_0^{\log 2} (e^x + 1)^{-1/2} e^{2x} dx.$$

After making the substitution $u = e^x + 1$, the integral can be evaluated in the following way:

- (a) solve for x in terms of u from the substitution, which yields

$$x = \log(u - 1);$$

- (b) compute dx in terms of u and du , which yields

$$dx = \frac{1}{u - 1} du;$$

^aSee Example 4 on page 11.

General Procedure for Substitution III (page 4–19, Chapter 2)

(c) rewrite the original integral in terms of the new variable:

$$\begin{aligned} \int_0^{\log 2} (e^x + 1)^{-1/2} e^{2x} dx &= \int_{e^0+1}^{e^{\log 2}+1} u^{-1/2} \underbrace{(u-1)^2}_{e^{2x}=(e^x)^2} \underbrace{\frac{1}{u-1} du}_{dx} \\ &= \int_2^3 u^{-1/2} (u-1) du, \end{aligned}$$

which yields the same integral as before.

Note that this procedure, though very general, works only for those problems for which the substitution equation $u = g(x)$ can be easily inverted. For problems involving complicated substitutions such as $u = x^3 + 3x + 1$ for which an inverse is difficult to obtain, the above procedure will not be applicable.

Integration by Parts I (page 38–39, Chapter 2)

- **Integration by parts** is a technique for simplifying integrals of the form

$$\int f(x)g(x) dx;$$

it is useful when

- (a) $f(x)$ can be differentiated repeatedly and $g(x)$ can be integrated repeatedly without difficulty, for example,

$$\int xe^x dx, \quad \text{where} \quad f(x) = x, g(x) = e^x;$$

- (b) each part of the integrand appears again after repeated differentiation or integration, for example,

$$\int e^x \sin x dx, \quad \text{where} \quad f(x) = e^x, g(x) = \sin x.$$

Integration by Parts II (page 38–39, Chapter 2)

- The method is essentially the product rule in integral form:

(a) if $u(x)$ and $v(x)$ are differentiable functions of x , then

$$[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x);$$

(b) integrating with respect to x , the equation becomes

$$\begin{aligned}\int [u(x)v(x)]' dx &= \int [u'(x)v(x) + u(x)v'(x)] dx \\ u(x)v(x) &= \int v(x)u'(x) dx + \int u(x)v'(x) dx,\end{aligned}$$

or in a more compact form,

$$\int u dv = uv - \int v du.$$

- A similar formula can be derived for definite integrals.

The ILATE Rule I (page 42–44, Chapter 2)

- A rule of thumb for the choice of u and dv in the integration by parts formula is to let u be whichever function that comes first in the **ILATE list**:

I : inverse trigonometric functions: $\tan^{-1} x$, $\sin^{-1} x$, etc.;

L : logarithmic functions: $\log x$, etc.;

A : algebraic functions: x^2 , $3x^5$, etc.;

T : trigonometric functions: $\sin x$, $\tan x$, etc.;

E : exponential functions: e^x , etc.;

and then let dv be the rest of the integrand (including dx).

The ILATE Rule II (page 45–51, Chapter 2)

- **Example.^a** As an example, consider the integral

$$I = \int x^4 \log x \, dx.$$

The ILATE rule suggests the choice

$$u = \log x, \quad dv = x^4 \, dx,$$

so integrating by parts yields

$$\begin{aligned} I &= \int \underbrace{\log x}_u \cdot \underbrace{x^4 \, dx}_{dv} = \int \underbrace{\log x}_u \, d\left(\underbrace{\frac{1}{5} x^5}_v\right) \\ &= \frac{1}{5} x^5 \log x - \int \frac{1}{5} x^5 \, d(\log x) = \frac{1}{5} x^5 \log x - \frac{1}{5} \int x^4 \, dx. \end{aligned}$$

^aSee Example 15 on page 45.

The ILATE Rule III (page 45–51, Chapter 2)

- **Example.^a** As another example, consider the integral

$$I = \int_0^{\pi/2} x \sin x \, dx.$$

The ILATE rule suggests the choice

$$u = x, \quad dv = \sin x \, dx,$$

so integrating by parts yields

$$\begin{aligned} I &= \int_0^{\pi/2} \underbrace{x}_u \underbrace{\sin x \, dx}_{dv} = \int_0^{\pi/2} \underbrace{x}_u d(\underbrace{-\cos x}_v) \\ &= -x \cos x \Big|_0^{\pi/2} - \int_0^{\pi/2} (-\cos x) \, dx. \end{aligned}$$

^aSee Example 18 on page 51.

Repeated Integration by Parts I (page 49–54, Chapter 2)

- **Example.**^a Sometimes we have to use integration by parts more than once. For example, consider the integral

$$I = \int x^2 e^{3x} dx.$$

- (a) The ILATE rule suggests the choice

$$u = x^2, \quad dv = e^{3x} dx,$$

so integrating by parts yields

$$\begin{aligned} I &= \int \underbrace{x^2}_u \underbrace{e^{3x} dx}_{dv} = \int \underbrace{x^2}_u d \left(\underbrace{\frac{1}{3} e^{3x}}_v \right) \\ &= \frac{1}{3} x^2 e^{3x} - \int \frac{1}{3} e^{3x} d(x^2) = \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \int x e^{3x} dx; \end{aligned}$$

^aSee Example 17 on page 49.

Repeated Integration by Parts II (page 49–54, Chapter 2)

- (b) to evaluate the last integral, we apply integration by parts one more time with

$$u = x, \quad dv = e^{3x} dx;$$

the result is

$$\int \underbrace{x}_u \underbrace{e^{3x} dx}_{dv} = \int \underbrace{x}_u d \left(\underbrace{\frac{1}{3} e^{3x}}_v \right) = \frac{1}{3} x e^{3x} - \int \frac{1}{3} e^{3x} dx;$$

- (c) substituting the result in (b) back into (a) then yields

$$I = \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \left(\frac{1}{3} x e^{3x} - \frac{1}{3} \int e^{3x} dx \right).$$

Repeated Integration by Parts III (page 49–54, Chapter 2)

- **Example.^a** As another example, consider the integral

$$I = \int e^{3x} \sin x \, dx.$$

- (a) The ILATE rule suggests the choice

$$u = \sin x, \quad dv = e^{3x} \, dx,$$

so integrating by parts yields

$$\begin{aligned} I &= \int \underbrace{\sin x}_u \cdot \underbrace{e^{3x} \, dx}_{dv} = \int \underbrace{\sin x}_u \, d\left(\underbrace{\frac{1}{3} e^{3x}}_v\right) \\ &= \frac{1}{3} e^{3x} \sin x - \frac{1}{3} \int e^{3x} \cos x \, dx; \end{aligned}$$

^aSee Example 19 on page 52.

Repeated Integration by Parts IV (page 49–54, Chapter 2)

(b) to evaluate the last integral, we apply integration by parts one more time with

$$u = \cos x, \quad dv = e^{3x} dx;$$

the result is

$$\begin{aligned} \int \underbrace{\cos x}_u \cdot \underbrace{e^{3x} dx}_{dv} &= \int \underbrace{\cos x}_u d \left(\underbrace{\frac{1}{3} e^{3x}}_v \right) \\ &= \frac{1}{3} e^{3x} \cos x + \frac{1}{3} \int e^{3x} \sin x dx = \frac{1}{3} e^{3x} \cos x + \frac{1}{3} I; \end{aligned}$$

(c) substituting the result in (b) back into (a) then yields

$$I = \frac{1}{3} e^{3x} \sin x - \frac{1}{3} \left(\frac{1}{3} e^{3x} \cos x + \frac{1}{3} I \right).$$

Integration by Parts with Substitution I (page 58–64, Chapter 2)

- **Example.**^a Sometimes we use a substitution before integration by parts to simplify the computation. For example, consider the integral

$$I = \int x \log(2\sqrt{x} + 1) dx.$$

Let's first try evaluating this integral using integration by parts:

- (a) the ILATE rule suggests the choice

$$u = \log(2\sqrt{x} + 1), \quad dv = x dx,$$

so integrating by parts yields

$$\begin{aligned} I &= \int \underbrace{\log(2\sqrt{x} + 1)}_u \cdot \underbrace{x dx}_{dv} = \int \underbrace{\log(2\sqrt{x} + 1)}_u d\left(\underbrace{\frac{1}{2} x^2}_v\right) \\ &= \frac{1}{2} x^2 \log(2\sqrt{x} + 1) - \frac{1}{2} \int \frac{x\sqrt{x}}{2\sqrt{x} + 1} dx; \end{aligned}$$

^aSee Example 21 on page 58.

Integration by Parts with Substitution II (page 58–64, Chapter 2)

(b) to evaluate the last integral, we

- (i) identify the “trouble maker” $1/(2\sqrt{x} + 1)$;
- (ii) try a substitution $w = 2\sqrt{x} + 1$;
- (iii) rewrite the rest of the integrand $x\sqrt{x}$ as a product of $w' = 1/\sqrt{x}$ and some function of $w = 2\sqrt{x} + 1$; the result is

$$\begin{aligned} \int \frac{x\sqrt{x}}{2\sqrt{x} + 1} dx &= \int \frac{1}{2\sqrt{x} + 1} x^2 \cdot \frac{1}{\sqrt{x}} dx \\ &= \int \frac{1}{\underbrace{2\sqrt{x} + 1}_w} \left(\underbrace{\sqrt{x}}_{\frac{1}{2}(w-1)} \right)^4 \cdot \underbrace{\frac{1}{\sqrt{x}} dx}_{w' dx} = \frac{1}{16} \int \frac{1}{w} (w - 1)^4 dw; \end{aligned}$$

(c) substituting the result in (b) back into (a) then yields

$$I = \frac{1}{2} x^2 \log(2\sqrt{x} + 1) - \frac{1}{32} \int \frac{1}{w} (w - 1)^4 dw.$$

Integration by Parts with Substitution III (page 58–64, Chapter 2)

An alternative way to proceed is to introduce the substitution $w = 2\sqrt{x} + 1$ first to simplify the integral:

$$\begin{aligned}
 I &= \int \log(\underbrace{2\sqrt{x} + 1}_w) \cdot x\sqrt{x} \cdot \frac{1}{\sqrt{x}} dx \\
 &= \int \log(\underbrace{2\sqrt{x} + 1}_w) \cdot \underbrace{\left(\sqrt{x}\right)^3}_{\frac{1}{2}(w-1)} \cdot \underbrace{\frac{1}{\sqrt{x}} dx}_{w' dx} = \frac{1}{8} \int \log w \cdot (w - 1)^3 dw.
 \end{aligned}$$

Then integrating by parts yields (the same result):

$$\begin{aligned}
 I &= \frac{1}{8} \int \underbrace{\log w}_u \cdot \underbrace{(w - 1)^3 dw}_{dv} = \frac{1}{8} \int \underbrace{\log w}_u d \left(\underbrace{\frac{1}{4} (w - 1)^4}_v \right) \\
 &= \frac{1}{32} (w - 1)^4 \log w - \frac{1}{32} \int \frac{1}{w} (w - 1)^4 dw.
 \end{aligned}$$

Integration by Parts with Substitution IV (page 58–64, Chapter 2)

- **Example.**^a As another example, consider the integral

$$I = \int_1^2 \frac{e^{1/x}}{x^3} dx.$$

Let's first try evaluating this integral using integration by parts:

- (a) at the first glance, the ILATE rule seems to suggest the choice

$$u = \frac{1}{x^3}, \quad dv = e^{1/x} dx; \quad (*)$$

the problem of this choice is that the integral of $e^{1/x}$, unlike that of e^x , cannot be easily found;

^aSee Example 22 on page 60.

Integration by Parts with Substitution V (page 58–64, Chapter 2)

(b) to remedy this situation, observe that

$$(e^{1/x})' = e^{1/x} \cdot \left(\frac{1}{x}\right)' = -\frac{1}{x^2} e^{1/x};$$

so instead of (*), we try the choice

$$u = -\frac{1}{x}, \quad dv = -\frac{1}{x^2} e^{1/x} dx;$$

then integrating by parts yields

$$\begin{aligned} I &= \int_1^2 \underbrace{\left(-\frac{1}{x}\right)}_u \cdot \underbrace{\left(-\frac{1}{x^2} e^{1/x} dx\right)}_{dv} = \int_1^2 \underbrace{\left(-\frac{1}{x}\right)}_u d(\underbrace{e^{1/x}}_v) \\ &= -\frac{1}{x} e^{1/x} \Big|_1^2 - \int_1^2 e^{1/x} d\left(-\frac{1}{x}\right) = -\frac{1}{x} e^{1/x} \Big|_1^2 + \int_1^2 1 d(e^{1/x}). \end{aligned}$$

Integration by Parts with Substitution VI (page 58–64, Chapter 2)

An alternative way to proceed is to introduce the substitution $w = 1/x$ first to simplify the integral:

$$\begin{aligned} I &= \int_1^2 \frac{1}{x} \exp(\textcolor{red}{1}/\textcolor{red}{x}) \cdot \frac{1}{x^2} dx \\ &= \int_1^2 \underbrace{\frac{1}{x}}_w \exp(\underbrace{\textcolor{red}{1}/\textcolor{red}{x}}_w) \cdot \underbrace{\frac{1}{x^2} dx}_{-w' dx} = \int_{1/1}^{\textcolor{red}{1/2}} w e^w \cdot (-dw). \end{aligned}$$

Then integrating by parts yields (the same result):

$$\begin{aligned} I &= - \int_1^{1/2} w e^w dw = \int_{1/2}^1 \underbrace{w}_u \underbrace{e^w dw}_{dv} \\ &= \int_{1/2}^1 \underbrace{w}_u d(\underbrace{e^w}_v) = w e^w \Big|_{1/2}^1 - \int_{1/2}^1 e^w dw. \end{aligned}$$

Tabular Integration I (page 67–71, Chapter 2)

- **Tabular integration** is a technique useful in evaluating integrals of the form

$$\int x^n e^x dx, \quad \int x^n \sin x dx, \quad \int x^n \cos x dx,$$

where repeated applications of integration by parts are needed, and where the derivatives of one part of the integrand (x^n) and the integrals of the other part of the integrand (e^x , $\sin x$, $\cos x$) remain simple throughout the computation^a.

^aIt complements the reduction method introduced on page 67–89 of Chapter 2.

Tabular Integration II (page 67–71, Chapter 2)

- **Example.** As an example, consider the integral

$$I = \int x^2 e^x dx.$$

With $f(x) = x^2$ and $g(x) = e^x$, we make the list

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	(+)	e^x
$2x$	(-)	e^x
2	(+)	e^x
0		e^x

Combining the products of the functions connected by the arrows according to the signs above the arrows then yields

$$I = x^2 e^x - 2x e^x + 2e^x + C.$$

Tabular Integration III (page 67–71, Chapter 2)

- **Example.** As another example, consider the integral

$$I = \int x^3 \sin x \, dx.$$

With $f(x) = x^3$ and $g(x) = \sin x$, we make the list

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	(+)	$\sin x$
$3x^2$	(-)	$-\cos x$
$6x$	(+)	$-\sin x$
6	(-)	$\cos x$
0		$\sin x$

Combining the products of the functions connected by the arrows according to the signs above the arrows then yields

$$I = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

Reduction Method with Substitution I (page 78–80, Chapter 2)

- **Example.**^a Since reduction formulas are derived using integration by parts, it is possible in some situations to use a substitution to simplify the computation. For example, consider the integral

$$I_n = \int x^n e^{x^2+1} dx.$$

The substitution $w = x^2 + 1$ transforms the integral into

$$\begin{aligned} I_n &= \int x^{n-1} \exp(\color{red}{x^2 + 1}) \cdot x \, dx \\ &= \int \underbrace{(x^2)}_{w-1}^{(n-1)/2} \exp(\underbrace{\color{red}{x^2 + 1}}_w) \cdot \underbrace{x \, dx}_{\frac{1}{2}w' \, dx} = \frac{1}{2} \int (w-1)^{(n-1)/2} e^w \, dw. \end{aligned}$$

^aSee Example 27 on page 78.

Reduction Method with Substitution II (page 78–80, Chapter 2)

Integrating by parts with

$$u = (w - 1)^{(n-1)/2}, \quad dv = e^w dw,$$

then yields

$$\begin{aligned} I_n &= \frac{1}{2} \int \underbrace{(w - 1)^{(n-1)/2}}_u \underbrace{e^w dw}_{dv} = \frac{1}{2} \int \underbrace{(w - 1)^{(n-1)/2}}_u d(\underbrace{e^w}_v) \\ &= \frac{1}{2} (w - 1)^{(n-1)/2} e^w - \frac{n-1}{2} \cdot \frac{1}{2} \int (w - 1)^{(n-3)/2} e^w dw \\ &= \frac{1}{2} (w - 1)^{(n-1)/2} e^w - \frac{n-1}{2} I_{n-2}. \end{aligned}$$

Trigonometric Integrals I (page 22–23, Chapter 2)

- To evaluate integrals of the form

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative (positive or zero) integers, we consider three (nonexclusive) cases:

- (a) if the exponent of cosine is odd ($n = 2k + 1$), save one factor of cosine and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining power of cosine as a function of sine; then substitute $u = \sin x$:

$$\begin{aligned} \int \sin^m x \cos^{2k+1} x \, dx &= \int (\sin x)^m (\cos^2 x)^k \cos x \, dx \\ &= \int \underbrace{(\sin x)}_u^m (1 - \underbrace{\sin^2 x}_{u^2})^k \underbrace{\cos x \, dx}_{u' \, dx} = \int u^m (1 - u^2)^k \, du; \end{aligned}$$

Trigonometric Integrals II (page 22–23, Chapter 2)

- (b) if the exponent of sine is odd ($m = 2k + 1$), save one factor of sine and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining power of sine as a function of cosine; then substitute $u = \cos x$:

$$\begin{aligned} \int \sin^{2k+1} x \cos^n x \, dx &= \int (\sin^2 x)^k (\cos x)^n \sin x \, dx \\ &= \int (1 - \underbrace{\cos^2 x}_{u^2})^k (\underbrace{\cos x}_u)^n \underbrace{\sin x \, dx}_{-u' \, dx} = - \int (1 - u^2)^k u^n \, du; \end{aligned}$$

- (c) if the exponents of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \frac{1}{2} (1 - \cos 2x), \quad \cos^2 x = \frac{1}{2} (1 + \cos 2x),$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Trigonometric Integrals III (page 22–23, Chapter 2)

- **Example.^a** As an example, let's use the above rules to compute

$$\begin{aligned}
 \int \sin^5 x \, dx &= \int (\sin^2 x)^2 \sin x \, dx \\
 &= \int (1 - \underbrace{\cos^2 x}_{u^2})^2 \underbrace{\sin x \, dx}_{-u' \, dx} = - \int (1 - u^2)^2 \, du, \\
 \int \sin^2 x \cos^3 x \, dx &= \int (\sin x)^2 (\cos^2 x) \cos x \, dx \\
 &= \int (\underbrace{\sin x}_u)^2 (1 - \underbrace{\sin^2 x}_{u^2}) \underbrace{\cos x \, dx}_{u' \, dx} = \int u^2 (1 - u^2) \, du, \\
 \int \sin^2 x \cos^2 x \, dx &= \int \frac{1}{2} (1 - \cos 2x) \cdot \frac{1}{2} (1 + \cos 2x) \, dx \\
 &= \frac{1}{4} \int (1 - \cos^2 2x) \, dx = \frac{1}{4} x - \frac{1}{4} \int \frac{1}{2} (1 + \cos 4x) \, dx.
 \end{aligned}$$

^aSee Example 10 on page 22 and Example 11 on page 23.

Trigonometric Integrals IV (page 55, Chapter 2)

- To evaluate integrals of the form

$$\int \tan^m x \sec^n x dx,$$

where m and n are nonnegative (positive or zero) integers, we consider three (nonexclusive) cases:

- (a) if the exponent of secant is even ($n = 2k$, $k \geq 1$), save one factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining power of secant as a function of tangent; then substitute $u = \tan x$:

$$\begin{aligned} \int \tan^m x \sec^{2k} x dx &= \int (\tan x)^m (\sec^2 x)^{k-1} \sec^2 x dx \\ &= \int \underbrace{(\tan x)^m}_u (1 + \underbrace{\tan^2 x}_{u^2})^{k-1} \underbrace{\sec^2 x dx}_{u' dx} = \int u^m (1 + u^2)^{k-1} du; \end{aligned}$$

Trigonometric Integrals V (page 55, Chapter 2)

- (b) if the exponent of tangent is odd ($m = 2k + 1$), save one factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining power of tangent as a function of secant; then substitute $u = \sec x$:

$$\begin{aligned} \int \tan^{2k+1} x \sec^n x \, dx &= \int (\tan^2 x)^k (\sec x)^{n-1} \sec x \tan x \, dx \\ &= \int (\underbrace{\sec^2 x}_{u^2} - 1)^k (\underbrace{\sec x}_{u})^{n-1} \underbrace{\sec x \tan x \, dx}_{u' \, dx} = \int (u^2 - 1)^k u^{n-1} \, du; \end{aligned}$$

- (c) if the exponent of secant is zero or odd and the exponent of tangent is even, use the identities

$$\tan^2 x = \sec^2 x - 1, \quad \sec^2 x = \tan^2 x + 1,$$

and integrate by parts when necessary to reduce the higher powers to lower powers.

Trigonometric Integrals VI (page 55, Chapter 2)

- **Example.^a** As an example, let's use the above rules to compute

$$\begin{aligned}
 \int \sec^3 x \, dx &= \int \sec x \sec^2 x \, dx \\
 &= \int \underbrace{\sec x}_u d(\underbrace{\tan x}_v) = \sec x \tan x - \int \sec x \tan^2 x \, dx \\
 &= \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx,
 \end{aligned}$$

$$\begin{aligned}
 \int \tan^4 x \, dx &= \int \tan^2 x \tan^2 x \, dx \\
 &= \int \tan^2 x (\sec^2 x - 1) \, dx = \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\
 &= \int \underbrace{\tan^2 x}_{u^2} \underbrace{\sec^2 x \, dx}_{u' \, dx} - \int (\sec^2 x - 1) \, dx.
 \end{aligned}$$

^aSee Example 20 on page 55.

