
EE3121
Differential Equations for Electrical Engineering

Lecture 7: Laplace Transform



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Laplace Transform

- For linear time-invariant (LTI) ordinary differential equations (ODEs) of any order, **Laplace transforms**
 - convert ODEs into algebraic ones;
 - solve ODEs for initial conditions without extra steps;
 - permit solution of nonhomogeneous equations without first solving homogeneous equations;
 - address impulsive forcing functions easily.
- A handy means of solving differential equations.
- A principal analytic tool for engineers.

7.1 Preliminaries

- Improper integral:

$$\int_a^{\infty} f(t)dt = \lim_{A \rightarrow \infty} \int_a^A f(t)dt.$$

- Piecewise continuous: A function is *piecewise continuous* in the interval $\alpha \leq t \leq \beta$ if it is continuous there except for a finite number of jump discontinuities.
- Exponential order: A function is of *exponential order* if there exist a , K , and M such that $f(t) < Ke^{at}$ if $t > M$.

7.2 Introducing the Laplace Transform

- The Laplace transform $F(s)$ of a function $f(t)$ is

$$F(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

- Existence: The transform $F(s)$ exists provided the function $f(t)$ is piecewise continuous and of exponential order.
- Uniqueness: If $f(t)$ is continuous and $F(s)$ is its transform, there is no other continuous function with that transform.
- Linearity: If the Laplace transform of $f_1(t)$ is $F_1(s)$, the Laplace transform of $f_2(t)$ is $F_2(s)$, and a and b are constants, then the Laplace transform of $af_1(t) + bf_2(t)$ is $aF_1(s) + bF_2(s)$.

7.3 Laplace Transforms of Some Common Functions

- The Laplace transform of $f(t) = e^{at}$:

$$\begin{aligned} L\{e^{at}\} &= \int_0^{\infty} e^{-st} e^{at} dt = \lim_{T \rightarrow \infty} \int_0^T e^{-st} e^{at} dt \\ &= \lim_{T \rightarrow \infty} \left[\frac{1}{s-a} \left(1 - e^{-(s-a)T} \right) \right] = \frac{1}{s-a} \end{aligned}$$

The integral is defined only for $s > a$.

7.3 Laplace Transforms of Some Common Functions

- The Laplace transform of $f(t) = t$:

$$\begin{aligned} L\{t\} &= \int_0^{\infty} e^{-st} t dt = \lim_{T \rightarrow \infty} \int_0^T e^{-st} t dt \\ &= \lim_{T \rightarrow \infty} \left(-\frac{T e^{-sT}}{s} + \frac{1}{s^2} (1 - e^{-sT}) \right) = \frac{1}{s^2}. \end{aligned}$$

The result is valid only for $s > 0$.

7.3 Laplace Transforms of Some Common Functions

- The Laplace transform of $f(t) = \sin(\omega t)$:

$$L\{\sin(\omega t)\} = \int_0^{\infty} e^{-st} \sin(\omega t) dt = \lim_{T \rightarrow \infty} \int_0^T e^{-st} \sin(\omega t) dt$$

Integrating by parts twice:

$$\begin{aligned} \int_0^T e^{-st} \sin(\omega t) dt &= -\frac{1}{s} (e^{-sT} \sin(\omega T)) + \frac{\omega}{s} \int_0^T e^{-st} \cos(\omega t) dt \\ &= -\frac{1}{s} (e^{-sT} \sin(\omega T)) + \frac{\omega}{s} \left[-\frac{1}{s} (e^{-sT} \cos(\omega T) - 1) - \frac{\omega}{s} \int_0^T e^{-st} \sin(\omega t) dt \right] \end{aligned}$$

$$\left(1 + \frac{\omega^2}{s^2}\right) \int_0^T e^{-st} \sin(\omega t) dt = -\frac{1}{s} (e^{-sT} \sin(\omega T)) - \frac{\omega}{s^2} (e^{-sT} \cos(\omega T) - 1)$$

7.3 Laplace Transforms of Some Common Functions

$$\int_0^T e^{-st} \sin(\omega t) dt = \frac{1}{s^2 + \omega^2} (\omega - e^{-sT} (s \sin(\omega T) + \omega \cos(\omega T))) .$$