Trigonometric Integrals VII (page 22, Chapter 1)

- To evaluate integrals of the form

$$\int \sin mx \cos nx \, dx, \quad \int \sin mx \sin nx \, dx, \quad \int \cos mx \cos nx \, dx,$$

where m and n are arbitrary real numbers, we use the product-to-sum identities:

$$\sin mx \cos nx = \frac{1}{2} [\sin(m+n)x + \sin(m-n)x],$$

$$\sin mx \sin nx = -\frac{1}{2} [\cos(m+n)x - \cos(m-n)x],$$

$$\cos mx \cos nx = \frac{1}{2} [\cos(m+n)x + \cos(m-n)x].$$

Trigonometric Substitution I (page 24–26, Chapter 2)

- **Trigonometric substitutions** can be effective in transforming integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly.
- The most common substitutions are

$$x = a \sin \theta, \quad x = a \tan \theta, \quad \text{and} \quad x = a \sec \theta;$$

- (a) to eliminate $\sqrt{a^2 - x^2}$, set $x = a \sin \theta$ where $-\pi/2 \leq \theta \leq \pi/2$; then (note that $\cos \theta \geq 0$ when $-\pi/2 \leq \theta \leq \pi/2$)

$$\begin{aligned} \sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} \\ &= \sqrt{a^2(1 - \sin^2 \theta)} = \sqrt{a^2 \cos^2 \theta} = a \cos \theta; \end{aligned}$$

Trigonometric Substitution II (page 24–26, Chapter 2)

- (b) to eliminate $\sqrt{a^2 + x^2}$, set $x = a \tan \theta$ where $-\pi/2 < \theta < \pi/2$; then (note that $\sec \theta > 0$ when $-\pi/2 < \theta < \pi/2$)

$$\begin{aligned}\sqrt{a^2 + x^2} &= \sqrt{a^2 + a^2 \tan^2 \theta} \\ &= \sqrt{a^2(1 + \tan^2 \theta)} = \sqrt{a^2 \sec^2 \theta} = a \sec \theta;\end{aligned}$$

- (c) to eliminate $\sqrt{x^2 - a^2}$, set $x = a \sec \theta$ where $0 \leq \theta < \pi/2$; then (note that $\tan \theta \geq 0$ when $0 \leq \theta < \pi/2$)

$$\begin{aligned}\sqrt{x^2 - a^2} &= \sqrt{a^2 \sec^2 \theta - a^2} \\ &= \sqrt{a^2(\sec^2 \theta - 1)} = \sqrt{a^2 \tan^2 \theta} = a \tan \theta.\end{aligned}$$

Trigonometric Substitution III (page 27–33, Chapter 2)

- **Example.^a** As an example, consider the integral

$$I = \int \sqrt{3 - x^2} dx.$$

Let's first try evaluating this integral using standard substitution:

- (a) identify the “trouble maker” $\sqrt{3 - x^2} = (3 - x^2)^{1/2}$;
- (b) try a substitution $u = 3 - x^2$;
- (c) rewrite the rest of the integrand 1 as a product of $u' = -2x$ and some function of $u = 3 - x^2$; unfortunately, the resulting integral still seems too complicated to be evaluated directly:

$$1 dx = \frac{1}{u'} u' dx = -\frac{1}{2\sqrt{3 - u}} du, \quad I = -\frac{1}{2} \int \frac{\sqrt{u}}{\sqrt{3 - u}} du.$$

^aSee Example 12 on page 27.

Trigonometric Substitution IV (page 27–33, Chapter 2)

To proceed, we make use of the trigonometric substitution

$$x = \sqrt{3} \sin \theta, \quad dx = \sqrt{3} \cos \theta \, d\theta.$$

This leads to the transformed integral

$$\begin{aligned} I &= \int \sqrt{3 - 3 \sin^2 \theta} \cdot \sqrt{3} \cos \theta \, d\theta = \int \sqrt{3} \cos \theta \cdot \sqrt{3} \cos \theta \, d\theta \\ &= 3 \int \cos^2 \theta \, d\theta = 3 \int \frac{1}{2} [1 + \cos 2\theta] \, d\theta. \end{aligned}$$

Trigonometric Substitution V (page 27–33, Chapter 2)

- **Example.^a** As another example, consider the integral

$$I = \int \frac{1}{x^3 \sqrt{x^2 - 1}} dx.$$

Let's first try evaluating this integral using standard substitution:

- (a) identify the “trouble maker” $1/\sqrt{x^2 - 1} = (x^2 - 1)^{-1/2}$;
- (b) try a substitution $u = x^2 - 1$;
- (c) rewrite the rest of the integrand $1/x^3$ as a product of $u' = 2x$ and some function of $u = x^2 - 1$; unfortunately, the resulting integral still seems too complicated to be evaluated directly:

$$\frac{1}{x^3} dx = \frac{1}{2(u + 1)^2} du, \quad I = \frac{1}{2} \int \frac{1}{(u + 1)^2 \sqrt{u}} du.$$

^aSee Example 13 on page 30.

Trigonometric Substitution VI (page 27–33, Chapter 2)

To proceed, we make use of the trigonometric substitution

$$x = \sec \theta, \quad dx = \sec \theta \tan \theta \, d\theta.$$

This leads to the transformed integral

$$\begin{aligned} I &= \int \frac{1}{\sec^3 \theta \sqrt{\sec^2 \theta - 1}} \cdot \sec \theta \tan \theta \, d\theta \\ &= \int \frac{1}{\sec^3 \theta \tan \theta} \cdot \sec \theta \tan \theta \, d\theta = \int \frac{1}{\sec^2 \theta} \, d\theta \\ &= \int \cos^2 \theta \, d\theta = \int \frac{1}{2} [1 + \cos 2\theta] \, d\theta. \end{aligned}$$

Trigonometric Substitution VII (page 35–36, Chapter 2)

- **Example.^a** Trigonometric substitutions can also be useful in evaluating integrals containing integer powers of quadratic polynomials. For example, consider the integral

$$I = \int \frac{1}{(1 + x^2)^2} dx.$$

Let's first try evaluating this integral using standard substitution:

- (a) identify the “trouble maker” $1/(1 + x^2)^2 = (1 + x^2)^{-2}$;
- (b) try a substitution $u = 1 + x^2$;

^aSee the Example on page 35.

Trigonometric Substitution VIII (page 35–36, Chapter 2)

- (c) rewrite the rest of the integrand 1 as a product of $u' = 2x$ and some function of $u = 1 + x^2$; unfortunately, the resulting integral still seems too complicated to be evaluated directly:

$$1 \, dx = \frac{1}{u'} u' \, dx = \frac{1}{2\sqrt{u-1}} \, du, \quad I = \frac{1}{2} \int \frac{1}{u^2 \sqrt{u-1}} \, du.$$

To proceed, we make use of the trigonometric substitution

$$x = \tan \theta, \quad dx = \sec^2 \theta \, d\theta.$$

This leads to the transformed integral

$$\begin{aligned} I &= \int \frac{1}{(1 + \tan^2 \theta)^2} \cdot \sec^2 \theta \, d\theta = \int \frac{1}{(\sec^2 \theta)^2} \cdot \sec^2 \theta \, d\theta \\ &= \int \frac{1}{\sec^2 \theta} \, d\theta = \int \cos^2 \theta \, d\theta = \int \frac{1}{2} [1 + \cos 2\theta] \, d\theta. \end{aligned}$$

Trigonometric Substitution IX (page 45 & 56, Chapter 3)

- **Example.**^a When applying trigonometric substitution to **definite integrals**, it is often advantageous to write the result as a function of the **original** variable and substitute in the **original** integration limits. For example, consider the integral

$$I = \int_0^2 \sqrt{1 + 4x^2} \, dx.$$

The trigonometric substitution

$$x = \frac{1}{2} \tan \theta, \quad dx = \frac{1}{2} \sec^2 \theta \, d\theta,$$

^aSee Example 19 and 22 on page 45 and 56 of Chapter 3.

Trigonometric Substitution X (page 45 & 56, Chapter 3)

leads to the transformed integral

$$\begin{aligned} I &= \int_0^{\tan^{-1} 4} \sqrt{1 + \tan^2 \theta} \cdot \frac{1}{2} \sec^2 \theta d\theta = \frac{1}{2} \int_0^{\tan^{-1} 4} \sec^3 \theta d\theta \\ &= \frac{1}{4} (\sec \theta \tan \theta + \log |\sec \theta + \tan \theta|)_0^{\tan^{-1} 4}. \end{aligned}$$

Instead of substituting in the new limits (0 and $\tan^{-1} 4$) for θ , we transform the result back to x and use the original limits (0 and 2) for x :

$$I = \frac{1}{4} \left(\underbrace{2x}_{\tan \theta} \underbrace{\sqrt{1 + 4x^2}}_{\sec \theta} + \log |2x + \sqrt{1 + 4x^2}| \right)_0^2.$$

The resulting expression is much easier to evaluate.

Matching Integrals to Basic Formulas I (page 15–32, Chapter 2)

- We summarize below a few algebraic techniques that are useful in matching a given integral to basic formulas:
 - (a) completing the square; for example^a:

$$\begin{aligned}\int \frac{1}{x^2 - 6x + 10} dx &= \int \frac{1}{(x - 3)^2 + 1} dx, \\ \int \frac{5}{2x^2 + 8x + 13} dx &= \int \frac{1}{\frac{2}{5}(x + 2)^2 + 1} dx, \\ \int \frac{1}{\sqrt{x^2 + 4x + 7}} dx &= \int \frac{1}{\sqrt{(x + 2)^2 + 3}} dx;\end{aligned}$$

^aSee Example 7 on page 15, Example 8 on page 18, and Example 14 on page 32.

Matching Integrals to Basic Formulas II (page 69, Chapter 3)

(b) eliminating a square root; for example^a:

$$\begin{aligned}\int_0^{2\pi} (1 - \cos x)^{3/2} dx &= \int_0^{2\pi} \left(2 \sin^2 \frac{x}{2}\right)^{3/2} dx \\ &= 2\sqrt{2} \int_0^{2\pi} \left|\sin^3 \frac{x}{2}\right| dx = 2\sqrt{2} \int_0^{2\pi} \sin^3 \frac{x}{2} dx, \\ \int_0^{\pi/4} \sqrt{1 + \cos 4x} dx &= \int_0^{\pi/4} \sqrt{2 \cos^2 2x} dx \\ &= \sqrt{2} \int_0^{\pi/4} |\cos 2x| dx = \sqrt{2} \int_0^{\pi/4} \cos 2x dx;\end{aligned}$$

^aSee Example 27 on page 69 of Chapter 3.

Matching Integrals to Basic Formulas III (page 47, Chapter 2)

(c) reducing an “improper” fraction; for example^a:

$$\begin{aligned}\int \frac{x^2}{1+x^2} dx &= \int 1 dx - \int \frac{1}{1+x^2} dx, \\ \int \frac{2x+3}{(x-4)^4} dx &= \int \frac{2}{(x-4)^3} dx + \int \frac{11}{(x-4)^4} dx, \\ \int \frac{4x+5}{\sqrt{2x+1}} dx &= \int 2\sqrt{2x+1} dx + \int \frac{3}{\sqrt{2x+1}} dx;\end{aligned}$$

^aSee Example 16 on page 47, Example 10 on page 20 of Chapter 1, and Review Example 3 on page 51 of Chapter 1.

Matching Integrals to Basic Formulas IV (page 15–32, Chapter 2)

(d) separating a fraction; for example^a:

$$\int \frac{2x-5}{\underbrace{x^2-6x+10}_{u'=2x-6}} dx = \int \frac{2x-6}{x^2-6x+10} dx + \int \frac{1}{x^2-6x+10} dx,$$

$$\int \frac{3x+11}{\underbrace{2x^2+8x+13}_{u'=4x+8}} dx = \int \frac{3x+6}{2x^2+8x+13} dx + \int \frac{5}{2x^2+8x+13} dx,$$

$$\int \frac{3x+2}{\underbrace{\sqrt{1-x^2}}_{u'=-2x}} dx = \int \frac{3x}{\sqrt{1-x^2}} dx + \int \frac{2}{\sqrt{1-x^2}} dx;$$

^aSee Example 7 on page 15 and Example 8 on page 18.

Matching Integrals to Basic Formulas V (page 15–32, Chapter 2)

(e) multiplying by a form of 1; for example^a:

$$\begin{aligned}\int \sec x \, dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx \\ &= \int \frac{1}{\underbrace{\sec x + \tan x}_u} \underbrace{(\sec^2 x + \sec x \tan x) \, dx}_{u' \, dx} = \int \frac{1}{u} \, du,\end{aligned}$$

$$\begin{aligned}\int \frac{x(x-3)}{\sqrt{x}-\sqrt{3}} \, dx &= \int \frac{x(x-3)}{\sqrt{x}-\sqrt{3}} \cdot \frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}} \, dx \\ &= \int \frac{x(x-3)}{x-3} \cdot (\sqrt{x}+\sqrt{3}) \, dx = \int x(\sqrt{x}+\sqrt{3}) \, dx.\end{aligned}$$

^aSee Example 9 on page 20 and Example 11 on page 21 of Chapter 1.

Method of Partial Fractions I (page 90–91, Chapter 2)

- The **method of partial fractions** is a technique for rewriting rational functions (quotients of polynomials) as a sum of simpler fractions, the so-called **partial fractions**.
- The general procedure for finding the partial fractions of a rational function $f(x) = p(x)/q(x)$ can be summarized as follows:
 - (a) if $f(x)$ is improper, that is, if $p(x)$ is of degree at least that of $q(x)$, divide $p(x)$ by $q(x)$ to obtain

$$f(x) = \text{quotient (a polynomial)} + \frac{N(x)}{D(x)};$$

- (b) factor $D(x)$ into a product of linear and irreducible^a quadratic factors with **real** coefficients;

^aA quadratic polynomial is said to be **irreducible** if it cannot be written as the product of two linear factors with **real** coefficients.

Method of Partial Fractions II (page 90–91, Chapter 2)

- (c) for each factor of the form $(ax + b)^k$, expect the decomposition to have the terms

$$\frac{A_1}{(ax + b)} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k};$$

- (d) for each factor of the form $(ax^2 + bx + c)^m$, expect the decomposition to have the terms

$$\frac{B_1x + C_1}{(ax^2 + bx + c)} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_mx + C_m}{(ax^2 + bx + c)^m};$$

Method of Partial Fractions III (page 90–91, Chapter 2)

- (e) set $N(x)/D(x)$ equal to the sum of all the terms found in (c)–(d);
- (f) multiply both sides of the equation found in (e) by $D(x)$ and solve for the unknown constants; this can be done by either:
 - (i) equating coefficients of like powers (most general);
 - (ii) assigning convenient values to the variables x (most effective for simple linear factors $(ax + b)$); or
 - (iii) using differentiation (most effective for repeated linear factors).

Method of Partial Fractions IV (page 97, Chapter 2)

- **Example.**^a As an example, consider the integral

$$I = \int \frac{5x^3 - 12x^2 + 5x - 4}{(2x + 1)(x - 1)^3} dx.$$

The partial fractions of the integrand are found as follows:

- (a) check the integrand, denoted by $p(x)/q(x)$, is proper:

$$\deg p(x) = 3 < \deg q(x) = 4;$$

- (b) make the ansatz

$$\frac{5x^3 - 12x^2 + 5x - 4}{(2x + 1)(x - 1)^3} = \frac{A}{2x + 1} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2} + \frac{D}{(x - 1)^3};$$

^aSee Example 32 on page 97.

Method of Partial Fractions V (page 97, Chapter 2)

(c) multiply both sides by $(2x + 1)(x - 1)^3$ to obtain

$$\begin{aligned} 5x^3 - 12x^2 + 5x - 4 &= A(x - 1)^3 \\ &+ B(x - 1)^2(2x + 1) + C(x - 1)(2x + 1) + D(2x + 1); \quad (*) \end{aligned}$$

(d) determine the coefficients A and D by setting x to $-\frac{1}{2}$ and 1 (the roots of the linear factors) in equation (*):

$$\begin{aligned} x = -\frac{1}{2} : 5\left(-\frac{1}{8}\right) - 12\left(\frac{1}{4}\right) + 5\left(-\frac{1}{2}\right) - 4 &= A\left(-\frac{3}{2}\right)^3, \\ x = 1 : 5(1) - 12(1) + 5(1) - 4 &= D(3); \end{aligned}$$

the result is $A = 3$, $D = -2$;

Method of Partial Fractions VI (page 97, Chapter 2)

(e) determine the coefficients B and C by differentiating equation (*)

$$15x^2 - 24x + 5 = 3(3)(x - 1)^2 + B[2(x - 1)(2x + 1) + 2(x - 1)^2] + C[(2x + 1) + 2(x - 1)] + 2(-2),$$

and setting x to 1 (the multiple root) and -1 (a small value):

$$x = 1 : 15(1) - 24(1) + 5 = C(3) - 4,$$

$$x = -1 : 15(1) - 24(-1) + 5 = 3(3)(4) + B(12) + C(-5) - 4;$$

the result is $C = 0$, $B = 1$; the values of B and C can also be determined by comparing the coefficient of x^3 and that of x^0 (the constant term), as shown on page 98 of Chapter 2.

Method of Partial Fractions VII (Homework 2)

- **Example.**^a As another example, consider the integral

$$I = \int \frac{x^5 + 2x^4 - x + 2}{x^3 + 2x^2 - x - 2} dx.$$

The partial fractions of the integrand are found as follows:

- (a) since the integrand, denoted by $p(x)/q(x)$, is **improper**:

$$\deg p(x) = 5 > \deg q(x) = 3,$$

a long division (see below) is done to decompose $p(x)/q(x)$ as

$$\frac{p(x)}{q(x)} = x^2 + 1 + \frac{4}{x^3 + 2x^2 - x - 2};$$

^aSee Problem 16(h) in Homework 2.

Method of Partial Fractions VIII (Homework 2)

$$\begin{array}{r}
 \phantom{x^3 + 2x^2 - x - 2) } x^2 \\
 \hline
 x^3 + 2x^2 - x - 2) x^5 + 2x^4 \\
 \phantom{x^3 + 2x^2 - x - 2) } - x^5 - 2x^4 + x^3 + 2x^2 \\
 \hline
 \phantom{x^3 + 2x^2 - x - 2) } x^3 + 2x^2 - x + 2 \\
 \phantom{x^3 + 2x^2 - x - 2) } - x^3 - 2x^2 + x + 2 \\
 \hline
 \phantom{x^3 + 2x^2 - x - 2) } 4
 \end{array}$$

- (b) to find the factors of the polynomial $x^3 + 2x^2 - x - 2$, make use of the fact that if the coefficients a_i in a polynomial $r(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ are all **integers**, then every **integer root** of $r(x)$ is a factor of a_0 ;

Method of Partial Fractions IX (Homework 2)

- (c) for the polynomial $x^3 + 2x^2 - x - 2$, this means that its integer roots, if there are any, must be ± 1 or ± 2 ; direct verification shows that $x = \pm 1, -2$ are the roots of the polynomial, and hence

$$x^3 + 2x^2 - x - 2 = (x - 1)(x + 1)(x + 2);$$

alternatively, one may observe that

$$\begin{aligned} x^3 + 2x^2 - x - 2 &= (x^3 - x) + (2x^2 - 2) \\ &= x(x^2 - 1) + 2(x^2 - 1) = (x + 2)(x^2 - 1), \end{aligned}$$

which leads to the same factorization;

Method of Partial Fractions X (Homework 2)

(d) for the fractional part of the integrand, make the ansatz

$$\frac{4}{(x-1)(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2};$$

(e) multiplying both sides by $(x-1)(x+1)(x+2)$:

$$4 = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1),$$

and setting $x = 1$, -1 , and -2 , respectively then yields

$$A = \frac{2}{3}, \quad B = -2, \quad C = \frac{4}{3}.$$

Areas between Curves I (page 9, Chapter 3)

- Consider a region that is bounded from above by the curve $y = f(x)$, from below by the curve $y = g(x)$, and on the left and right by the lines $x = a$ and $x = b$.

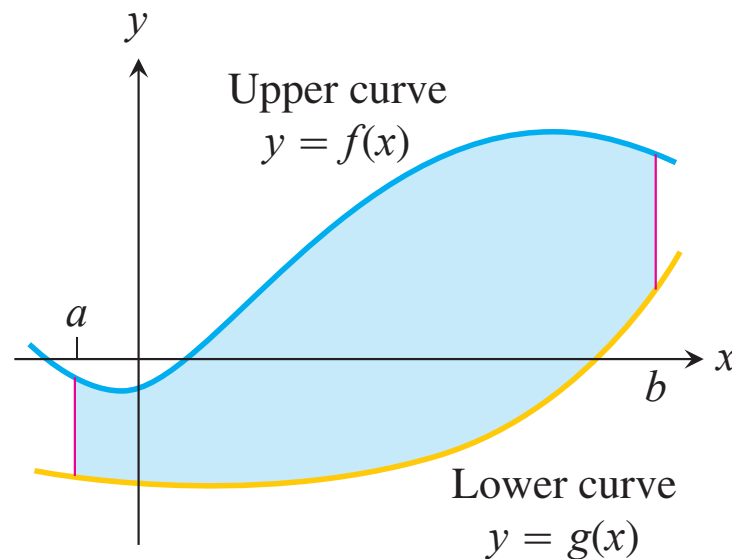


Figure 8: The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

Areas between Curves II (page 9, Chapter 3)

- To find the area of the region:
 - (a) we first approximate the region with n vertical rectangles based on a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$;

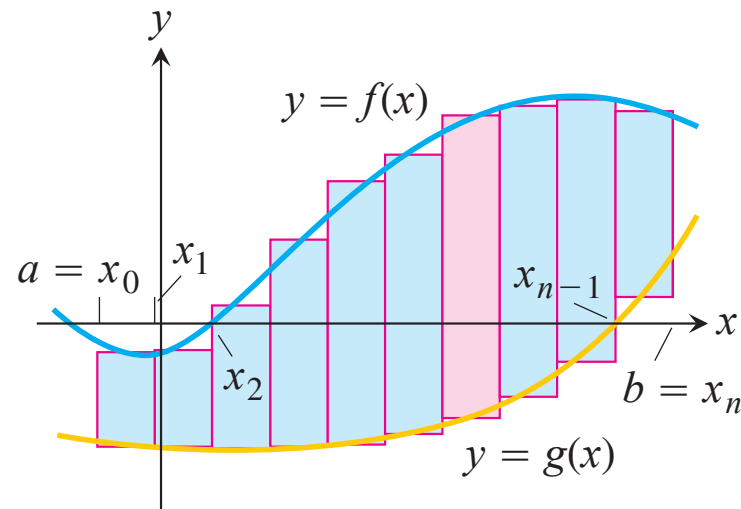


Figure 9: Approximating the region with rectangles perpendicular to the x -axis.

Areas between Curves III (page 9, Chapter 3)

(b) with the choice of the points c_i , the area of the i -th rectangle is

$$\Delta A_i = \text{height} \times \text{width} = [f(c_i) - g(c_i)] \Delta x_i;$$

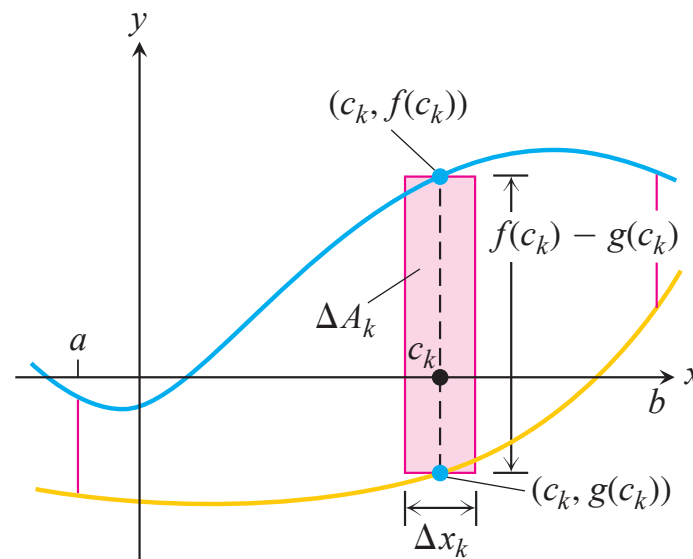


Figure 10: The area ΔA_i of the i -th rectangle is the product of its height, $f(c_i) - g(c_i)$, and its width, Δx_i .

Areas between Curves IV (page 9, Chapter 3)

- (c) the area A of the region is then approximated by the total area of the n rectangles, which corresponds to the Riemann sum:

$$A \approx \sum_{i=1}^n \Delta A_i = \sum_{i=1}^n [f(c_i) - g(c_i)] \Delta x_i;$$

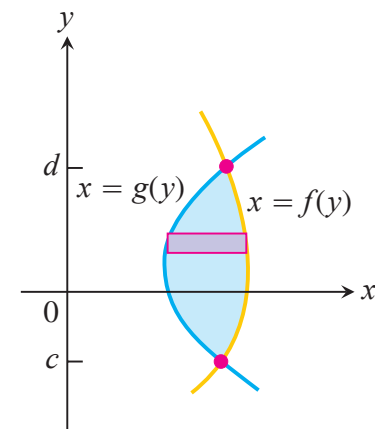
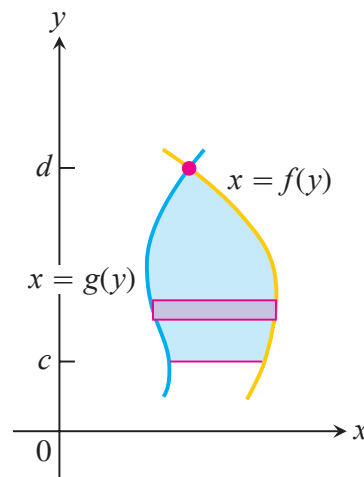
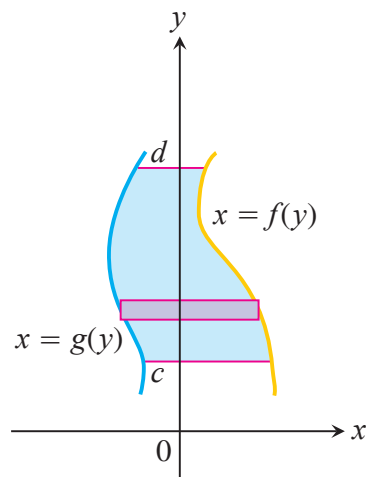
- (d) assuming $f(x)$ and $g(x)$ are continuous (hence Riemann integrable; see Theorem 5), the above Riemann sums converge as $\|P\| \rightarrow 0$; the limiting integral is taken to be the area of the region:

$$A = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n [f(c_i) - g(c_i)] \Delta x_i = \int_a^b [f(x) - g(x)] dx;$$

Areas between Curves V (page 9, Chapter 3)

- (e) if the region's bounding curves are described by functions of y , the approximating rectangles are **horizontal** instead of vertical, and the basic formula has y in place of x :

$$A = \int_c^d [f(y) - g(y)] dy.$$



Areas between Curves VI (page 9, Chapter 3)

- To apply the above formula to calculate the area of a region lying between two curves, we take the following steps:
 - (a) sketch the region and a typical rectangle;
 - (b) find a formula for h , the height of a typical rectangle;
 - (c) find the limits of integration;
 - (d) integrate h using the Fundamental Theorem.
- Note that h is
 - (a) a function of x if the rectangle is vertical;
 - (b) a function of y if the rectangle is horizontal.

Areas between Curves VII (page 9, Chapter 3)

- **Example.^a** As an example, consider the region that is bounded from above by $y = \sqrt{x}$ and from below by the x -axis and the line $y = x - 2$.

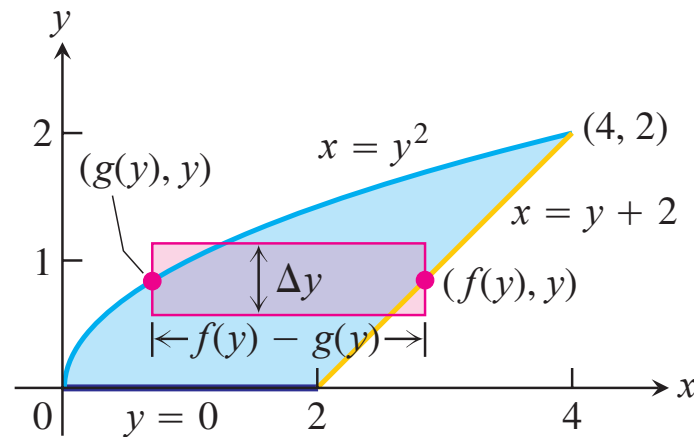


Figure 11: The region in the first quadrant bounded from above by $y = \sqrt{x}$ and from below by the x -axis and the line $y = x - 2$.

^aSee also Example 4, 5, 6, and 7 on page 10, 11, 13, and 15.

Areas between Curves VIII (page 9, Chapter 3)

To find the area of the region:

- (a) sketch the region and a typical horizontal rectangle based on a partition of an interval of y -values (see Figure 11);
- (b) find a formula for $h(y)$, the height of a typical rectangle:

$$h(y) = \text{right curve} - \text{left curve} = (y + 2) - y^2;$$

note that both bounding curves must be written as **functions of y** ;

- (c) find the limits of integration: the lower limit is obviously $y = 0$, and the upper limit can be found by solving $x = y + 2$ and $x = y^2$ simultaneously for y , which yields $y = -1$ (discard) and $y = 2$;
- (d) the area of the region is then

$$A = \int_0^2 (y + 2 - y^2) dy.$$

Areas between Curves IX (page 9, Chapter 3)

The same area can also be computed by integrating with respect to x , as shown in the following figure. The disadvantage of this approach is that it requires splitting the integral into two parts.

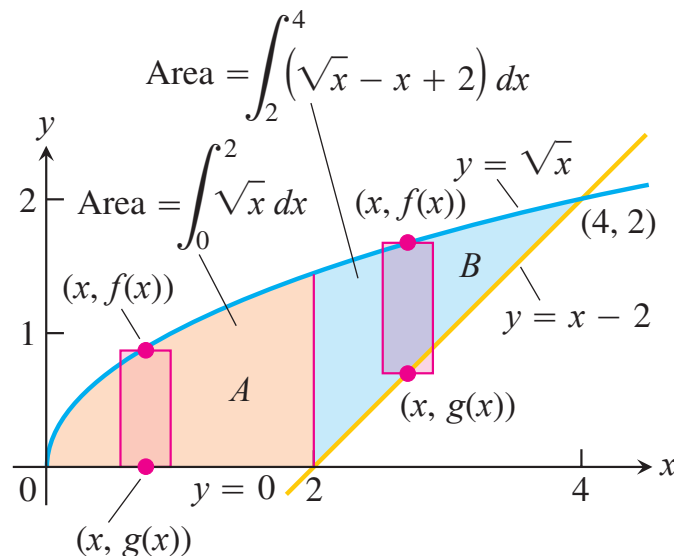


Figure 12: Computing the same area as shown in Figure 11 by integrating with respect to x .

Areas between Curves X (page 9, Chapter 3)

For this particular problem, the best solution is to combine calculus and geometry, which yields

$$A = \int_0^4 \sqrt{x} \, dx - \frac{1}{2} (2)(2).$$

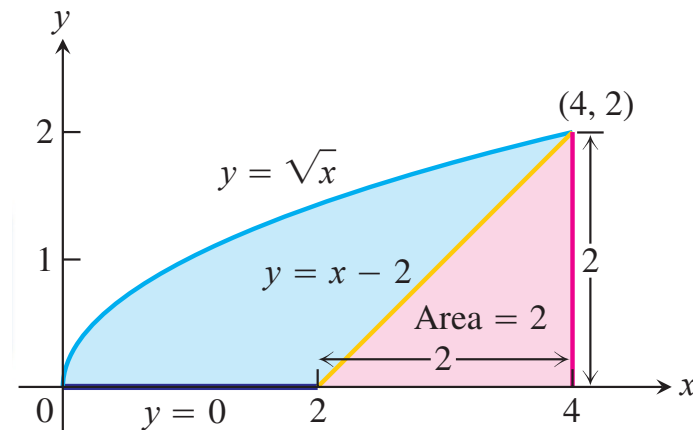


Figure 13: The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle.

Volumes by Slicing I (page 18, Chapter 3)

- Consider a solid whose cross-sections are planar regions.

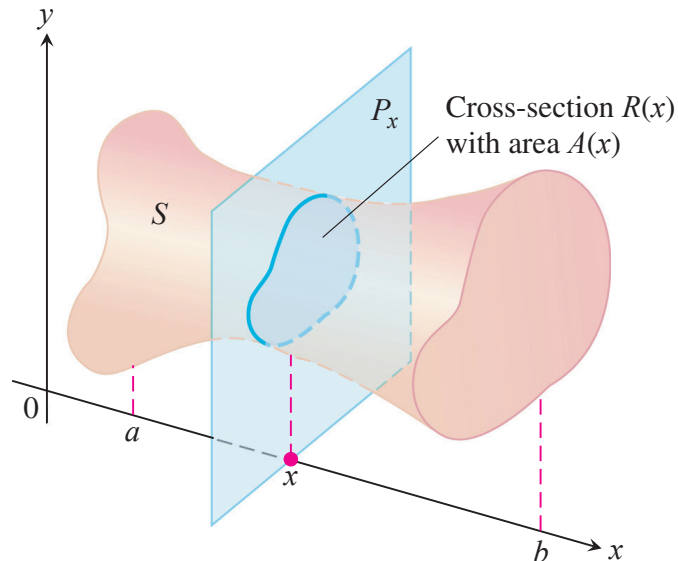


Figure 14: A cross-section of the solid S formed by intersecting S with a plane P_x perpendicular to the x -axis through the point x .

Volumes by Slicing II (page 18, Chapter 3)

- To find the volume of the solid:
 - (a) consider first an arbitrary cylindrical solid with a known base area A and height h ; its volume is given by

$$\text{Volume} = \text{base area} \times \text{height} = A \cdot h;$$

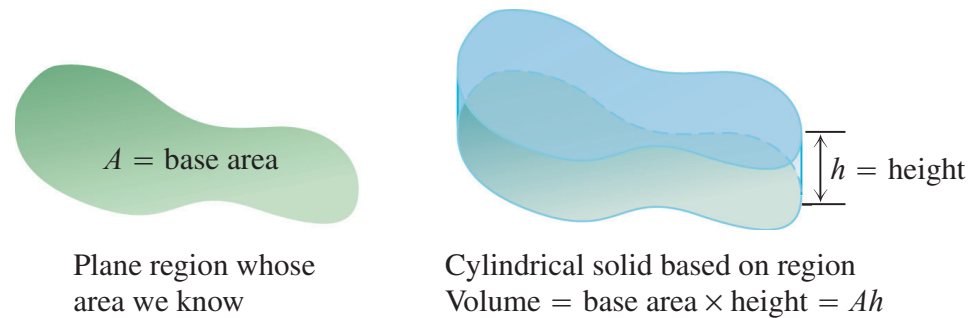


Figure 15: The volume of a cylindrical solid is always defined to be its base area times its height.

Volumes by Slicing III (page 18, Chapter 3)

- (b) the above formula forms the basis for defining the volume of a solid by the **method of slab**; to begin with, assume that the cross-section of the solid at each point x in the interval $[a, b]$ is a region $R(x)$ of area $A(x)$, and $A(x)$ is a continuous function;
- (c) partition $[a, b]$ into subintervals of width Δx_i and slice the solid, as we would do to a loaf of bread, by planes perpendicular to the x -axis at the partition points $a = x_0 < x_1 < \cdots < x_n = b$;

Volumes by Slicing IV (page 18, Chapter 3)

- (d) the planes P_{x_i} perpendicular to the x -axis at the partition points slice the solid into thin “slabs” (like thin slices of a loaf of bread);

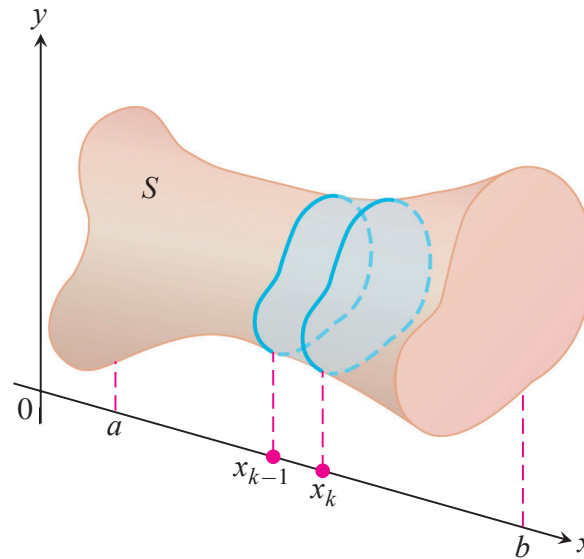


Figure 16: A typical thin slab in the solid S .

Volumes by Slicing V (page 18, Chapter 3)

- (e) approximate the slab between the planes at x_{i-1} and x_i by a cylindrical solid with volume $\Delta V_i = A(x_i)\Delta x_i$;

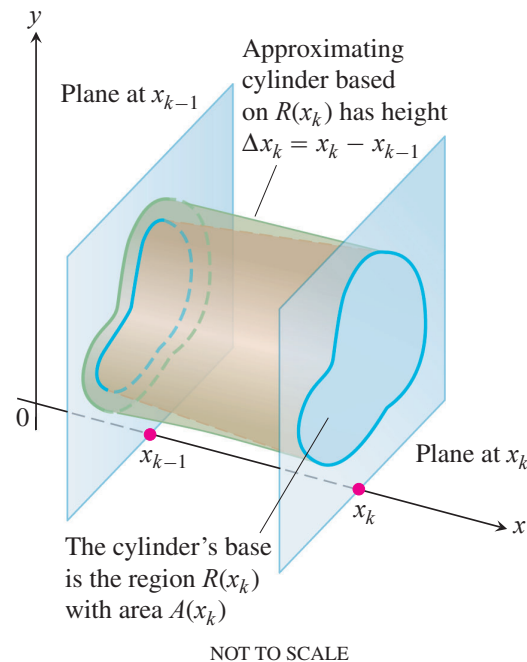


Figure 17: The solid thin slab in Figure 16 is approximated by a cylindrical solid with base area $A(x_i)$ and height $\Delta x_i = x_i - x_{i-1}$.

Volumes by Slicing VI (page 18, Chapter 3)

- (f) the volume V of the entire solid is therefore approximated by the total volume of the n cylindrical solids, which corresponds to the Riemann sum:

$$V \approx \sum_{i=1}^n \Delta V_i = \sum_{i=1}^n A(x_i) \Delta x_i;$$

- (g) since $A(x)$ is assumed to be continuous (hence Riemann integrable; see Theorem 5), the above Riemann sums converge as $\|P\| \rightarrow 0$; the limiting integral is taken to be the volume of the solid:

$$V = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n A(x_i) \Delta x_i = \int_a^b A(x) dx;$$

- (h) if the solid is sliced by planes perpendicular to the y - (or z -) axis, its volume is given by the same formula with x replaced by y (or z).

Volumes by Slicing VII (page 18, Chapter 3)

- To apply the above formula to calculate the volume of a solid, we take the following steps:
 - (a) sketch the solid and a typical cross-section;
 - (b) find a formula for A , the area of a typical cross-section;
 - (c) find the limits of integration;
 - (d) integrate A using the Fundamental Theorem.
- Note that A is
 - (a) a function of x if the cross-section is perpendicular to the x -axis;
 - (b) a function of y if the cross-section is perpendicular to the y -axis;
 - (c) a function of z if the cross-section is perpendicular to the z -axis.

Volumes by Slicing VIII (page 18, Chapter 3)

- **Example.** As an example, consider the wedge cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder, and the other crosses the axis at an angle of 45° .

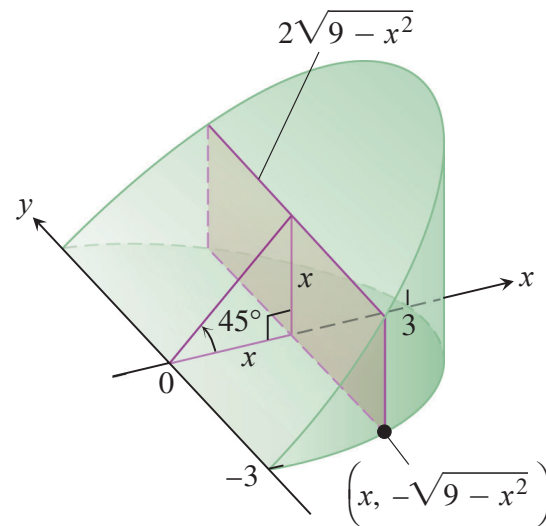


Figure 18: The wedge cut from a cylinder by two planes, sliced perpendicular to the x -axis; the cross-sections are rectangles.

Volumes by Slicing IX (page 18, Chapter 3)

To find the volume of the wedge:

- (a) sketch the wedge and a typical cross-section (see Figure 18);
- (b) find a formula for $A(x)$, the area of a typical cross-section:

$$A(x) = \text{width} \times \text{height} = x(2\sqrt{9 - x^2}) = 2x\sqrt{9 - x^2};$$

- (c) find the limits of integration, which are $x = 0$ and $x = 3$;
- (d) the volume of the wedge is then

$$V = \int_0^3 2x\sqrt{9 - x^2} dx = \int_{9-0^2}^{9-3^2} \underbrace{(9 - x^2)}_u^{1/2} \cdot \underbrace{2x dx}_{-u' dx}.$$

Solids of Revolution: Disk Method I (page 20, Chapter 3)

- The solid generated by rotating a planar region about an axis in its plane is called a **solid of revolution**; to find the volume of a solid like the one shown in Figure 19:
 - (a) observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance from the planar region's boundary to the axis of revolution;
 - (b) the area is then

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2,$$

and the volume of the solid is given by

$$V = \int_a^b A(x) dx = \int_a^b \pi[R(x)]^2 dx;$$

Solids of Revolution: Disk Method II (page 20, Chapter 3)

- (c) if the axis of revolution is vertical (as in Figure 20), the volume of the solid is given by the same formula with x replaced by y :

$$V = \int_c^d A(y) dy = \int_c^d \pi [R(y)]^2 dy,$$

where again $R(y)$ is the distance from the planar region's boundary to the axis of revolution.

- This method for calculating the volume of a solid of revolution is often called the **disk method**.

Solids of Revolution: Disk Method III (page 20, Chapter 3)

- To apply the disk method to calculate the volume of a solid of revolution, we take the following steps:
 - (a) draw the planar region and sketch a typical line segment across it **perpendicular** to the axis of revolution;
 - (b) find the radius R of the disk that would be swept out by the line segment if it were revolved about the axis along with the region;
 - (c) find the limits of integration;
 - (d) integrate πR^2 using the Fundamental Theorem.
- Note that R is
 - (a) a function of x if the disk is perpendicular to the x -axis;
 - (b) a function of y if the disk is perpendicular to the y -axis.

Solids of Revolution: Disk Method IV (page 22–24, Chapter 3)

- **Example.^a** As an example, consider the solid generated by revolving the region bounded by $y = \sqrt{x}$, $y = 1$, and $x = 4$ about the line $y = 1$.

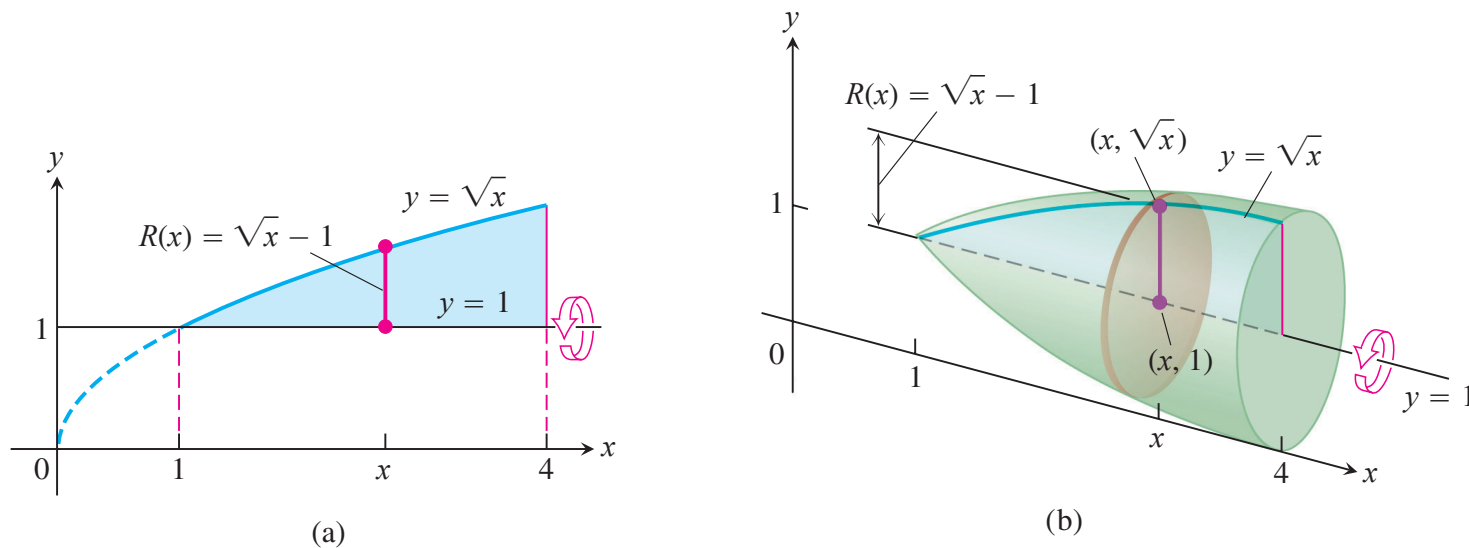


Figure 19: (a) The region bounded by $y = \sqrt{x}$, $y = 1$, and $x = 4$, and (b) the solid generated by revolving the region about the line $y = 1$.