■ Therefore,

$$L\{\sin(\omega t)\} = \lim_{T \rightarrow \infty} \int_0^T e^{-st} \sin(\omega t) dt = \frac{\omega}{s^2 + \omega^2}$$

■ The Laplace transform of $f(t)e^{-\lambda t}$:

$$L\{f(t)e^{-\lambda t}\} = \int_0^\infty e^{-st} f(t) e^{-\lambda t} dt = \int_0^\infty e^{-(s+\lambda)t} f(t) dt = F(s + \lambda).$$

7.4 Laplace Transforms of Derivatives

- Theorem: Let $f(t)$ be continuous and of exponential order and let $\frac{df}{dt}$ be piecewise continuous on any interval $0 \leq t \leq A$. Then the Laplace transform of $\frac{df}{dt}$ exists and

$$L\left\{\frac{df}{dt}\right\} = sL\{f(t)\} - f(0) = sF(s) - f(0).$$

- Corollary:

$$L\left\{\frac{d^2f}{dt^2}\right\} = s^2L\{f(t)\} - sf(0) - \frac{df}{dt}(0)$$

$$L\left\{\frac{d^3f}{dt^3}\right\} = s^3L\{f(t)\} - s^2f(0) - s\frac{df}{dt}(0) - \frac{d^2f}{dt^2}(0)$$

$\vdots$

$$L\left\{\frac{d^nf^n}{dt^n}\right\} = s^nL\{f(t)\} - s^{n-1}f(0) - \sum_{i=1}^{n-1} s^{n-1-i} \frac{d^i f}{dt^i}(0).$$

7.5 Short Table of Laplace Transforms

TABLE 7.1
Short Table of Laplace Transforms
(Initial Version)

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$e^{-\lambda t}$	$\frac{1}{s + \lambda}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$te^{-\lambda t}$	$\frac{1}{(s + \lambda)^2}$
$e^{-\lambda t} \sin \omega t$	$\frac{\omega}{(s + \lambda)^2 + \omega^2}$
$e^{-\lambda t} \cos \omega t$	$\frac{s + \lambda}{(s + \lambda)^2 + \omega^2}$

7.6 Laplace Transforms in Homogeneous Ordinary Differential Equations

- The general approach to solving ODEs with Laplace transforms is a **four** step process:
 - Transform each element of the given ODE from a function of time into an algebraic function of s .
 - Solve the resulting algebraic equation.
 - Manipulate the algebraic solution into a linear combination of forms that appear in our short table of Laplace transforms.
 - Use the table to determine the function of time corresponding to the algebraic solution.

7.6.1 Example 1

- **Problem:** Using Laplace transforms, solve the first-order ODE

$$\frac{dy}{dt} + 2y = 0, \quad y(0) = 5.$$

- **Solution:**

- $L\{\frac{dy}{dt} + 2y\} = L\{0\} \Rightarrow L\{\frac{dy}{dt}\} + 2L\{y\} = 0.$
- Defining $L\{y(t)\} = Y(s).$
- $L\{\frac{dy}{dt}\} = sL\{y\} - y(0) = sY - 5 \Rightarrow sY - 5 + 2Y = 0.$
- $Y(s) = \frac{5}{s+2}$ is the solution to the ODE in the Laplace transform domain (frequency domain).
- Refer to the short table and get the inverse Laplace transform,

$$y(t) = L^{-1}\{Y(s)\} = 5e^{-2t}.$$

7.6.2 Example 2

- **Problem:** Using Laplace transforms, solve the second-order ODE

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 0, \quad y(0) = 4, \quad \frac{dy}{dt}(0) = -1.$$

- **Solution:**

- $L\{\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y\} = L\{0\} \Rightarrow L\{\frac{d^2y}{dt^2}\} + 4L\{\frac{dy}{dt}\} + 13L\{y\} = 0.$
- Defining $L\{y(t)\} = Y(s).$
- $L\{\frac{dy}{dt}\} = sL\{y\} - y(0) = sY - 4.$
- $L\{\frac{d^2y}{dt^2}\} = sL\{\frac{dy}{dt}\} - \frac{dy}{dt}(0) = s(sY - 4) - (-1) = s^2Y - 4s + 1.$
- $L\{\frac{d^2y}{dt^2}\} + 4L\{\frac{dy}{dt}\} + 13L\{y\} = s^2Y - 4s + 1 + 4(sY - 4) + 13Y = 0.$
- $Y(s) = \frac{4s+15}{s^2+4s+13}$ is the solution to the ODE in the Laplace transform domain.