^aSee also Example 8 on page 22 and Example 9 on page 23.

Solids of Revolution: Disk Method V (page 22–24, Chapter 3)

To find the volume of the solid:

- (a) draw the planar region and sketch a typical line segment across it **perpendicular** to the axis of revolution (see Figure 19);
- (b) find the radius $R(x)$ of the disk that would be swept out by the line segment if it were revolved about $y = 1$ along with the region:

$$R(x) = \text{distance from bounding curve to axis} = \sqrt{x} - 1;$$

- (c) find the limits of integration: the upper limit is obviously $x = 4$, and the lower limit can be found by solving $y = \sqrt{x}$ and $y = 1$ simultaneously for x , which yields $x = 1$;
- (d) the volume of the solid is then

$$V = \int_1^4 \pi(\sqrt{x} - 1)^2 dx.$$

Solids of Revolution: Disk Method VI (page 22–24, Chapter 3)

- **Example.** As another example, consider the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

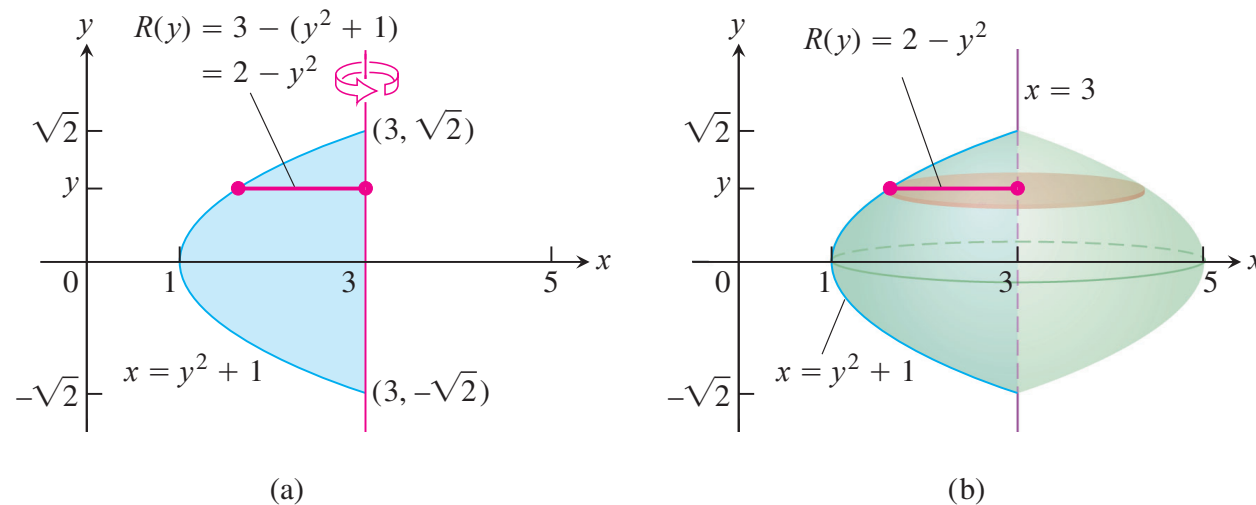


Figure 20: (a) The region bounded by $x = y^2 + 1$ and $x = 3$, and (b) the solid generated by revolving the region about the line $x = 3$.

Solids of Revolution: Disk Method VII (page 22–24, Chapter 3)

To find the volume of the solid:

- (a) draw the planar region and sketch a typical line segment across it **perpendicular** to the axis of revolution (see Figure 20);
- (b) find the radius $R(y)$ of the disk that would be swept out by the line segment if it were revolved about $x = 3$ along with the region:

$$R(y) = \text{distance from bounding curve to axis} = 3 - (y^2 + 1);$$

- (c) find the limits of integration: both the lower and the upper limit can be found by solving $x = y^2 + 1$ and $x = 3$ simultaneously for y , which yields $y = -\sqrt{2}$ and $y = \sqrt{2}$;
- (d) the volume of the solid is then

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi(2 - y^2)^2 dy.$$

Solids of Revolution: Washer Method I (page 20, Chapter 3)

- If the region that is revolved to generate a solid does not border on or cross the axis of revolution, the solid will have a hole in it.

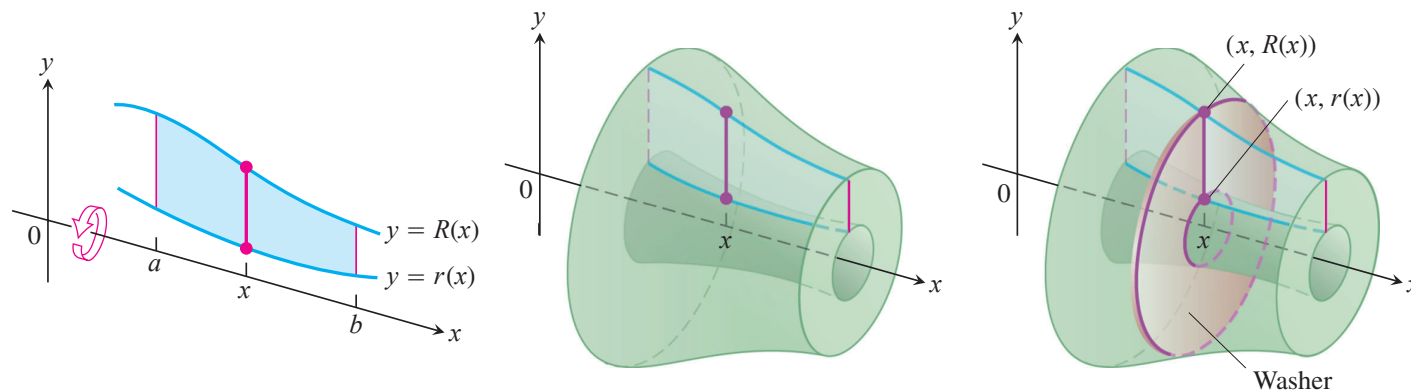


Figure 21: The solid generated by revolving a region that does not border on or cross the axis of revolution; the cross-sections of the solid are washers, not disks.

Solids of Revolution: Washer Method II (page 20, Chapter 3)

- To find the volume of a solid like the one shown in Figure 21:
 - (a) observe that the cross-sectional area $A(x)$ is the area of a washer of inner radius $r(x)$ and outer radius $R(x)$, where $r(x)$ and $R(x)$ are distances from the region's boundaries to the axis of revolution;
 - (b) the area is then

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi[R^2(x) - r^2(x)],$$

and the volume of the solid is given by

$$V = \int_a^b A(x) dx = \int_a^b \pi[R^2(x) - r^2(x)] dx;$$

Solids of Revolution: Washer Method III (page 20, Chapter 3)

- (c) if the axis of revolution is vertical (as in Figure 23), the volume of the solid is given by the same formula with x replaced by y :

$$V = \int_c^d A(y) dy = \int_c^d \pi [R^2(y) - r^2(y)] dy,$$

where again $r(y)$ and $R(y)$ are distances from the region's boundaries to the axis of revolution.

- This method for calculating the volume of a solid of revolution is often called the **washer method**.

Solids of Revolution: Washer Method IV (page 20, Chapter 3)

- To apply the washer method to calculate the volume of a solid of revolution, we take the following steps:
 - (a) draw the planar region and sketch a typical line segment across it **perpendicular** to the axis of revolution;
 - (b) find the inner and outer radii r and R of the washer that would be swept out by the line segment if it were revolved about the axis along with the region;
 - (c) find the limits of integration;
 - (d) integrate $\pi(R^2 - r^2)$ using the Fundamental Theorem.
- Note that r and R are
 - (a) functions of x if the washer is perpendicular to the x -axis;
 - (b) functions of y if the washer is perpendicular to the y -axis.

Solids of Revolution: Washer Method V (page 24–27, Chapter 3)

- **Example.^a** As an example, consider the solid generated by revolving the region bounded by $y = x^2 + 1$ and $y = -x + 3$ about the x -axis.

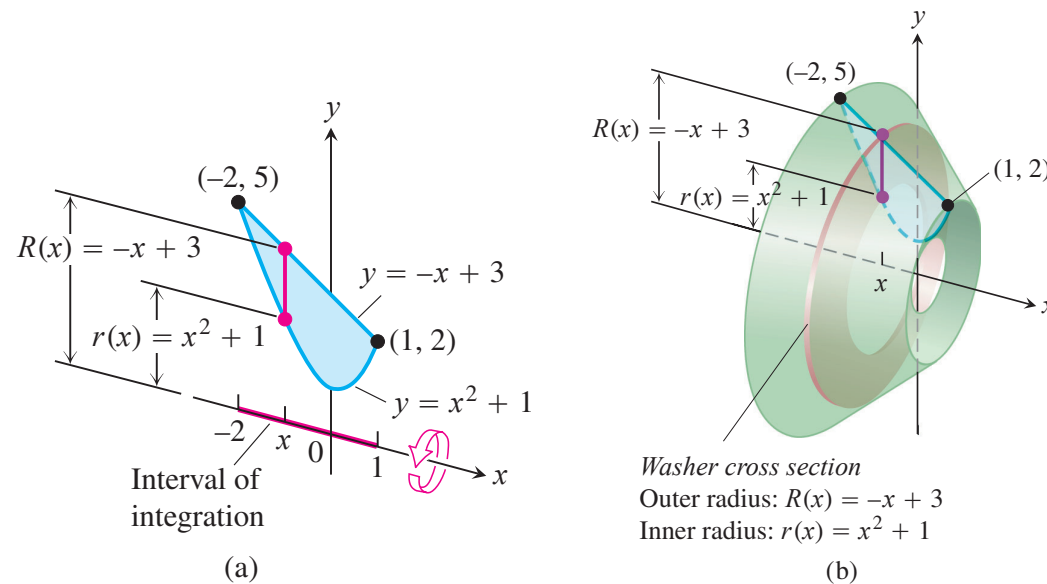


Figure 22: (a) The region bounded by $y = x^2 + 1$ and $y = -x + 3$, and (b) the solid generated by revolving the region about the x -axis.

^aSee also Example 10 on page 24 and Example 11 on page 26.

Solids of Revolution: Washer Method VI (page 24–27, Chapter 3)

To find the volume of the solid:

- (a) draw the planar region and sketch a typical line segment across it **perpendicular** to the axis of revolution (see Figure 22);
- (b) find the inner and outer radii $r(x)$ and $R(x)$ of the washer that would be swept out by the line segment if it were revolved about the x -axis along with the region:

$$r(x) = x^2 + 1, \quad R(x) = -x + 3;$$

- (c) find the limits of integration: both the lower and the upper limit can be found by solving $y = x^2 + 1$ and $y = -x + 3$ simultaneously for x , which yields $x = -2$ and $x = 1$;
- (d) the volume of the solid is then

$$V = \int_{-2}^1 \pi [(-x + 3)^2 - (x^2 + 1)^2] dx.$$

Solids of Revolution: Washer Method VII (page 28, Chapter 3)

- **Example.^a** As another example, consider the solid generated by revolving the region bounded by $y = x^2$ and $y = 2x$ about the y -axis.

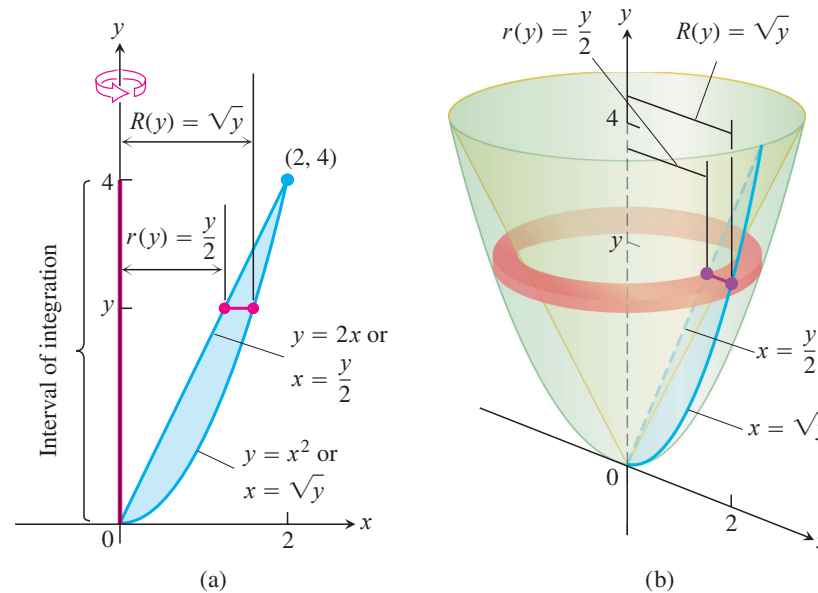


Figure 23: (a) The region bounded by $y = x^2$ and $y = 2x$, and (b) the solid generated by revolving the region about the y -axis.

^aSee also Example 12 on page 28.

Solids of Revolution: Washer Method VIII (page 28, Chapter 3)

To find the volume of the solid:

- (a) draw the planar region and sketch a typical line segment across it **perpendicular** to the axis of revolution (see Figure 23);
- (b) find the inner and outer radii $r(y)$ and $R(y)$ of the washer that would be swept out by the line segment if it were revolved about the y -axis along with the region:

$$r(y) = y/2, \quad R(y) = \sqrt{y};$$

- (c) find the limits of integration: both the lower and the upper limit can be found by solving $x = \sqrt{y}$ and $x = y/2$ simultaneously for y , which yields $y = 0$ and $y = 4$;
- (d) the volume of the solid is then

$$V = \int_0^4 \pi [(\sqrt{y})^2 - (y/2)^2] dy.$$

Solids of Revolution: Shell Method I (page 32, Chapter 3)

- When computing the volume of a solid of revolution:
 - (a) both the disk and the washer methods work by slicing through the solid with a **plane perpendicular** to the axis of revolution;
 - (b) in some situations, it is more advantageous to slice through the solid using **circular cylinders** of increasing radii (like cookie cutters), whose axes are **parallel** to the axis of revolution.

Solids of Revolution: Shell Method II (page 32, Chapter 3)

- Consider a solid generated by revolving the region bounded by a nonnegative continuous function $y = f(x)$ and the x -axis over the interval $[a, b]$ about the line $x = L \leq a$ (see Figure 24).
- To find the volume of the solid:
 - (a) let P be a partition of the interval $[a, b]$ by the points

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and let c_i be the midpoint of the i -th subinterval $[x_{i-1}, x_i]$;

- (b) approximate the region in Figure 24(a) with rectangles based on this partition of $[a, b]$; a typical approximating rectangle has height $f(c_i)$ and width $\Delta x_i = x_i - x_{i-1}$;

Solids of Revolution: Shell Method III (page 32, Chapter 3)

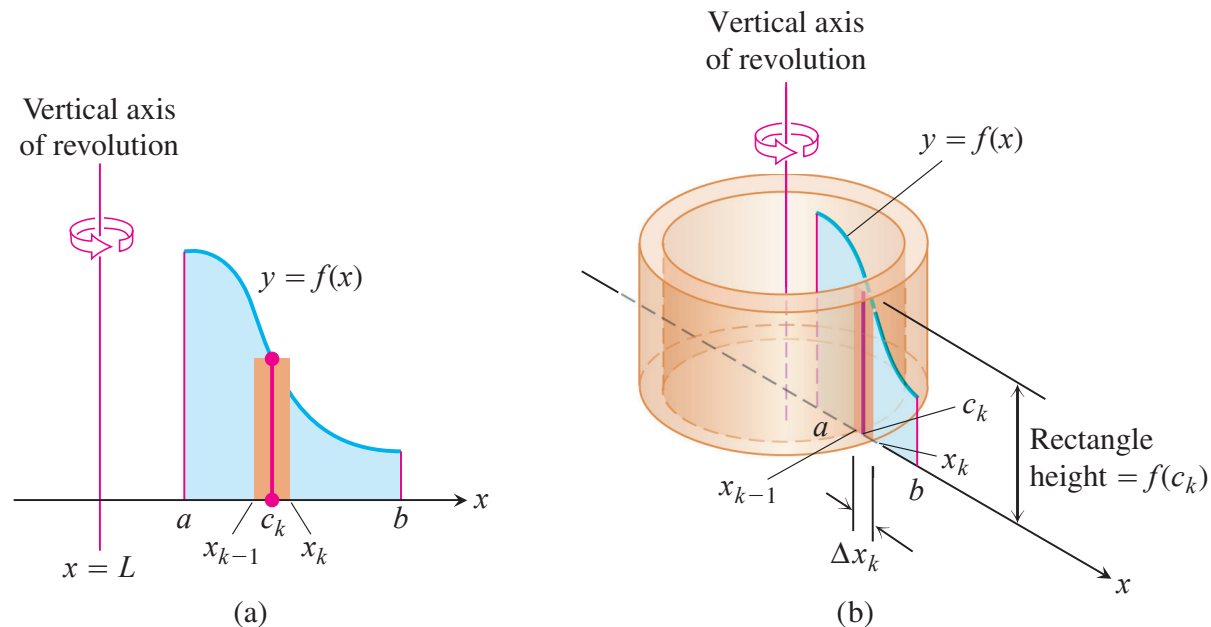


Figure 24: (a) The region bounded by $y = f(x) \geq 0$ and the x -axis, and (b) a typical cylindrical shell generated by revolving a thin slice parallel to the line $x = L$ about the line $x = L$, where $L \leq a$.

Solids of Revolution: Shell Method IV (page 32, Chapter 3)

- (c) if this rectangle is rotated about the line $x = L$, then a shell is swept out, as in Figure 24(b); the volume of this shell is (see Figure 25)

$$\begin{aligned}\Delta V_i &= 2\pi \times \text{average shell radius} \times \text{shell height} \times \text{thickness} \\ &= 2\pi \cdot (c_i - L) \cdot f(c_i) \cdot \Delta x_i;\end{aligned}$$

- (d) the volume V of the entire solid is therefore approximated by the total volume of the n cylindrical shells, which corresponds to the Riemann sum:

$$V \approx \sum_{i=1}^n \Delta V_i = \sum_{i=1}^n 2\pi(c_i - L)f(c_i)\Delta x_i;$$

Solids of Revolution: Shell Method V (page 32, Chapter 3)

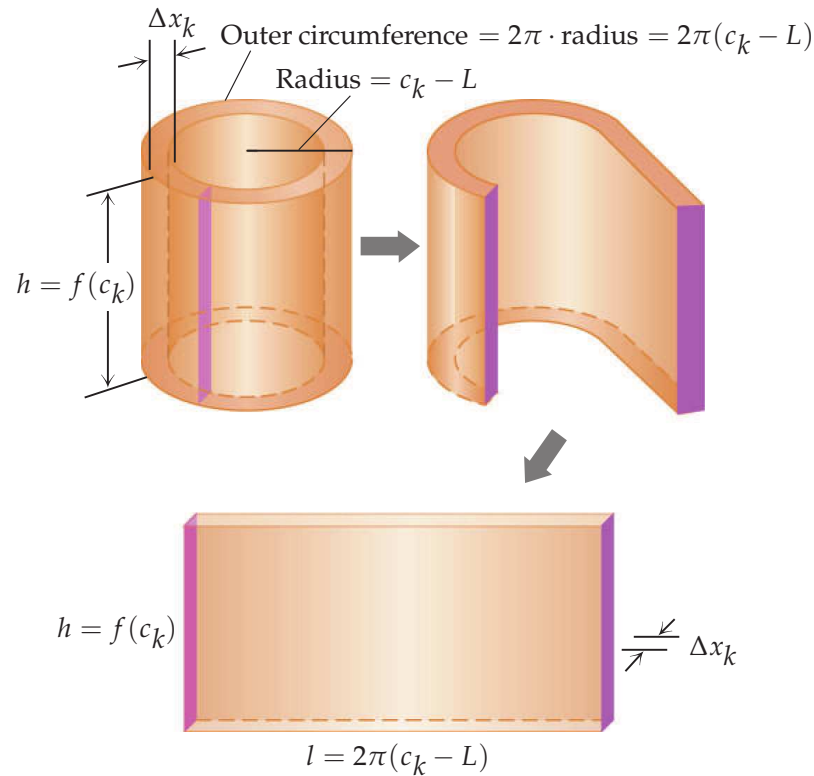


Figure 25: Imagine cutting and unrolling a cylindrical shell to get a (nearly) flat rectangular solid.

Solids of Revolution: Shell Method VI (page 32, Chapter 3)

(e) since $(x - L)f(x)$ is assumed to be continuous (hence integrable; see Theorem 5), the above Riemann sums converge as $\|P\| \rightarrow 0$; the limiting integral is taken to be the volume of the solid:

$$\begin{aligned} V &= \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n 2\pi(c_i - L)f(c_i)\Delta x_i \\ &= \int_a^b 2\pi(x - L)f(x) dx \\ &= \int_a^b 2\pi(\text{shell radius})(\text{shell height}) dx; \end{aligned}$$

Solids of Revolution: Shell Method VII (page 32, Chapter 3)

- (f) if the axis of revolution is a horizontal line $y = L$, and $L \leq c$ where $[c, d]$ is the interval over which the function $x = f(y)$ is defined (as in Figure 27), the volume of the solid is given by the same formula with x replaced by y :

$$\begin{aligned} V &= \int_c^d 2\pi(\text{shell radius})(\text{shell height}) dy \\ &= \int_c^d 2\pi(y - L)f(y) dy. \end{aligned}$$

- This method for calculating the volume of a solid of revolution is often called the **shell method**.

Solids of Revolution: Shell Method VIII (page 32, Chapter 3)

- To apply the shell method to calculate the volume of a solid of revolution, we take the following steps:
 - (a) draw the planar region and sketch a typical line segment across it **parallel** to the axis of revolution;
 - (b) find the radius r and height h of the shell that would be swept out by the line segment if it were revolved about the axis along with the region;
 - (c) find the limits of integration;
 - (d) integrate $2\pi rh$ using the Fundamental Theorem.
- Note that r and h are
 - (a) functions of x if the axis of the shell is parallel to the y -axis;
 - (b) functions of y if the axis of the shell is parallel to the x -axis.

Solids of Revolution: Shell Method IX (page 32–35, Chapter 3)

- **Example.^a** As an example, consider the solid generated by revolving the region bounded by $y = \sqrt{x}$, the x -axis, and $x = 4$ about the y -axis.

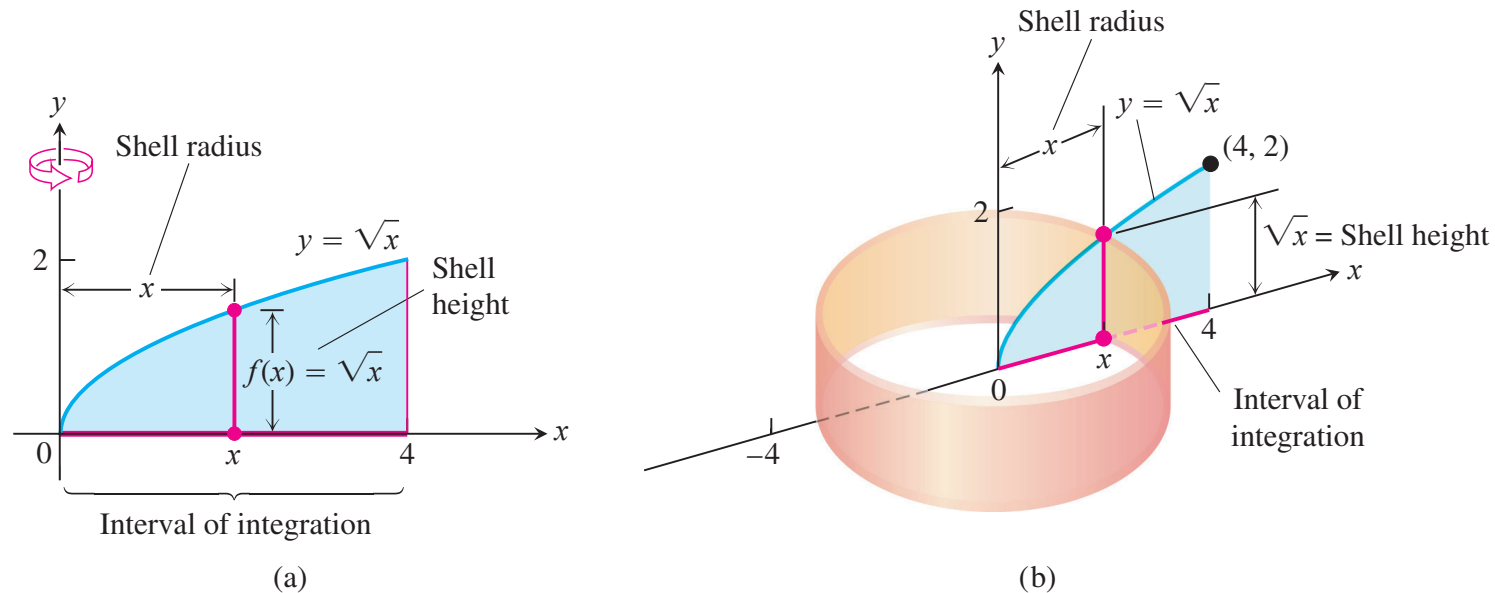


Figure 26: (a) The region bounded by $y = \sqrt{x}$, the x -axis, and $x = 4$, and (b) the solid generated by revolving the region about the y -axis.

^aSee also Example 14 on page 32 and Example 15 on page 34.

Solids of Revolution: Shell Method X (page 32–35, Chapter 3)

To find the volume of the solid:

- (a) draw the planar region and sketch a typical line segment across it **parallel** to the axis of revolution (see Figure 26);
- (b) find the radius $r(x)$ and height $h(x)$ of the shell that would be swept out by the line segment if it were revolved about the y -axis along with the region:

$$r(x) = x, \quad h(x) = \sqrt{x};$$

- (c) find the limits of integration: the upper limit is obviously $x = 4$, and the lower limit can be found by solving $y = 0$ (the x -axis) and $y = \sqrt{x}$ simultaneously for x , which yields $x = 0$;
- (d) the volume of the solid is then

$$V = \int_0^4 2\pi(x)(\sqrt{x}) dx.$$

Solids of Revolution: Shell Method XI (page 35, Chapter 3)

- **Example.^a** As another example, consider the solid generated by revolving the same region as in Figure 26(a) about the x -axis.

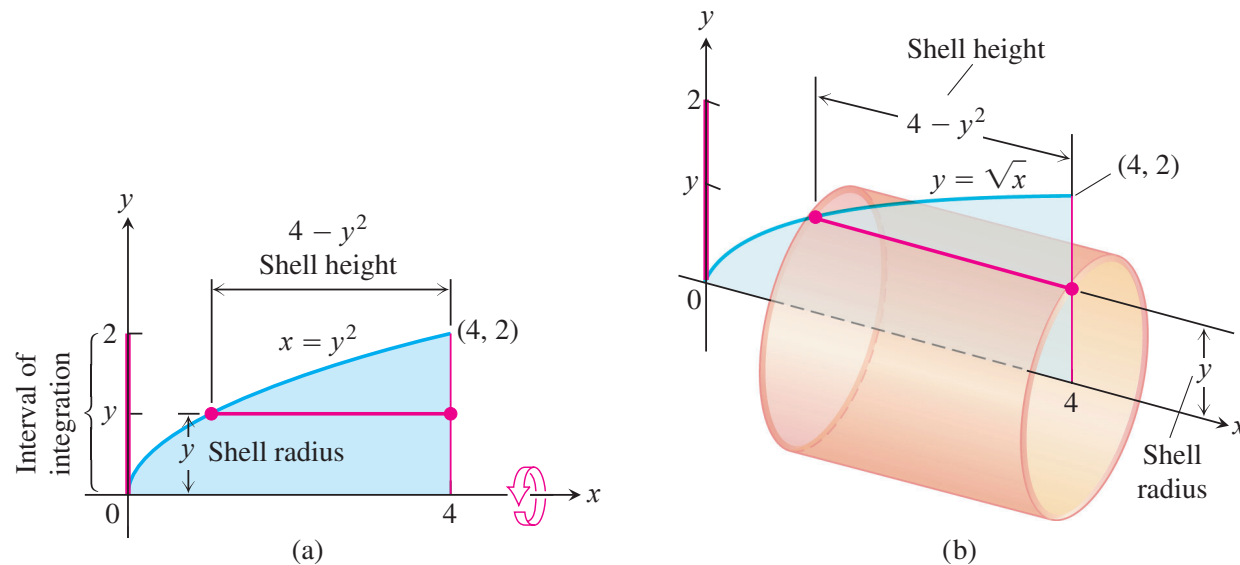


Figure 27: (a) The region bounded by $y = \sqrt{x}$, the x -axis, and $x = 4$, and (b) the solid generated by revolving the region about the x -axis.

^aSee also Example 16 on page 35.

Solids of Revolution: Shell Method XII (page 35, Chapter 3)

To find the volume of the solid:

- (a) draw the planar region and sketch a typical line segment across it **parallel** to the axis of revolution (see Figure 27);
- (b) find the radius $r(y)$ and height $h(y)$ of the shell that would be swept out by the line segment if it were revolved about the x -axis along with the region:

$$r(y) = y, \quad h(y) = 4 - y^2;$$

- (c) find the limits of integration: the lower limit is obviously $y = 0$, and the upper limit can be found by solving $x = y^2$ and $x = 4$ simultaneously for y , which yields $y = -2$ (discard) and $y = 2$;
- (d) the volume of the solid is then

$$V = \int_0^2 2\pi(y)(4 - y^2) dy.$$

Lengths of Plane Curves I (page 61, Chapter 3)

- Consider a curve given parametrically by the equations

$$x = f(t), \quad y = g(t), \quad a \leq t \leq b.$$

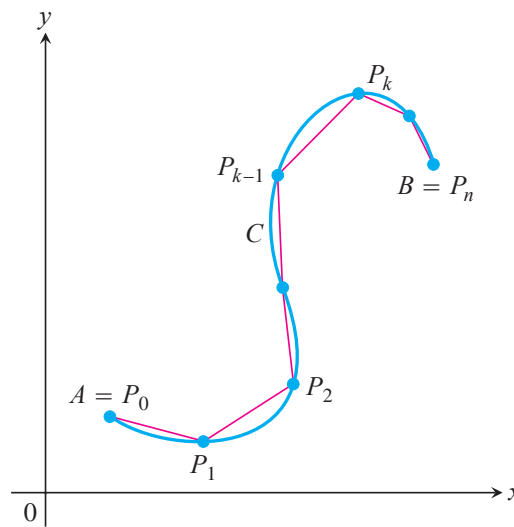


Figure 28: The curve C defined parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$; the length of the curve is approximated by the sum of the lengths of the polygonal path $P_0P_1 \cdots P_n$.

Lengths of Plane Curves II (page 61, Chapter 3)

- To find the length of the curve:
 - (a) assume that the functions $f(t)$ and $g(t)$ have continuous derivatives, i.e. are **continuously differentiable**, on the interval $[a, b]$, and that their derivatives are not simultaneously zero;
 - (b) subdivide the path (or arc) AB , where $A = (f(a), g(a))$ and $B = (f(b), g(b))$, into n pieces at points $A = P_0, P_1, P_2, \dots, P_{n-1}, P_n = B$; these points correspond to a partition of the interval $[a, b]$ by

$$a = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = b,$$

where $P_i = (f(t_i), g(t_i))$;

Lengths of Plane Curves III (page 61, Chapter 3)

- (c) join successive points of this subdivision by straight line segments; a representative line segment has length (see Figure 29)

$$\begin{aligned}\Delta L_i &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &= \sqrt{[f(t_i) - f(t_{i-1})]^2 + [g(t_i) - g(t_{i-1})]^2};\end{aligned}$$

if $\Delta t_i = t_i - t_{i-1}$ is small, the length ΔL_i will be a good approximation to the length of the arc $P_{i-1}P_i$;

- (d) since $f(t)$ and $g(t)$ are continuously differentiable, the Mean Value Theorem implies that there exist some ξ_i, η_i in (t_{i-1}, t_i) such that

$$\begin{aligned}\Delta x_i &= f(t_i) - f(t_{i-1}) = f'(\xi_i)(t_i - t_{i-1}) = f'(\xi_i)\Delta t_i, \\ \Delta y_i &= g(t_i) - g(t_{i-1}) = g'(\eta_i)(t_i - t_{i-1}) = g'(\eta_i)\Delta t_i;\end{aligned}$$

Lengths of Plane Curves IV (page 61, Chapter 3)

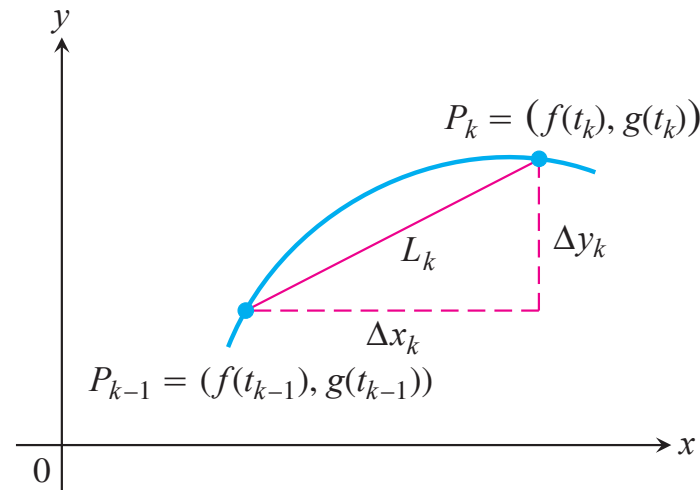


Figure 29: The arc $P_{i-1}P_i$ is approximated by a straight line segment, which has length $\Delta L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$.

Lengths of Plane Curves V (page 61, Chapter 3)

- (e) assuming that the path from A to B is traversed exactly once as t increases from $t = a$ to $t = b$, with no doubling back or retracing, the length L of the entire curve can be approximated by the total length of the n line segments:

$$\begin{aligned} L &\approx \sum_{i=1}^n \Delta L_i = \sum_{i=1}^n \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &= \sum_{i=1}^n \sqrt{[f'(\xi_i)]^2 + [g'(\eta_i)]^2} \Delta t_i, \end{aligned}$$

which **roughly** corresponds to the Riemann sum

$$L \approx \sum_{i=1}^n \sqrt{[f'(\xi_i)]^2 + [g'(\xi_i)]^2} \Delta t_i;$$

Lengths of Plane Curves VI (page 61, Chapter 3)

(f) since the function

$$\sqrt{[f'(t)]^2 + [g'(t)]^2}$$

is assumed to be continuous (hence Riemann integrable; see Theorem 5), the above Riemann sums converge as $\|P\| \rightarrow 0$; the limiting integral is taken to be the length of the curve:

$$\begin{aligned} L &= \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n \sqrt{[f'(\xi_i)]^2 + [g'(\xi_i)]^2} \Delta t_i \\ &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt; \end{aligned}$$

Lengths of Plane Curves VII (page 39–41, Chapter 3)

- (g) if the curve is described by the graph of a continuously differentiable function $y = f(x)$, $a \leq x \leq b$, we can assign $x = t$ and obtain the following parametric representation of the curve:

$$x = t, \quad y = f(t), \quad a \leq t \leq b;$$

the length of the curve is then given by

$$\begin{aligned} L &= \int_a^b \sqrt{(t')^2 + [f'(t)]^2} dt \\ &= \int_a^b \sqrt{1 + [f'(t)]^2} dt = \int_a^b \sqrt{1 + [f'(x)]^2} dx; \end{aligned}$$

- (h) similarly, if the curve is described by the graph of a continuously differentiable function $x = g(y)$, $c \leq y \leq d$, its length is given by

$$L = \int_c^d \sqrt{1 + [g'(y)]^2} dy.$$

Lengths of Plane Curves VIII (page 62–70, Chapter 3)

- **Example.**^a As an example, consider the **astroid**

$$x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \leq t \leq 2\pi.$$

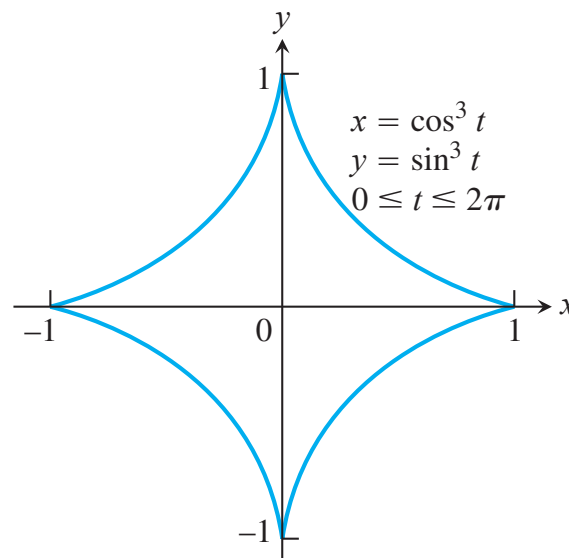


Figure 30: The astroid $x = \cos^3 t$, $y = \sin^3 t$, $0 \leq t \leq 2\pi$.

^aSee also Example 24, 25, 26, and 27 on page 62, 64, 66, and 68.

Lengths of Plane Curves IX (page 62–70, Chapter 3)

To find the length of the curve, we compute:

$$[x'(t)]^2 = [3 \cos^2 t (-\sin t)]^2 = 9 \cos^4 t \sin^2 t,$$

$$[y'(t)]^2 = [3 \sin^2 t (\cos t)]^2 = 9 \sin^4 t \cos^2 t,$$

$$\begin{aligned}\sqrt{[x'(t)]^2 + [y'(t)]^2} &= \sqrt{9 \sin^2 t \cos^2 t (\sin^2 t + \cos^2 t)} \\ &= \sqrt{9 \sin^2 t \cos^2 t} = 3|\sin t \cos t|.\end{aligned}$$

Then by symmetry,

$$\begin{aligned}L &= \int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt = 4 \int_0^{\pi/2} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt \\ &= 4 \int_0^{\pi/2} 3|\sin t \cos t| dt = 12 \int_0^{\pi/2} \sin t \cos t dt = 6 \int_0^{\pi/2} \sin 2t dt.\end{aligned}$$

Lengths of Plane Curves X (page 42–46, Chapter 3)

- **Example.^a** As another example, consider the curve

$$y = (x/2)^{2/3}, \quad 0 \leq x \leq 2.$$

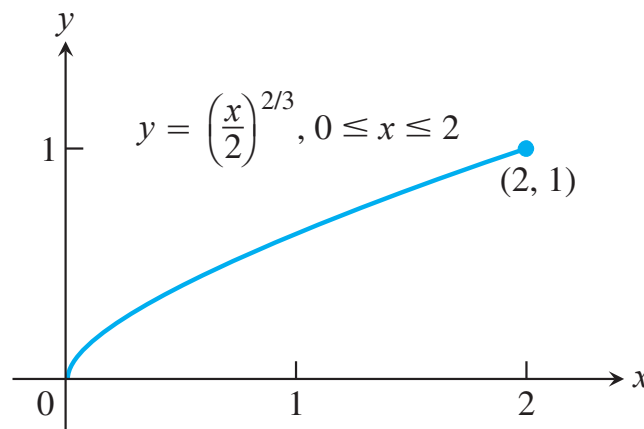


Figure 31: The curve $y = (x/2)^{2/3}$, $0 \leq x \leq 2$, which also has the representation $x = 2y^{3/2}$, $0 \leq y \leq 1$.

^aSee also Example 18 on page 42 and Example 19 on page 45.

Lengths of Plane Curves XI (page 42–46, Chapter 3)

Since the derivative

$$y'(x) = [(x/2)^{2/3}]' = \frac{2}{3} \left(\frac{x}{2}\right)^{-1/3} \frac{1}{2} = \frac{1}{3} \left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x = 0$, the length of the curve cannot be found using the formula^a

$$L = \int_0^2 \sqrt{1 + [y'(x)]^2} dx.$$

^aStrictly speaking, it can, by using the notion of **improper integrals**.

Lengths of Plane Curves XII (page 42–46, Chapter 3)

Instead, we rewrite the equation to express x in terms of y :

$$\begin{aligned}y &= (x/2)^{2/3}, \\y^{3/2} &= x/2, \\x &= 2y^{3/2}.\end{aligned}$$

Since the derivative

$$x'(y) = [2y^{3/2}]' = \frac{3}{2} \cdot 2y^{1/2} = 3y^{1/2}$$

is continuous on $[0, 1]$, the length of the curve can be computed using the formula

$$L = \int_0^1 \sqrt{1 + [x'(y)]^2} dy = \int_0^1 \sqrt{1 + 9y} dy.$$

Lengths of Plane Curves XIII (page 41 & 61, Chapter 3)

- The curve length formulas can be conveniently remembered by
(a) introducing the **arc length differential**

$$ds = \sqrt{dx^2 + dy^2},$$

which is an infinitesimal analog of the arc length increment

$$\Delta L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

introduced in the Riemann sum $\sum_{i=1}^n \Delta L_i$ (see Figure 32);

Lengths of Plane Curves XIV (page 41 & 61, Chapter 3)

(b) recognizing from the law of differential that

$$ds = \sqrt{\left(\frac{dx}{dt} dt\right)^2 + \left(\frac{dy}{dt} dt\right)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

if $x = x(t)$ and $y = y(t)$ are functions of t , that

$$ds = \sqrt{dx^2 + \left(\frac{dy}{dx} dx\right)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

if $y = y(x)$ is a function of x , and that

$$ds = \sqrt{\left(\frac{dx}{dy} dy\right)^2 + dy^2} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy,$$

if $x = x(y)$ is a function of y ;

Lengths of Plane Curves XV (page 41 & 61, Chapter 3)

(c) writing the curve length formulas collectively as

$$L = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n \Delta L_i = \int ds = \int \sqrt{dx^2 + dy^2},$$

and substituting in appropriate representations for ds depending on the specific problem at hand.

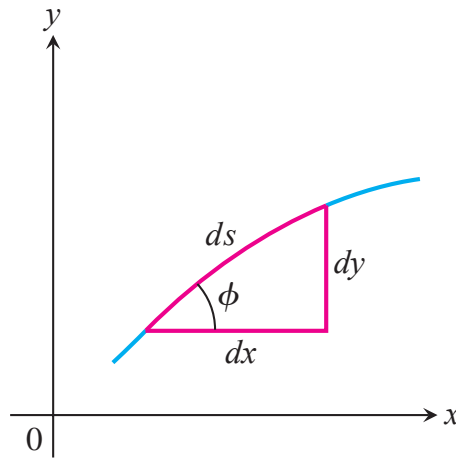


Figure 32: The arc length differential $ds = \sqrt{dx^2 + dy^2}$.

Areas of Surfaces of Revolution I (page 47, Chapter 3)

- Consider a surface swept out by revolving the graph of a non-negative continuously differentiable function $y = f(x)$, $a \leq x \leq b$, about the x -axis.

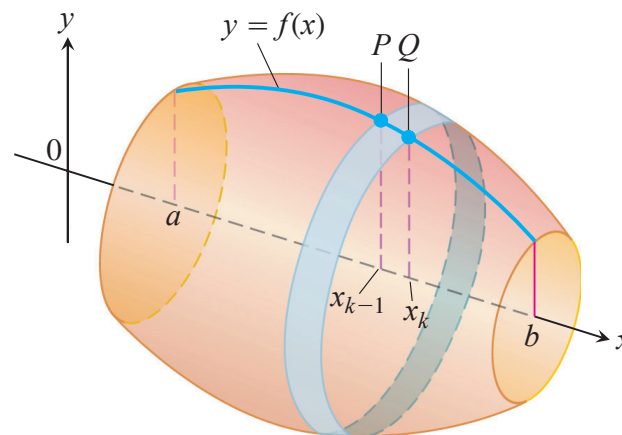


Figure 33: The surface generated by revolving the graph of a nonnegative function $y = f(x)$, $a \leq x \leq b$, about the x -axis; the surface is a union of bands like the one swept out by the arc PQ .

Areas of Surfaces of Revolution II (page 47, Chapter 3)

- To find the area of the surface:
 - (a) consider first the surface generated by rotating a horizontal line segment AB of length Δx about the x -axis; it is a cylinder with surface area $2\pi y\Delta x$;

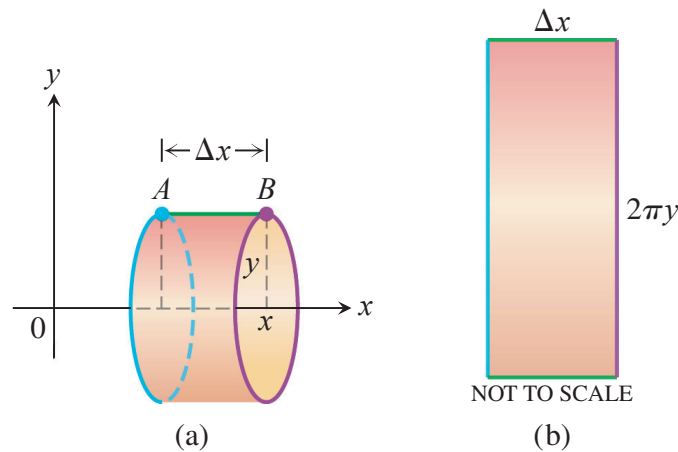


Figure 34: (a) The cylindrical surface generated by rotating a horizontal line segment AB of length Δx about the x -axis has area $2\pi y\Delta x$, and (b) the cut and rolled-out surface as a rectangle.