7.6.2 Example 2

■ Solution:

- $Y(s) = \frac{4s+15}{s^2+4s+13}$ is the solution to the ODE in the Laplace transform domain.

■

$$Y(s) = \frac{4s + 15}{(s + 2)^2 + 3^2} = c_1 \frac{(s + 2)}{(s + 2)^2 + 3^2} + c_2 \frac{3}{(s + 2)^2 + 3^2}.$$

- Refer to the short table and get the inverse Laplace transform,
 $y(t) = L^{-1}\{Y(s)\} = c_1 e^{-2t} \cos(3t) + c_2 e^{-2t} \sin(3t).$
- $c_1 = 4, \quad 3(c_1 + c_2) = 15 \Rightarrow c_2 = \frac{7}{3}.$

■

$$y(t) = 4e^{-2t} \cos(3t) + \frac{7}{3}e^{-2t} \sin(3t).$$

7.6.3 Example 3

- **Problem:** Using Laplace transforms, solve the third-order ODE

$$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0, \quad y(0) = 1, \quad \frac{dy}{dt}(0) = 2, \quad \frac{d^2y}{dt^2}(0) = 3.$$

- **Solution:**

- Defining $L\{y(t)\} = Y(s)$.
- $L\{\frac{dy}{dt}\} = sL\{y\} - y(0) = sY - 1$.
- $L\{\frac{d^2y}{dt^2}\} = sL\{\frac{dy}{dt}\} - \frac{dy}{dt}(0) = s(sY - 1) - 2 = s^2Y - s - 2$.
- $L\{\frac{d^3y}{dt^3}\} = sL\{\frac{d^2y}{dt^2}\} - \frac{d^2y}{dt^2}(0) = s(s^2Y - s - 2) - 3 = s^3Y - s^2 - 2s - 3$.
- $Y(s) = \frac{s^2+5s+13}{s^3+3s^2+4s+2}$ is the solution to the ODE in the Laplace transform domain.

7.6.3 Example 3

■ Solution:

- $Y(s) = \frac{s^2+5s+13}{s^3+3s^2+4s+2}$ is the solution to the ODE in the Laplace transform domain.

■

$$Y(s) = \frac{s^2 + 5s + 13}{s^3 + 3s^2 + 4s + 2} = \frac{s^2 + 5s + 13}{(s + 1)((s + 1)^2 + 1)}$$

- Partial fraction expansion:

$$\begin{aligned} Y(s) &= \frac{c_1}{s + 1} + \frac{c_2s + c_3}{(s + 1)^2 + 1} \\ &= \frac{(c_1 + c_2)s^2 + (2c_1 + c_2 + c_3)s + (2c_1 + c_3)}{(s + 1)((s + 1)^2 + 1)}. \end{aligned}$$

7.6.3 Example 3

■ Solution:

- $c_1 + c_2 = 1, 2c_1 + c_2 + c_3 = 5, 2c_1 + c_3 = 13 \Rightarrow c_1 = 9, c_2 = -8, c_3 = -5.$
- $Y(s) = \frac{9}{s+1} - \frac{8s+5}{(s+1)^2+1} = \frac{9}{s+1} - \frac{8(s+1)}{(s+1)^2+1} + \frac{3}{(s+1)^2+1}.$
- Refer to the short table and get the inverse Laplace transform,

$$y(t) = L^{-1}\{Y(s)\} = 9e^{-t} - 8e^{-t} \cos(t) + 3e^{-t} \sin(t).$$

7.7 Laplace Transforms in Nonhomogeneous Ordinary Differential Equations

- Consider ODEs of the form

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \dots + a_n y = g(t),$$

$$y(0) = y_0, \frac{dy}{dt}(0) = \dot{y}_0, \dots, \frac{d^{n-1} y}{dt^{n-1}}(0) = y_0^{(n-1)}.$$