Areas of Surfaces of Revolution III (page 47, Chapter 3)

- (b) if the line segment AB has length Δs and is **slanted**, then it generates a frustum of a cone when rotated about the x -axis; the surface area of this frustum is $2\pi y^* \Delta s$, where $y^* = (y_1 + y_2)/2$;

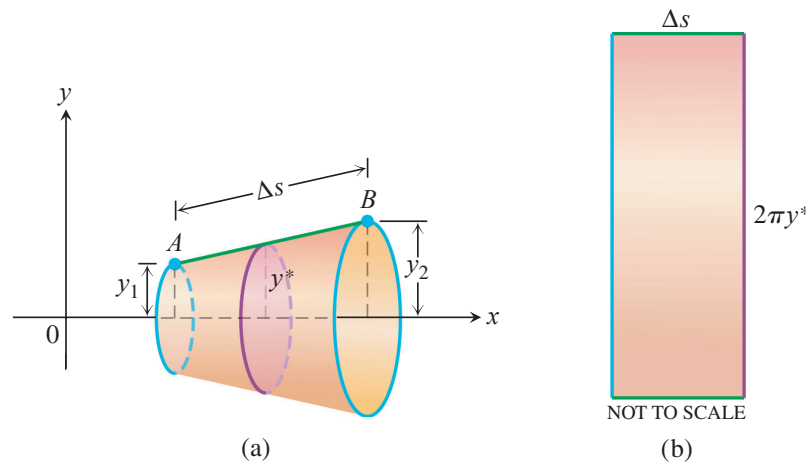


Figure 35: (a) The frustum of a cone generated by rotating a slanted line segment AB of length Δs about the x -axis has area $2\pi y^* \Delta s$, and (b) the area of the rectangle for $y^* = (y_1 + y_2)/2$.

Areas of Surfaces of Revolution IV (page 47, Chapter 3)

- (c) to find the area of the surface shown in Figure 33, partition the closed interval $[a, b]$ in the usual way and use the points in the partition to subdivide the curve into short arcs;
- (d) consider a typical arc PQ and the frustum of a cone swept out by revolving the **line segment** joining P and Q about the x -axis (see Figure 36); the surface area $2\pi y^* \Delta L_i$ of this frustum approximates the surface area of the band swept out by the arc PQ ;

Areas of Surfaces of Revolution V (page 47, Chapter 3)

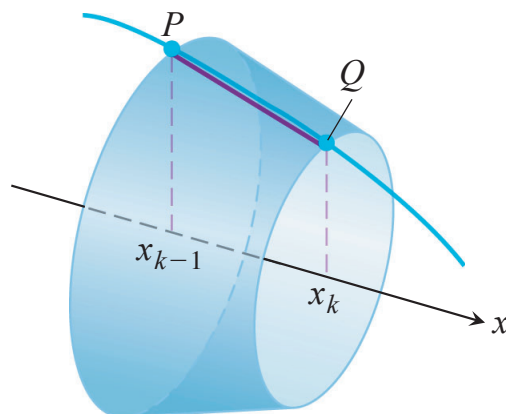


Figure 36: The frustum of a cone swept out by the line segment joining P and Q .

Areas of Surfaces of Revolution VI (page 47, Chapter 3)

- (e) the average height y^* and the slant length ΔL_i of the line segment joining P and Q are (see Figure 37)

$$y^* = \frac{1}{2} [f(x_{i-1}) + f(x_i)], \quad \Delta L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2};$$

therefore, the area of the frustum is given by

$$\begin{aligned} \Delta S_i &= 2\pi \cdot \frac{1}{2} [f(x_{i-1}) + f(x_i)] \cdot \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &= \pi [f(x_{i-1}) + f(x_i)] \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}; \end{aligned}$$

- (f) since $f(x)$ is continuously differentiable, the Mean Value Theorem implies that there exists some ξ_i in (x_{i-1}, x_i) such that

$$\Delta y_i = f(x_i) - f(x_{i-1}) = f'(\xi_i)(x_i - x_{i-1}) = f'(\xi_i)\Delta x_i;$$

Areas of Surfaces of Revolution VII (page 47, Chapter 3)

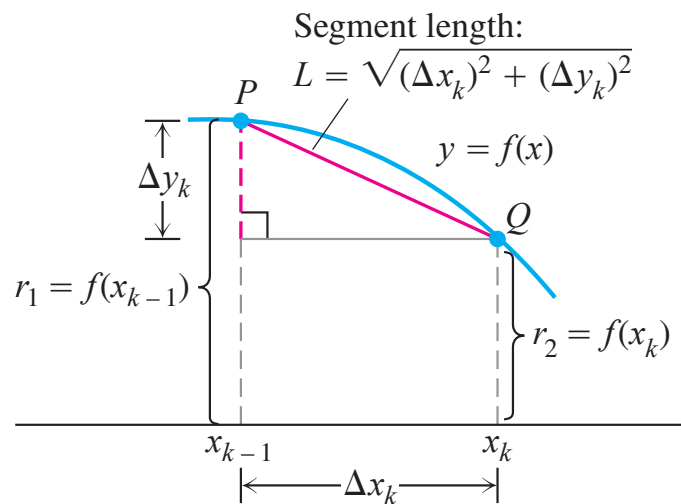


Figure 37: Dimensions associated with the arc PQ and the line segment joining P and Q .

Areas of Surfaces of Revolution VIII (page 47, Chapter 3)

(g) the area S of the entire surface is therefore approximated by the total area of the n frustums:

$$\begin{aligned} S &\approx \sum_{i=1}^n \Delta S_i = \sum_{i=1}^n \pi [f(x_{i-1}) + f(x_i)] \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &= \sum_{i=1}^n \pi [f(x_{i-1}) + f(x_i)] \sqrt{1 + [f'(\xi_i)]^2} \Delta x_i, \end{aligned}$$

which **roughly** corresponds to the Riemann sum

$$S \approx \sum_{i=1}^n 2\pi f(\xi_i) \sqrt{1 + [f'(\xi_i)]^2} \Delta x_i;$$

Areas of Surfaces of Revolution IX (page 47, Chapter 3)

(h) since the function

$$2\pi f(x) \sqrt{1 + [f'(x)]^2}$$

is assumed to be continuous (hence Riemann integrable; see Theorem 5), the above Riemann sums converge as $\|P\| \rightarrow 0$; the limiting integral is taken to be the area of the surface:

$$\begin{aligned} S &= \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n 2\pi f(\xi_i) \sqrt{1 + [f'(\xi_i)]^2} \Delta x_i \\ &= \int_a^b 2\pi \underbrace{f(x)}_{\text{radius } y} \underbrace{\sqrt{1 + [f'(x)]^2} dx}_{\text{band width } ds}; \end{aligned}$$

Areas of Surfaces of Revolution X (page 47, Chapter 3)

- (i) if the curve is described by the graph of a nonnegative continuously differentiable function $x = g(y)$, $c \leq y \leq d$, and it is revolved about the y -axis (as in Figure 39), the area of the surface thus generated is given by the same formula with x replaced by y :

$$S = \int_c^d 2\pi \underbrace{g(y)}_{\text{radius } x} \underbrace{\sqrt{1 + [g'(y)]^2}}_{\text{band width } ds} dy;$$

Areas of Surfaces of Revolution XI (page 47, Chapter 3)

- (j) if the curve is described by a pair of continuously differentiable parametric equations $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, and it is traversed exactly once as t increases from a to b (as in Figure 40), then the area of the surface generated by revolving the curve is

$$S = \int_a^b 2\pi \underbrace{g(t)}_{\text{radius } y} \underbrace{\sqrt{[f'(t)]^2 + [g'(t)]^2} dt}_{\text{band width } ds},$$

if the revolution is about the x -axis (assuming $y = g(t) \geq 0$), and

$$S = \int_a^b 2\pi \underbrace{f(t)}_{\text{radius } x} \underbrace{\sqrt{[f'(t)]^2 + [g'(t)]^2} dt}_{\text{band width } ds},$$

if the revolution is about the y -axis (assuming $x = f(t) \geq 0$);

Areas of Surfaces of Revolution XII (page 47, Chapter 3)

- (k) all these formulas for calculating areas of surfaces of revolution can be written in a unified form as

$$S = \int 2\pi(\text{radius})(\text{band width}) = \int 2\pi\rho \, ds,$$

where ρ is the radius from the axis of revolution to an element of arc length ds (see Figure 38);

- (l) for any given problem, the task is then to express the radius function ρ and the arc length differential ds in terms of a common variable, and to supply limits of integration for that variable.

Areas of Surfaces of Revolution XIII (page 47, Chapter 3)

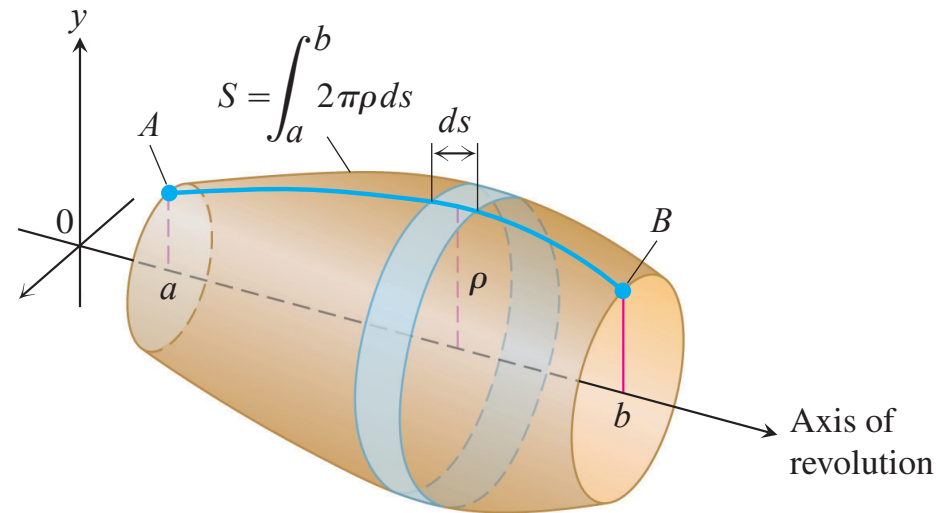


Figure 38: The area of the surface swept out by revolving an arc AB about the axis shown here is $\int_a^b 2\pi\rho ds$; the exact expression depends on the formulas for ρ and ds .

Areas of Surfaces of Revolution XIV (page 47, Chapter 3)

- In summary, to calculate the area of a surface of revolution, we take the following steps:
 - (a) find the radius function ρ and the arc length differential ds ; as a general rule, $\rho = x$ if the axis of revolution is the y -axis, and $\rho = y$ if the axis of revolution is the x -axis;
 - (b) choose the variable of integration; this can be x , y , or a parameter t , if the curve that is used to generate the surface is described by a function of x , y , or a pair of parametric equations in t ;
 - (c) express the radius function ρ and the arc length differential ds as functions of the variable of integration;
 - (d) find the limits of integration;
 - (e) integrate $2\pi\rho ds$ using the Fundamental Theorem.

Areas of Surfaces of Revolution XV (page 49–57, Chapter 3)

- **Example.^a** As an example, consider the cone generated by revolving the line segment $x = 1 - y$, $0 \leq y \leq 1$ about the y -axis.

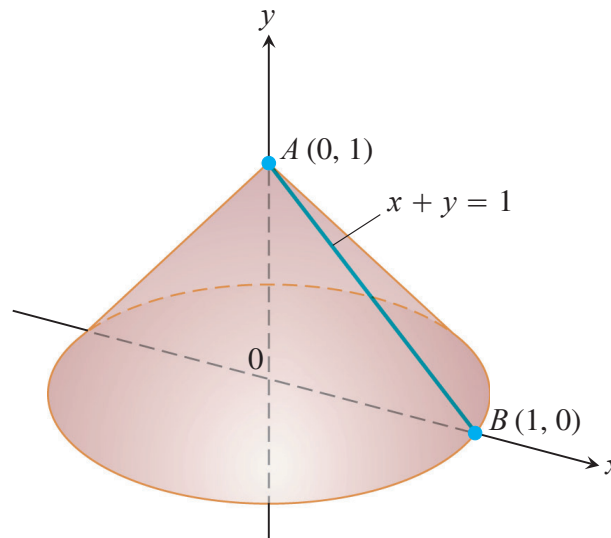


Figure 39: The cone generated by revolving the line segment $x = 1 - y$, $0 \leq y \leq 1$ about the y -axis.

^aSee also Example 20, 21, and 22 on page 49, 52, and 56.

Areas of Surfaces of Revolution XVI (page 49–57, Chapter 3)

To find the area of the cone:

- (a) find the radius function ρ and the arc length differential ds :

$$\rho = x, \quad ds = \sqrt{dx^2 + dy^2};$$

- (b) choose the variable of integration: since the line segment is described by a function of y , it is convenient to choose y as the variable of integration;

Areas of Surfaces of Revolution XVII (page 49–57, Chapter 3)

- (c) express the radius function ρ and the arc length differential ds as functions of the variable of integration y :

$$\rho = x = 1 - y,$$

$$ds = \sqrt{\left(\frac{dx}{dy} dy\right)^2 + dy^2} = \sqrt{1 + [x'(y)]^2} dy = \sqrt{2} dy;$$

- (d) find the limits of integration, which are $y = 0$ and $y = 1$;
(e) the area of the surface is then

$$S = \int_0^1 2\pi(1 - y) \cdot \sqrt{2} dy.$$

Areas of Surfaces of Revolution XVIII (page 49–57, Chapter 3)

- **Example.** As another example, consider the surface generated by revolving the circle $x = \cos t$, $y = 1 + \sin t$, $0 \leq t \leq 2\pi$, centered at the point $(0, 1)$ in the xy -plane about the x -axis.

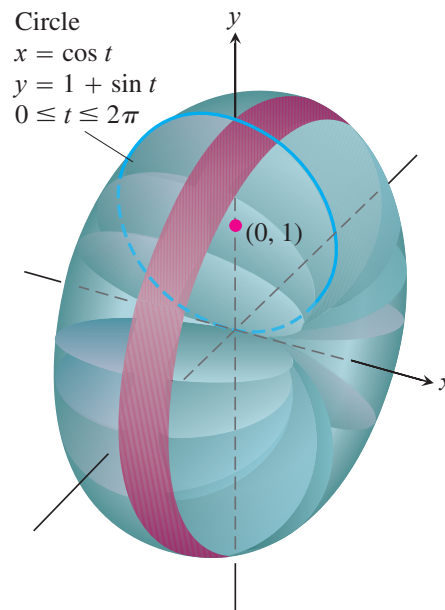


Figure 40: The surface generated by revolving the circle $x = \cos t$, $y = 1 + \sin t$, $0 \leq t \leq 2\pi$, about the x -axis.

Areas of Surfaces of Revolution XIX (page 49–57, Chapter 3)

To find the area of the surface:

- (a) find the radius function ρ and the arc length differential ds :

$$\rho = y, \quad ds = \sqrt{dx^2 + dy^2};$$

- (b) choose the variable of integration: since the circle is described by a pair of parametric equations in t , it is convenient to choose t as the variable of integration;

Areas of Surfaces of Revolution XX (page 49–57, Chapter 3)

- (c) express the radius function ρ and the arc length differential ds as functions of the variable of integration t :

$$\rho = y = 1 + \sin t,$$

$$\begin{aligned} ds &= \sqrt{\left(\frac{dx}{dt} dt\right)^2 + \left(\frac{dy}{dt} dt\right)^2} = \sqrt{[x'(t)]^2 + [y'(t)]^2} dt \\ &= \sqrt{(-\sin t)^2 + (\cos t)^2} dt = \sqrt{\sin^2 t + \cos^2 t} dt = dt; \end{aligned}$$

- (d) find the limits of integration, which are $t = 0$ and $t = 2\pi$;
(e) the area of the surface is then

$$S = \int_0^{2\pi} 2\pi(1 + \sin t) \cdot dt.$$

Argand Diagrams I (page 13, Chapter 5)

- There are two geometric representations of a complex number $z = x + iy$ (see Figure 41):
 - (a) as the point $P(x, y)$ in the xy -plane;
 - (b) as the vector $\overrightarrow{OP}$ from the origin to P .
- In each representation:
 - (a) the x -axis is called the **real axis**, and
 - (b) the y -axis is called the **imaginary axis**;both representations are **Argand diagrams** for $x + iy$.
- If we identify $z = x + iy$ with the vector $\overrightarrow{OP} = \langle x, y \rangle$, then
 - (a) the modulus $r = \sqrt{x^2 + y^2}$ of z is simply the length of $\overrightarrow{OP}$, and
 - (b) the argument θ of z is simply the angle between $\overrightarrow{OP}$ and $\vec{i} = \langle 1, 0 \rangle$.

Argand Diagrams II (page 13, Chapter 5)

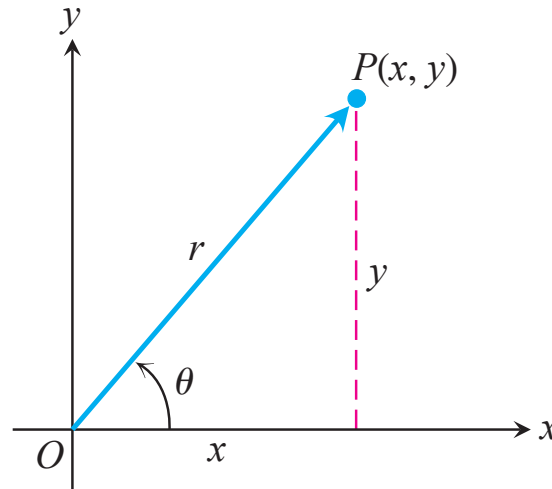


Figure 41: The Argand diagram represents the complex number $z = x + iy$ both as a point $P(x, y)$ and as a vector $\vec{OP}$.

Euler's Formula (page 31, Chapter 5)

- Geometrically, Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$ represents the complex number $e^{i\theta}$ as a unit vector that makes an angle θ with the positive x -axis.

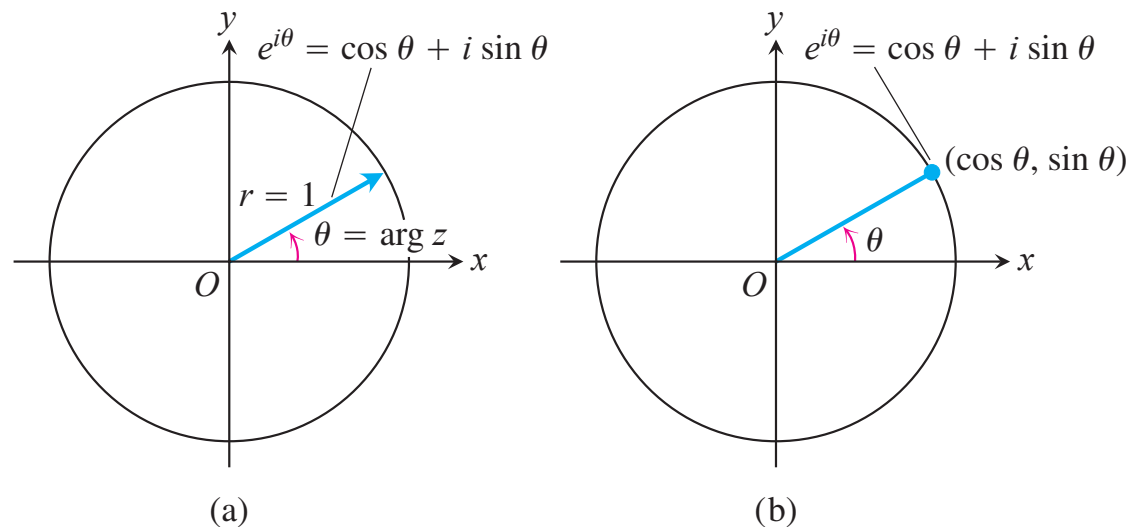


Figure 42: Argand diagrams for $e^{i\theta} = \cos \theta + i \sin \theta$ (a) as a vector and (b) as a point.

Expressing Sums in Euler's Form I (page 36, Chapter 5)

- **Example.^a** To express the complex number $z = e^{i\theta} + e^{i\phi}$ in Euler's form, one could factor out the “average-angle exponential” $e^{i(\theta+\phi)/2}$:

$$\begin{aligned} z &= e^{i\theta} + e^{i\phi} = e^{i(\theta+\phi)/2} [e^{i\theta-i(\theta+\phi)/2} + e^{i\phi-i(\theta+\phi)/2}] \\ &= e^{i(\theta+\phi)/2} [e^{i(\theta-\phi)/2} + e^{-i(\theta-\phi)/2}] = e^{i(\theta+\phi)/2} \cdot 2 \cos \frac{\theta - \phi}{2}, \end{aligned}$$

as was done in the lecture notes, or proceed as follows:

$$\begin{aligned} z &= (\cos \theta + \cos \phi) + i(\sin \theta + \sin \phi) \\ &= 2 \cos \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} + 2i \sin \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} \\ &= 2 \cos \frac{\theta - \phi}{2} \left(\cos \frac{\theta + \phi}{2} + i \sin \frac{\theta + \phi}{2} \right) = 2 \cos \frac{\theta - \phi}{2} \cdot e^{i(\theta+\phi)/2}. \end{aligned}$$

^aSee Example 11 on page 36.

Expressing Sums in Euler's Form II (page 39, Chapter 5)

- **Example.^a** To express the complex numbers $-\cos \theta + i \sin \theta$ and $1 - \cos \theta + i \sin \theta$ in Euler's form, one could invoke the generic substitution

$$\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \quad \sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}),$$

as was done in the lecture notes, or proceed as follows:

$$\begin{aligned} -\cos \theta + i \sin \theta &= -(\cos \theta - i \sin \theta) \\ &= (-1)[\cos(-\theta) + i \sin(-\theta)] = e^{i\pi} e^{-i\theta} = e^{i(\pi-\theta)}, \end{aligned}$$

$$\begin{aligned} 1 - \cos \theta + i \sin \theta &= 1 + (-\cos \theta + i \sin \theta) \\ &= e^{i0} + e^{i(\pi-\theta)} = e^{i(\pi-\theta)/2} [e^{-i(\pi-\theta)/2} + e^{i(\pi-\theta)/2}] \\ &= e^{i(\pi-\theta)/2} \cdot 2 \cos \frac{\pi - \theta}{2} = 2 \sin \frac{\theta}{2} \cdot e^{i(\pi-\theta)/2}. \end{aligned}$$

^aSee Example 12 on page 39.

Products of Complex Numbers (page 26, Chapter 5)

- Geometrically, the product of two complex numbers is a vector whose length is the product of the lengths of the two factors and whose argument is the sum of their arguments.

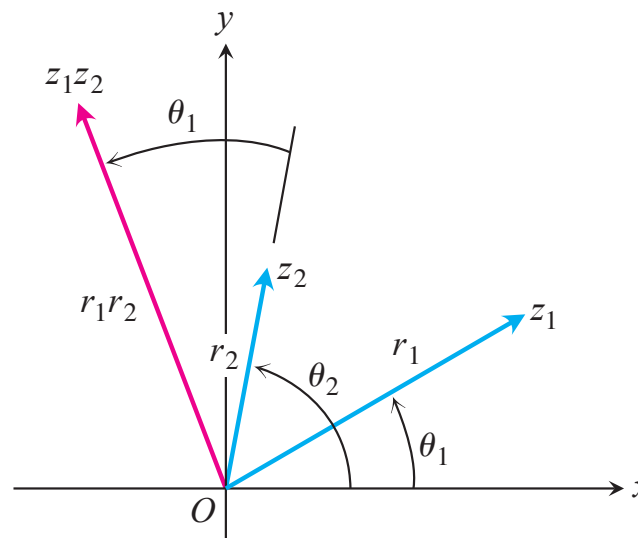


Figure 43: The product of $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ satisfies $|z_1 z_2| = r_1 r_2$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$.

Powers and Roots of Complex Numbers (page 43, Chapter 5)

- The n -th power of a complex number z , where n is a positive integer, can be easily obtained from the Euler form $re^{i\theta}$ of z :

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta).$$

- Likewise, the n -th root of z , where n is a positive integer, can be easily obtained from the **generalized** Euler form of z :

$$z = re^{i\theta} \cdot \mathbf{1} = re^{i\theta} \mathbf{e^{i2k\pi}} = re^{i(\theta+2k\pi)}, \quad k = 0, \pm 1, \pm 2, \dots,$$

$$z^{1/n} = [re^{i(\theta+2k\pi)}]^{1/n} = r^{1/n} e^{i(\theta+2k\pi)/n}, \quad k = 0, 1, 2, \dots, n-1;$$

note that

$$e^{i2(k+ln)\pi/n} = e^{i2k\pi/n + i2ln\pi/n} = e^{i2k\pi/n} \mathbf{e^{i2l\pi}} = e^{i2k\pi/n} \cdot \mathbf{1},$$

for all integers l , so $e^{i2k\pi/n}$ takes on only n distinct values.

Roots of Complex Numbers I (page 49–53, Chapter 5)

- Geometrically:
 - (a) all the n -th roots of $re^{i\theta}$ lie on a circle centered at the origin and having radius equal to the real, positive n -th root of r ;
 - (b) one of them has argument θ/n , and
 - (c) the others are uniformly spaced around the circle, each being separated from its neighbors by an angle equal to $2\pi/n$.
- Figure 44 illustrates the placement of the three cube roots, w_0 , w_1 , and w_2 , of the complex number $z = re^{i\theta}$.

Roots of Complex Numbers II (page 49–53, Chapter 5)

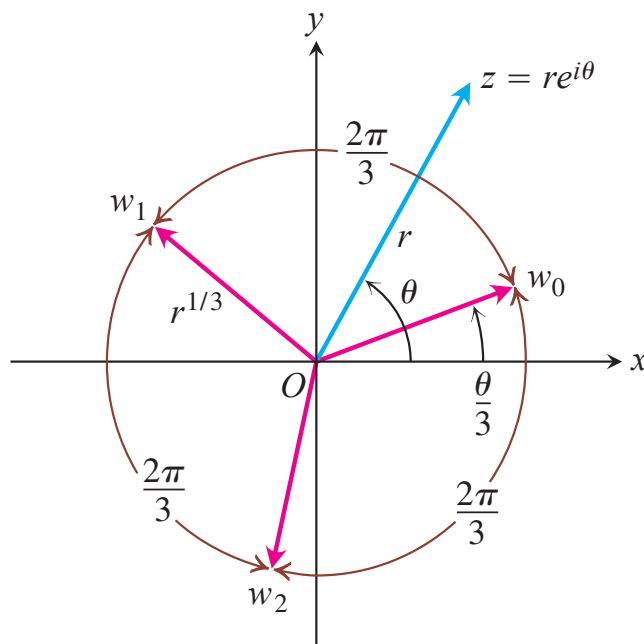


Figure 44: The three cube roots of $z = re^{i\theta}$.

Roots of Complex Numbers III (page 54–57, Chapter 5)

- **Example.^a** To compute $\sqrt[4]{1-i}$, write $1-i$ in Euler's form:

$$1-i = \sqrt{2}e^{i(-\pi/4)} = \sqrt{2}e^{i(-\pi/4+2k\pi)},$$

and then raise the result to the $1/4$ -th power:

$$\begin{aligned}\sqrt[4]{1-i} &= [\sqrt{2}e^{i(-\pi/4+2k\pi)}]^{1/4} = (\sqrt{2})^{1/4}[e^{i(-\pi/4+2k\pi)}]^{1/4} \\ &= 2^{1/8}e^{i(-\pi/4+2k\pi)/4}, \quad k = 0, 1, 2, 3.\end{aligned}$$

Note that, while the extra factor $2k\pi$ in the argument of $1-i$ does not change the value of the number, it does play a role in its fourth root $(1-i)^{1/4}$. The reason is that $2k\pi/4$ is, in general, no longer a multiple of 2π , which means that the extra exponential $e^{i2k\pi/4}$ multiplying the “principal root” $[\sqrt{2}e^{i(-\pi/4)}]^{1/4}$ is no longer 1.

^aSee Example 16 on page 54.

Fractional Powers of Complex Numbers I (page 58, Chapter 5)

- **Example.**^a To find all possible values of $(-\sqrt{3} + i)^{2/5}$, write $-\sqrt{3} + i$ in Euler's form:

$$-\sqrt{3} + i = 2e^{i5\pi/6},$$

square it:

$$(-\sqrt{3} + i)^2 = (2e^{i5\pi/6})^2 = 2^2 e^{i(5\pi/6)2} = 4e^{i(5\pi/3)} = 4e^{i(5\pi/3 + 2k\pi)},$$

and then raise the result to the $1/5$ -th power:

$$\begin{aligned} (-\sqrt{3} + i)^{2/5} &= [(-\sqrt{3} + i)^2]^{1/5} = [4e^{i(5\pi/3 + 2k\pi)}]^{1/5} \\ &= 4^{1/5} e^{i(5\pi/3 + 2k\pi)/5}, \quad k = 0, 1, 2, 3, 4. \end{aligned}$$

^aSee Example 18 on page 58.

Fractional Powers of Complex Numbers II (page 58, Chapter 5)

Alternatively, one may directly raise the Euler form of $-\sqrt{3} + i$:

$$-\sqrt{3} + i = 2e^{i(5\pi/6)} = 2e^{i(5\pi/6+2k\pi)}$$

to the $2/5$ -th power, obtaining

$$\begin{aligned} (-\sqrt{3} + i)^{2/5} &= [2e^{i(5\pi/6+2k\pi)}]^{2/5} = 2^{2/5}e^{i(5\pi/6+2k\pi)(2/5)} \\ &= 4^{1/5}e^{i(5\pi/3+4k\pi)/5}, \quad k = 0, 1, 2, 3, 4. \end{aligned}$$

The two approaches indeed yield the same answers, since the values that $4k\pi/5$ take when k ranges from 0 to 4:

$$0, \frac{4\pi}{5}, \frac{8\pi}{5}, \frac{12\pi}{5} = \frac{2\pi}{5} \pmod{2\pi}, \frac{16\pi}{5} = \frac{6\pi}{5} \pmod{2\pi},$$

are the same as those that $2k\pi/5$ take (after modulo 2π).

Irrational Powers of Complex Numbers (page 59, Chapter 5)

- **Example.** To raise a complex number z to an **irrational** power α , one makes use of the identity $z^\alpha = e^{\log z^\alpha}$ and the properties of the log function. For example, for $z = re^{i\theta}$ and $\alpha = \sqrt{2}$, one computes

$$\begin{aligned}
 z^{\sqrt{2}} &= e^{\log z^{\sqrt{2}}} = e^{\sqrt{2} \log z} = e^{\sqrt{2} \log[re^{i(\theta+2k\pi)}]} = e^{\sqrt{2}[\log r + \log e^{i(\theta+2k\pi)}]} \\
 &= e^{\sqrt{2}[\log r + i(\theta+2k\pi)]} = e^{\sqrt{2} \log r + i\sqrt{2}(\theta+2k\pi)} = e^{\sqrt{2} \log r} e^{i\sqrt{2}(\theta+2k\pi)} \\
 &= e^{\log r^{\sqrt{2}}} e^{i\sqrt{2}\theta} e^{i2k\sqrt{2}\pi} = r^{\sqrt{2}} e^{i\sqrt{2}\theta} e^{i2k\sqrt{2}\pi}.
 \end{aligned}$$

Since $\sqrt{2}$ is irrational, the extra exponential $e^{i2k\sqrt{2}\pi}$ multiplying the “principal power” $(re^{i\theta})^{\sqrt{2}}$ takes on **infinitely many** distinct values. It follows that $z^{\sqrt{2}}$, too, has infinitely many distinct values. The same conclusion holds for any other irrational powers z^α of z .

Solving Algebraic Equations I (page 60, Chapter 5)

- **Example.^a** To solve the equation

$$(1 + z)^8 = (2 - z)^8,$$

divide both sides by $(2 - z)^8$:

$$\frac{(1 + z)^8}{(2 - z)^8} = \left(\frac{1 + z}{2 - z} \right)^8 = 1,$$

and then take the 8-th root to obtain

$$\frac{1 + z}{2 - z} = 1^{1/8} = [e^{i(0+2k\pi)}]^{1/8} = e^{i(2k\pi)/8} = e^{ik\pi/4}.$$

This shows that

$$1 + z = e^{ik\pi/4}(2 - z), \quad \text{or} \quad z = \frac{2e^{ik\pi/4} - 1}{1 + e^{ik\pi/4}}.$$

^aSee Example 20 on page 60.

Solving Algebraic Equations II (page 60, Chapter 5)

To write the result in Cartesian form, observe that

$$\begin{aligned} z &= \frac{2e^{ik\pi/4} - 1}{1 + e^{ik\pi/4}} = \frac{2[(e^{ik\pi/4} + 1) - 1] - 1}{1 + e^{ik\pi/4}} \\ &= \frac{2(e^{ik\pi/4} + 1)}{1 + e^{ik\pi/4}} - \frac{3}{1 + e^{ik\pi/4}} = 2 - \frac{3}{1 + e^{ik\pi/4}}. \end{aligned}$$

To simplify the last fraction, write its components in Euler's form:

$$\begin{aligned} 3 &= 3e^{i0}, \\ 1 + e^{ik\pi/4} &= e^{i0} + e^{ik\pi/4} = e^{i(0+k\pi/4)/2} [e^{i(0-k\pi/4)/2} + e^{i(k\pi/4-0)/2}] \\ &= e^{ik\pi/8} [e^{-ik\pi/8} + e^{ik\pi/8}] = e^{ik\pi/8} \cdot 2 \cos(k\pi/8), \end{aligned}$$

and then divide:

$$\frac{3}{1 + e^{ik\pi/4}} = \frac{3e^{i0}}{2 \cos(k\pi/8) e^{ik\pi/8}} = \frac{3}{2 \cos(k\pi/8)} e^{i(0-k\pi/8)}.$$

Solving Algebraic Equations III (page 60, Chapter 5)

It follows that

$$\begin{aligned} z &= 2 - \frac{3}{1 + e^{ik\pi/4}} = 2 - \frac{3}{2 \cos(k\pi/8)} e^{-ik\pi/8} \\ &= 2 - \frac{3}{2 \cos(k\pi/8)} [\cos(k\pi/8) - i \sin(k\pi/8)] = \frac{1}{2} + \frac{3}{2} i \tan(k\pi/8). \end{aligned}$$

Note that this is the same answer as the one given in the lecture notes (page 62), since the double angle formulas imply that

$$\begin{aligned} \frac{3 \sin(k\pi/4)}{2[1 + \cos(k\pi/4)]} &= \frac{3 \cdot 2 \sin(k\pi/8) \cos(k\pi/8)}{2[1 + 2 \cos^2(k\pi/8) - 1]} \\ &= \frac{3 \sin(k\pi/8) \cos(k\pi/8)}{2 \cos^2(k\pi/8)} = \frac{3}{2} \tan(k\pi/8). \end{aligned}$$

Properties of Complex Conjugate (page 77, Chapter 5)

- In Euler's form, the complex conjugate of $z = re^{i\theta}$ is given by

$$\begin{aligned}\bar{z} &= \overline{re^{i\theta}} = \overline{r \cos \theta + ir \sin \theta} \\ &= r \cos \theta - ir \sin \theta = r [\cos(-\theta) + i \sin(-\theta)] = re^{-i\theta}.\end{aligned}$$

- The various properties of complex conjugate can be readily verified using the definition; for example, if $z_1 = x_1 + iy_1 = r_1 e^{i\theta_1}$ and $z_2 = x_2 + iy_2 = r_2 e^{i\theta_2}$, then

$$\begin{aligned}\overline{z_1 z_2} &= \overline{r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}} = \overline{r_1 r_2 e^{i(\theta_1 + \theta_2)}} \\ &= r_1 r_2 e^{-i(\theta_1 + \theta_2)} = r_1 e^{-i\theta_1} \cdot r_2 e^{-i\theta_2} = \bar{z}_1 \bar{z}_2, \\ \overline{z_1 \pm z_2} &= \overline{(x_1 + iy_1) \pm (x_2 + iy_2)} = \overline{(x_1 \pm x_2) + i(y_1 \pm y_2)} \\ &= (x_1 \pm x_2) - i(y_1 \pm y_2) = (x_1 - iy_1) \pm (x_2 - iy_2) = \bar{z}_1 \pm \bar{z}_2,\end{aligned}$$

which are properties 8 and 9.

Linear Independence I (page 71, Chapter 4)

- **Proof of the theorem.** It suffices to prove that the vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ are linearly **dependent** if and only if the equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{0}, \quad (*)$$

has **at least one nontrivial** (nonzero) solution. To begin with, suppose that the vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ are linearly dependent. Then by definition, there exists some k between 1 and n such that the vector $\vec{a}_k$ can be expressed as a linear combination of the other $\vec{a}_j$'s:

$$\vec{a}_k = c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_{k-1}\vec{a}_{k-1} + c_{k+1}\vec{a}_{k+1} + \cdots + c_n\vec{a}_n.$$

Subtracting $\vec{a}_k$ from both sides yields

$$c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_{k-1}\vec{a}_{k-1} + (-1)\vec{a}_k + c_{k+1}\vec{a}_{k+1} + \cdots + c_n\vec{a}_n = \vec{0},$$

which shows that the equation $(*)$ has one nontrivial solution.

Linear Independence II (page 71, Chapter 4)

Conversely, suppose that the equation (*) has one nontrivial solution $x_1, x_2, \dots, x_n$. Then there exists some k between 1 and n such that $x_k \neq 0$. Subtracting $x_k \vec{a}_k$ from both sides of (*) and dividing through the resulting equation by $(-x_k) \neq 0$ yields

$$\vec{a}_k = -\frac{x_1}{x_k} \vec{a}_1 - \frac{x_2}{x_k} \vec{a}_2 - \dots - \frac{x_{k-1}}{x_k} \vec{a}_{k-1} - \frac{x_{k+1}}{x_k} \vec{a}_{k+1} - \dots - \frac{x_n}{x_k} \vec{a}_n,$$

which shows that $\vec{a}_k$ is a linear combination of the other $\vec{a}_j$'s. In other words, the vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ are linearly dependent. $\square$

Properties of Matrix Addition (page 4, Chapter 6)

- For $m \times n$ matrices A , B , and C , the following properties hold:
 - (a) closure under addition: $A + B$ is again an $m \times n$ matrix;
 - (b) associative law: $(A + B) + C = A + (B + C)$;
 - (c) commutative law: $A + B = B + A$;
 - (d) existence of **additive identity**^a: there exists one and only one element in $\mathbb{C}^{m \times n}$, i.e. the zero matrix 0 consisting of all zeros, such that $A + 0 = A$ for all $A \in \mathbb{C}^{m \times n}$;
 - (e) existence of **additive inverse**: for each $A \in \mathbb{C}^{m \times n}$, there exists one and only one element in $\mathbb{C}^{m \times n}$, i.e. the matrix $(-A)$ obtained by negating each entry of A , such that $A + (-A) = 0$.

^aIn what follows, $\mathbb{C}^{m \times n}$ denotes the set of all $m \times n$ complex matrices, i.e. matrices whose entries are complex numbers.

Matrix Multiplication I (page 5, Chapter 6)

- Before proceeding to defining matrix multiplication, let's consider the problem of composing two linear functions

$$f(x) = f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix},$$

$$g(x) = g \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_{11}x_1 + b_{12}x_2 \\ b_{21}x_1 + b_{22}x_2 \end{pmatrix};$$

(a) first, a direct computation shows that

$$h(x) := f(g(x)) = \begin{pmatrix} (a_{11}b_{11} + a_{12}b_{21})x_1 + (a_{11}b_{12} + a_{12}b_{22})x_2 \\ (a_{21}b_{11} + a_{22}b_{21})x_1 + (a_{21}b_{12} + a_{22}b_{22})x_2 \end{pmatrix};$$

Matrix Multiplication II (page 5, Chapter 6)

- (b) if the functions f, g, h are represented by matrices F, G, H of their coefficients, then

$$F = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad G = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix};$$

- (c) with this association, it is natural to call H the **composition** (or **product**) of F and G , and to write

$$FG = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} = H;$$

- (d) in other words, the product of two matrices represents the composition of the two associated linear functions.

Matrix Multiplication III (page 5, Chapter 6)

- To further develop the concept of matrix multiplication, let's begin by composing a single row with a single column; if

$$R = \begin{pmatrix} r_1 & r_2 & \cdots & r_n \end{pmatrix}, \quad C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

the **inner product** of R with C is defined to be the scalar

$$RC = r_1c_1 + r_2c_2 + \cdots + r_nc_n = \sum_{i=1}^n r_ic_i.$$

Matrix Multiplication IV (page 5, Chapter 6)

- The previous example shows that the (i, j) -entry in the product $H = [h_{ij}]$ of two 2×2 matrices

$$F = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad G = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

can be described as the inner product of the i -th row of F with the j -th column in G , that is,

$$h_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} = F_{i*}G_{*j} = \begin{pmatrix} a_{i1} & a_{i2} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \end{pmatrix}.$$

- This is exactly the way that the general definition of matrix multiplication is formulated.

Matrix Multiplication V (page 5, Chapter 6)

- **Definition 10.** *Matrices A and B are said to be **conformable** for multiplication in the order AB if A has exactly as many columns as B has rows, i.e. A is $m \times p$ and B is $p \times n$. For conformable matrices $A_{m \times p} = [a_{ij}]$ and $B_{p \times n} = [b_{ij}]$, the **matrix product** AB is defined to be the $m \times n$ matrix whose (i, j) -entry is the inner product of the i -th row of A with the j -th column in B , that is,*

$$[AB]_{ij} = A_{i*}B_{*j} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^p a_{ik}b_{kj}.$$

In case A and B fail to be conformable, then no product AB is defined.

Properties of Matrix Multiplication I (page 5, Chapter 6)

- For conformable matrices A , B , and C , there holds:
 - (a) left-hand distributive law: $A(B + C) = AB + AC$;
 - (b) right-hand distributive law: $(A + B)C = AC + BC$;
 - (c) associative law: $A(BC) = (AB)C$.
- **Proof.** To prove (a), assume A is $m \times p$ and B , C are $p \times n$. Then

$$\begin{aligned}
 [A(B + C)]_{ij} &= \sum_{k=1}^p [A]_{ik}[B + C]_{kj} = \sum_{k=1}^p [A]_{ik}([B]_{kj} + [C]_{kj}) \\
 &= \sum_{k=1}^p ([A]_{ik}[B]_{kj} + [A]_{ik}[C]_{kj}) = \sum_{k=1}^p [A]_{ik}[B]_{kj} + \sum_{k=1}^p [A]_{ik}[C]_{kj} \\
 &= [AB]_{ij} + [AC]_{ij} = [AB + AC]_{ij}.
 \end{aligned}$$

So $A(B + C) = AB + AC$, as claimed.

Properties of Matrix Multiplication II (page 5, Chapter 6)

The proof of (b) is similar. As for (c), assume A is $m \times p$, B is $p \times q$, and C is $q \times n$. Then

$$\begin{aligned}
 [A(BC)]_{ij} &= \sum_{k=1}^p [A]_{ik} [BC]_{kj} = \sum_{k=1}^p [A]_{ik} \sum_{l=1}^q [B]_{kl} [C]_{lj} \\
 &= \sum_{k=1}^p \sum_{l=1}^q [A]_{ik} [B]_{kl} [C]_{lj} = \sum_{l=1}^q \left(\sum_{k=1}^p [A]_{ik} [B]_{kl} \right) [C]_{lj} \\
 &= \sum_{l=1}^q [AB]_{il} [C]_{lj} = [(AB)C]_{ij}.
 \end{aligned}$$

So $A(BC) = (AB)C$, as claimed. □

Properties of Scalar Multiplication (page 13, Chapter 6)

- For $m \times n$ matrices A, B and scalars α, β , the following holds:
 - (a) closure under scalar multiplication: αA is again an $m \times n$ matrix;
 - (b) associative law: $(\alpha\beta)A = \alpha(\beta A)$;
 - (c) distributive law I: $\alpha(A + B) = \alpha A + \alpha B$;
 - (d) distributive law II: $(\alpha + \beta)A = \alpha A + \beta A$;
 - (e) scalar identity: $1A = A$ for all $A \in \mathbb{C}^{m \times n}$.

Properties of Matrix Transposition (page 14, Chapter 6)

- For $m \times n$ matrices A , B and scalar α , the following holds:

(a) $(A^T)^T = A$;

(b) $(A + B)^T = A^T + B^T$;

(c) $(\alpha A)^T = \alpha A^T$.

- **Proof.** (a) is obvious. To prove (b) and (c), note that

$$\begin{aligned} [(A + B)^T]_{ij} &= [A + B]_{ji} = [A]_{ji} + [B]_{ji} \\ &= [A^T]_{ij} + [B^T]_{ij} = [A^T + B^T]_{ij}, \\ [(\alpha A)^T]_{ij} &= [\alpha A]_{ji} = \alpha [A]_{ji} = \alpha [A^T]_{ij} = [\alpha A^T]_{ij}. \end{aligned}$$

So $(A + B)^T = A^T + B^T$ and $(\alpha A)^T = \alpha A^T$, as claimed. □

Matrix Inversion (page 21, Chapter 6)

- Consider the scalar equation $\alpha x = \beta$ where α is a nonzero scalar:

(a) a solution of this equation is $x = \alpha^{-1}\beta$ since

$$\alpha(\alpha^{-1}\beta) = (\alpha\alpha^{-1})\beta = (1)\beta = \beta;$$

(b) this solution is unique since if x_1, x_2 are both solutions, then

$$\begin{aligned}\alpha x_1 = \beta = \alpha x_2 &\implies (\alpha^{-1}\alpha)x_1 = (\alpha^{-1}\alpha)x_2 \\ &\implies (1)x_1 = (1)x_2 \implies x_1 = x_2.\end{aligned}$$

- This simple exercise shows that:
 - (a) the properties $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$ is the key to the unique solvability of the scalar equation $\alpha x = \beta$;
 - (b) to solve matrix equations $Ax = b$ in the same fashion as the scalar equation is solved, a matrix analogue of these properties is needed.

Properties of Matrix Inversion (page 27, Chapter 6)

- The operation of transposition has an interesting effect upon a matrix product: a **reversal of order** occurs; more precisely, if A is $m \times p$ and B is $p \times n$, then:

$$(AB)^T = B^T A^T.$$

- **Proof.** By definition,

$$[(AB)^T]_{ij} = [AB]_{ji} = \sum_{k=1}^p [A]_{jk} [B]_{ki} = \sum_{k=1}^p [B^T]_{ik} [A^T]_{kj} = [B^T A^T]_{ij},$$

so $(AB)^T = B^T A^T$, as claimed. □

- **Proof of Property 4.** To show $(A^T)^{-1} = (A^{-1})^T$, simply note that

$$A^T (A^{-1})^T = (A^{-1} A)^T = I^T = I. \quad \square$$

Properties of Determinants I (page 50–52, Chapter 6)

- **Theorem 11** (Effects of row operations). *Let B be the matrix obtained from $A_{n \times n}$ by one of the three elementary row operations:*

- (i) type I: interchange rows i and j ;*
- (ii) type II: multiply row i by $\alpha \neq 0$;*
- (iii) type III: add α times row i to row j .*

Then $\det(B)$ is given by:

- (a) $\det(B) = -\det(A)$ for type I operations;*
- (b) $\det(B) = \alpha \det(A)$ for type II operations;*
- (c) $\det(B) = \det(A)$ for type III operations.*

Properties of Determinants II (page 50–52, Chapter 6)

- **Proof.** We illustrate the proof of the theorem on a 3×3 matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

- (1) Let B be obtained by interchanging two rows, say the first and the second rows, of A . Then

$$\begin{aligned} \det(B) &= \det \begin{pmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \\ &= c_1 \det \begin{pmatrix} b_2 & b_3 \\ a_2 & a_3 \end{pmatrix} - c_2 \det \begin{pmatrix} b_1 & b_3 \\ a_1 & a_3 \end{pmatrix} + c_3 \det \begin{pmatrix} b_1 & b_2 \\ a_1 & a_2 \end{pmatrix}. \end{aligned}$$

Properties of Determinants III (page 50–52, Chapter 6)

For any 2×2 matrix $(a_{ij})_{i,j=1}^2$, it is easy to verify that

$$\det \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix} = a_{21}a_{12} - a_{11}a_{22} = -\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Thus the right-hand side of the above equation equals

$$\begin{aligned} & -\left\{ c_1 \det \begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix} - c_2 \det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix} + c_3 \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \right\} \\ & \qquad \qquad \qquad = -\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \textcolor{red}{c}_1 & \textcolor{red}{c}_2 & \textcolor{red}{c}_3 \end{pmatrix} = -\det(A), \end{aligned}$$

which is what's to be shown.

Properties of Determinants IV (page 50–52, Chapter 6)

- (2) Let B be obtained by multiplying some row, say the first row, of A by $\alpha \neq 0$. Then

$$\begin{aligned}
 \det(B) &= \det \begin{pmatrix} \alpha a_1 & \alpha a_2 & \alpha a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \\
 &= \alpha a_1 \det \begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix} - \alpha a_2 \det \begin{pmatrix} b_1 & b_3 \\ c_1 & c_3 \end{pmatrix} + \alpha a_3 \det \begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \\
 &= \alpha \left\{ a_1 \det \begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix} - a_2 \det \begin{pmatrix} b_1 & b_3 \\ c_1 & c_3 \end{pmatrix} + a_3 \det \begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \right\} \\
 &= \alpha \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \alpha \det(A).
 \end{aligned}$$

Properties of Determinants V (page 50–52, Chapter 6)

- (3) Let B be obtained by adding a multiple of some row, say the first row, of A to some other row, say the second row, of A . Then

$$\begin{aligned} \det(B) &= \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 + \alpha a_1 & b_2 + \alpha a_2 & b_3 + \alpha a_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \\ &= -(b_1 + \alpha a_1) \det \begin{pmatrix} a_2 & a_3 \\ c_2 & c_3 \end{pmatrix} \\ &\quad + (b_2 + \alpha a_2) \det \begin{pmatrix} a_1 & a_3 \\ c_1 & c_3 \end{pmatrix} - (b_3 + \alpha a_3) \det \begin{pmatrix} a_1 & a_2 \\ c_1 & c_2 \end{pmatrix} \end{aligned}$$

Properties of Determinants VI (page 50–52, Chapter 6)

$$\begin{aligned}
 &= -b_1 \det \begin{pmatrix} a_2 & a_3 \\ c_2 & c_3 \end{pmatrix} + b_2 \det \begin{pmatrix} a_1 & a_3 \\ c_1 & c_3 \end{pmatrix} - b_3 \det \begin{pmatrix} a_1 & a_2 \\ c_1 & c_2 \end{pmatrix} \\
 &\quad - \alpha a_1 \det \begin{pmatrix} a_2 & a_3 \\ c_2 & c_3 \end{pmatrix} + \alpha a_2 \det \begin{pmatrix} a_1 & a_3 \\ c_1 & c_3 \end{pmatrix} - \alpha a_3 \det \begin{pmatrix} a_1 & a_2 \\ c_1 & c_2 \end{pmatrix} \\
 &= \det \begin{pmatrix} a_1 & a_2 & a_3 \\ \textcolor{red}{b}_1 & \textcolor{red}{b}_2 & \textcolor{red}{b}_3 \\ c_1 & c_2 & c_3 \end{pmatrix} + \det \begin{pmatrix} a_1 & a_2 & a_3 \\ \textcolor{red}{\alpha a}_1 & \textcolor{red}{\alpha a}_2 & \textcolor{red}{\alpha a}_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \\
 &= \det(A) + \alpha \det \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.
 \end{aligned}$$

Properties of Determinants VII (page 50–52, Chapter 6)

Note that, according to part (a) of the theorem,

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \stackrel{R_1 \leftrightarrow R_2}{=} \det \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

which implies that^a

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = 0.$$

Thus

$$\det(B) = \det(A) + \alpha \cdot 0 = \det(A).$$

^aMore generally, the determinant of any matrix with two identical rows is zero.

Properties of Determinants VIII (page 50–52, Chapter 6)

- Since

$$\det(A^T) = \det(A)$$

for all $A_{n \times n}$, it is not necessary to distinguish between rows and columns when discussing properties of determinant.

- In particular, Theorem 11 still holds when all row operations are replaced by the corresponding column operations; for example:
(a) type I column operation:

$$\det \begin{pmatrix} a_2 & a_1 & a_3 \\ b_2 & b_1 & b_3 \\ c_2 & c_1 & c_3 \end{pmatrix} \stackrel{C_1 \leftrightarrow C_2}{=} - \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} ;$$

Properties of Determinants IX (page 50–52, Chapter 6)

(b) type II column operation:

$$\det \begin{pmatrix} \alpha a_1 & a_2 & a_3 \\ \alpha b_1 & b_2 & b_3 \\ \alpha c_1 & c_2 & c_3 \end{pmatrix} = \alpha \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix};$$

(c) type III column operation:

$$\det \begin{pmatrix} a_1 & a_2 + \alpha a_1 & a_3 \\ b_1 & b_2 + \alpha b_1 & b_3 \\ c_1 & c_2 + \alpha c_1 & c_3 \end{pmatrix} = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Gaussian Elimination I (page 69, Chapter 6)

- The general procedure for Gaussian elimination applied to the solution of a linear system can be summarized as follows:
 - (a) consider a system $Ax = b$ of m equations in n unknowns; represent the system using the augmented matrix $[A|b]$;
 - (b) for $i = 1, 2, \dots, m$, apply Gaussian elimination to the i -th row of the augmented matrix $[A|b]$ by using the following procedure:
 - (i) focus on the submatrix of $[A|b]$ starting from the i -th row and i -th column; choose a **pivot column** from this submatrix, which is the first column in the submatrix that contains a nonzero entry; for example, if

$$[A|b] = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & \boxed{4} & 5 & 2 \\ 0 & 6 & 7 & 3 \end{array} \right),$$

Gaussian Elimination II (page 69, Chapter 6)

and $i = 2$, i.e. the second row is being processed, then the pivot column is $j = 2$, and the **pivot** is thus the (2,2)-entry; note that, when $i = 2$, only the submatrix starting from the second row and second column is considered;

- (ii) if the entry at the pivot position is zero, then perform row interchanges to bring a nonzero entry **below the pivot** into the pivot position; for example, if

$$[A|b] = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & \boxed{0} & 5 & 2 \\ 0 & 6 & 7 & 3 \end{array} \right),$$

and $i = 2$, i.e. the second row is being processed, then the pivot column is again $j = 2$, and to bring a nonzero entry into the (2,2)-pivot position, interchange row 2 with row 3 to obtain:

Gaussian Elimination III (page 69, Chapter 6)

$$[A|b] \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & \boxed{6} & 7 & 3 \\ 0 & 0 & 5 & 2 \end{array} \right);$$

- (iii) after the pivot is determined on the i -th row, eliminate all the entries below the pivot by adding a suitable multiple of row i to the rows below it; this completes the processing of the i -th row;
- (iv) if $i < m$, then move on to row $i + 1$ and repeat the above procedure; otherwise stop;
- (c) once Gaussian elimination is done, the original augmented matrix $[A|b]$ is transformed into a reduced matrix $[U|c]$ where U is in **row echelon form**; see below for the typical structure of a matrix in row echelon form;

Gaussian Elimination IV (page 69, Chapter 6)

- The typical structure of a matrix in row echelon form is illustrated below with the pivots boxed:

$$\begin{pmatrix} \boxed{*} & * & * & * & * & * & * & * \\ 0 & 0 & \boxed{*} & * & * & * & * & * \\ 0 & 0 & 0 & \boxed{*} & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{*} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Gaussian Elimination V (page 69, Chapter 6)

- (d) if $[U|c]$ has a row of the form $(0\ 0\ \cdots\ 0|\alpha)$ where $\alpha \neq 0$ is a nonzero number, then the original system $Ax = b$ has no solutions; this is true because the equation corresponding to this row says

$$0 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n = \alpha, \quad \text{or simply} \quad 0 = \alpha,$$

which cannot possibly be satisfied since $\alpha \neq 0$ is nonzero;

- (e) on the other hand, any rows of the form $(0\ 0\ \cdots\ 0|0)$ in $[U|c]$ are equivalent to the equation

$$0 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n = 0, \quad \text{or simply} \quad 0 = 0,$$

which is trivially satisfied; in other words, these rows can be safely ignored from $[U|c]$;

Gaussian Elimination VI (page 69, Chapter 6)

- (f) if no rows of the form $(0 \ 0 \ \cdots \ 0 | \alpha \neq 0)$ appear in $[U|c]$, then the original system $Ax = b$ has at least one solution; furthermore:
- (i) if each column of U contains a pivot, then the system has a unique solution, which can be found using back substitution;
 - (ii) if at least one column of U contains no pivot, which means

$$\begin{aligned} \#(\text{unknowns}) &= \#(\text{columns}) \\ &> \#(\text{pivots}) = \#(\text{nonzero rows in } U) = \#(\text{equations}), \end{aligned}$$

then the system has infinitely many solutions.

Gauss-Jordan Method I (page 69, Chapter 6)

- The **Gauss-Jordan method** is a variation of Gaussian elimination and is distinct in the following two aspects:
 - (a) at each step, the pivot element is forced to be 1;
 - (b) at each step, all terms **above** and below the pivot are eliminated.
- The row echelon form produced by Gauss-Jordan contains a reduced number of nonzero entries, and hence entails a special name: **reduced row echelon form**.

Gauss-Jordan Method II (page 69, Chapter 6)

- The typical structure of a matrix in reduced row echelon form is illustrated below with the pivots boxed:

$$\begin{pmatrix} \boxed{1} & * & 0 & 0 & * & * & 0 & * \\ 0 & 0 & \boxed{1} & 0 & * & * & 0 & * \\ 0 & 0 & 0 & \boxed{1} & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Gauss-Jordan Method III (page 69, Chapter 6)

- **Example.** As an example, let's apply the Gauss-Jordan method to the following 3×3 system:

$$2x_1 + 2x_2 + 6x_3 = 4$$

$$2x_1 + x_2 + 7x_3 = 6$$

$$-2x_1 - 6x_2 - 7x_3 = -1$$

The elimination procedure yields:

$$\left(\begin{array}{ccc|c} 2 & 2 & 6 & 4 \\ 2 & 1 & 7 & 6 \\ -2 & -6 & -7 & -1 \end{array} \right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right),$$

hence the solution is

$$x_1 = 0, \quad x_2 = -1, \quad x_3 = 1.$$

Systems with Infinitely Many Solutions I (page 77, Chapter 6)

- **Example.^a** Consider the linear system

$$\begin{aligned}x + y - 2z &= 2 \\ 2x + 2y - 3z &= 1\end{aligned}$$

Applying Gaussian elimination to the augmented matrix yields

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 2 \\ 2 & 2 & -3 & 1 \end{array} \right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 2 \\ 0 & 0 & \boxed{1} & -3 \end{array} \right).$$

Since the number of pivots (2) is less than the number of unknowns (3), the system has infinitely many solutions. To find the general solution:

- (a) identify the **pivot variables**, which are variables corresponding to the pivot columns of the coefficient matrix (x and z);

^aSee Example 38 on page 77.

Systems with Infinitely Many Solutions II (page 77, Chapter 6)

- (b) identify the **free variables**, which are variables corresponding to the non-pivot columns of the coefficient matrix (y);
- (c) solve the pivot variables x and z in terms of the free variable y :

$$z = -3, \quad x = 2 - y + 2z = 2 - y + 2(-3) = -4 - y;$$

- (d) it is convenient to express the solution set by writing

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 - y \\ y \\ -3 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ -3 \end{pmatrix} + y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

with the understanding that y is a free variable that can range over all real numbers; this representation is called the **general solution** of the linear system.

Systems with Infinitely Many Solutions III (page 77, Chapter 6)

Note that:

- (a) the vector $(-4, 0, -3)^T$ is a **particular solution** of the linear system, corresponding to the particular choice $y = 0$ of the free variable;
- (b) the vector $(-1, 1, 0)^T$ multiplying the free variable y is a solution of the associated homogeneous system

$$\begin{aligned}x + y - 2z &= 0 \\ 2x + 2y - 3z &= 0\end{aligned};$$

indeed, if we write the linear system as $Ax = b$ where

$$A = \begin{pmatrix} 1 & 1 & -2 \\ 2 & 2 & -3 \end{pmatrix}, \quad x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

Systems with Infinitely Many Solutions IV (page 77, Chapter 6)

and write the general solution as $x = x_p + yh$ where

$$x_p = \begin{pmatrix} -4 \\ 0 \\ -3 \end{pmatrix}, \quad h = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

then the fact that x_p is a particular solution of the linear system implies that

$$y \cdot Ah = A(yh) = A(x - x_p) = Ax - Ax_p = b - b = 0;$$

since y is an arbitrary real number, it must then follow that

$$Ah = 0;$$

in other words, the vector h must be a solution of the homogeneous system $Ax = 0$, as claimed.

Block Matrix Multiplication I (page 93, Chapter 6)

- A useful technique in matrix multiplication is to partition one or both factors into **submatrices**, or simply **blocks**: if

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{sr} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1t} \\ B_{21} & B_{22} & \cdots & B_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ B_{r1} & B_{r2} & \cdots & B_{rt} \end{pmatrix},$$

are two **conformably partitioned** block matrices in which each pair (A_{ik}, B_{kj}) is conformable^a, then the (i, j) -block in AB is

$$A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{ir}B_{rj}.$$

^aMatrices A and B are said to be **conformable** for multiplication in the order AB if A has exactly as many columns as B has rows, i.e. A is $m \times p$ and B is $p \times n$.

Block Matrix Multiplication II (page 93, Chapter 6)

- **Example.** Block multiplication is particularly useful when there are patterns in the matrices to be multiplied. Consider the block matrices

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} C & I \\ I & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \end{pmatrix} = \begin{pmatrix} I & 0 \\ C & C \end{pmatrix},$$

Block Matrix Multiplication III (page 93, Chapter 6)

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Using block multiplication, the product AB is easily computed to be

$$AB = \begin{pmatrix} 2C & C \\ I & 0 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 1 & 2 \\ 6 & 8 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Finding the Inverse of a Matrix I (page 93, Chapter 6)

- To find the inverse X of an $n \times n$ matrix A , note that:
 - (a) X is the solution of the matrix equation $AX = I_n$, where I_n is the $n \times n$ identity matrix;
 - (b) the equation $AX = I_n$ can be written in block form as

$$\underbrace{A}_{n \times n} \left[\underbrace{x_1}_{n \times 1} \mid x_2 \mid \cdots \mid x_n \right] = [e_1 \mid e_2 \mid \cdots \mid e_n],$$

where x_j is the j -th column of X and e_j is the j -th **unit vector** (vector that contains a 1 in the j -th position and 0's elsewhere);

Finding the Inverse of a Matrix II (page 93, Chapter 6)

- (c) using block multiplication, the equation in (b) can be written as

$$[Ax_1 | Ax_2 | \cdots | Ax_n] = [e_1 | e_2 | \cdots | e_n],$$

or equivalently,

$$Ax_j = e_j, \quad j = 1, 2, \dots, n;$$

thus finding the inverse X of A is equivalent to solving n simultaneous linear systems $Ax_j = e_j$;

- (d) the n linear systems $Ax_j = e_j$ can be solved **simultaneously**, instead of one after one, by applying Gaussian elimination to the augmented matrix $[A | e_1 | e_2 | \cdots | e_n] = [A | I_n]$;

Finding the Inverse of a Matrix III (page 93, Chapter 6)

- (e) alternatively, the n systems $Ax_j = e_j$ can be solved simultaneously by applying the **Gauss-Jordan method** to the augmented matrix $[A|I_n]$, which reduces $[A|I_n]$ to its **reduced** row echelon form;
- (f) if A is nonsingular, then for each j Gauss-Jordan reduces $[A|e_j]$ to $[I|x_j]$ where x_j is precisely the j -th column of X ; hence

$$[A|I_n] \xrightarrow{\text{Gauss-Jordan}} [I|X]$$

where $X = A^{-1}$ is the inverse of A ;

- (g) the only way for Gauss-Jordan to fail is for a row of zeros to emerge in the left-hand side of $[A|I]$ at some point during the elimination process, and this occurs if and only if A is singular.

Testing for Linear Independence I (page 104, Chapter 6)

- A useful way to check whether the vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ are linearly independent is to check whether the equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{0}$$

has only the trivial solution, or equivalently, to check whether the homogeneous system

$$\vec{a}_1x_1 + \vec{a}_2x_2 + \cdots + \vec{a}_nx_n = \underbrace{[\vec{a}_1 \mid \vec{a}_2 \mid \cdots \mid \vec{a}_n]}_{n \times 1} \begin{pmatrix} \underbrace{x_1}_{1 \times 1} \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \vec{0}$$

has only the trivial solution $x_1 = x_2 = \cdots = x_n = 0$.

Testing for Linear Independence II (page 105, Chapter 6)

- **Example.**^a To determine whether the vectors

$$\vec{a} = 3\vec{i} + 2\vec{j}, \quad \vec{b} = \vec{i} - \vec{k}, \quad \vec{c} = 2\vec{i} + 3\vec{j} + 4\vec{k},$$

are linearly independent:

- (a) consider the homogeneous linear system $Ax = 0$ where

$$A = [\vec{a} | \vec{b} | \vec{c}] = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 0 & 3 \\ 0 & -1 & 4 \end{pmatrix};$$

^aSee Example 47 on page 105.

Testing for Linear Independence III (page 105, Chapter 6)

(b) Gaussian eliminating the augmented matrix^a $[A|0]$ yields

$$[A|0] = \left(\begin{array}{ccc|c} 3 & 1 & 2 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \end{array} \right) \xrightarrow{\text{G.E.}} \left(\begin{array}{ccc|c} 1 & 1/3 & 2/3 & 0 \\ 0 & 1 & -5/2 & 0 \\ 0 & 0 & 3/2 & 0 \end{array} \right);$$

(c) note that the system $Ax = 0$ is consistent^b, since there is no row of the form $(0 \ 0 \ 0 \mid b)$ with $b \neq 0$ (in fact, b is **always** 0);

^aIndeed, when solving homogeneous linear systems of the form $Ax = 0$, it suffices to apply Gaussian elimination to the matrix A alone, since the right-hand side remains to be 0 in the entire elimination process.

^bMore generally, every homogeneous linear system $Ax = 0$ is consistent since $x = 0$ is **always** a solution.

Testing for Linear Independence IV (page 105, Chapter 6)

- (d) since every column of A contains a pivot, the system $Ax = 0$ has a unique solution, i.e. the trivial solution $x = 0$; this shows that the equation

$$x_1\vec{a} + x_2\vec{b} + x_3\vec{c} = [\vec{a}|\vec{b}|\vec{c}] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0}$$

has only the trivial solution $x_1 = x_2 = x_3 = 0$; thus the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly independent.