7.7.1 General Solution Using Laplace Transforms

■ **Theorem:** The solution to the above ODE is

$$y(t) = L^{-1}\{Y(s)\},$$

where

$$Y(s) = \frac{p_1 s^{n-1} + p_2 s^{n-2} + \dots + p_{n-1} s + p_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n} + \frac{G(s)}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n},$$

$$p_1 = y_0, \quad p_2 = \dot{y}_0 + a_1 y_0, \quad p_3 = \ddot{y}_0 + a_1 \dot{y}_0 + a_2 y_0, \dots,$$

$$p_n = y_0^{n-1} + a_1 y_0^{n-2} + \dots + a_{n-2} \dot{y}_0 + a_{n-1} y_0$$

and $G(s) = L\{g(t)\}$.

7.7.1 General Solution Using Laplace Transforms

■ Proof:

- Step 1: Transform each element of the ODE:

$$L\{y(t)\} = Y(s), \quad L\left\{\frac{dy}{dt}\right\} = sY(s) - y(0) = sY(s) - y_0, \dots,$$
$$L\left\{\frac{d^n y}{dt^n}\right\} = s^n Y(s) - s^{n-1}y_0 - \dots - sy_0^{(n-2)} - y_0^{(n-1)}$$

- Step 2: Substitute the transformed elements into the ODE:

$$\begin{aligned} & (s^n Y(s) - s^{n-1}y_0 - \dots - sy_0^{(n-2)} - y_0^{(n-1)}) \\ & + a_1(s^{n-1}Y(s) - s^{n-2}y_0 - \dots - sy_0^{(n-3)} - y_0^{(n-2)}) \dots \\ & + a_{n-2}(s^2Y(s) - sy_0 - \dot{y}_0) + a_{n-1}(sY(s) - y_0) + a_n Y(s) = G(s). \end{aligned}$$

7.7.1 General Solution Using Laplace Transforms

■ Proof:

- Step 3: Collect all terms involving $Y(s)$ on the left-hand side of the equation, everything else on the other:

$$\begin{aligned} (s^n + a_1 s^{(n-1)} + \dots + a_{n-1} s + a_n) Y(s) &= s^{n-1} y_0 + \dots + y_0^{(n-1)} \\ &+ a_1 (s^{n-2} y_0 + \dots + y_0^{(n-2)}) + \dots + a_{n-2} (s y_0 + \dot{y}_0) + a_{n-1} y_0 + G(s). \end{aligned}$$

7.7.1 General Solution Using Laplace Transforms

■ Proof:

- Step 4: On the right-hand side, collect coefficients of each power of s :

$$\begin{aligned} & (s^n + a_1 s^{(n-1)} + \dots + a_{n-1} s + a_n) Y(s) \\ &= y_0 s^{(n-1)} + (\dot{y}_0 + a_1 y_0) s^{(n-2)} + (\ddot{y}_0 + a_1 \dot{y}_0 + a_2 y_0) s^{n-3} + \dots \\ &\quad + (y_0^{(n-2)} + a_1 y_0^{(n-3)} + \dots + a_{n-2} y_0) s \\ &\quad + (y_0^{(n-1)} + a_1 y_0^{(n-2)} + \dots + a_{n-1} y_0) + G(s) \\ &= p_1 s^{n-1} + p_2 s^{n-2} + \dots + p_{n-1} s + p_n + G(s). \end{aligned}$$

- Step 5: Divide through by $s^n + a_1 s^{(n-1)} + \dots + a_{n-1} s + a_n$.

7.7.2 Example

- **Problem:** Using Laplace transforms, solve the ODE

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 15y = 2t, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = \dot{y}_0 = 0.$$

- **Solution:**

- Defining $L\{y(t)\} = Y(s)$.
- $L\{\frac{dy}{dt}\} = sL\{y\} - y(0) = sY$.
- $L\{\frac{d^2y}{dt^2}\} = sL\{\frac{dy}{dt}\} - \frac{dy}{dt}(0) = s^2Y$.
- $L\{2t\} = \frac{2}{s^2}$.
- Substituting the transformed elements into the ODE:

$$(s^2 + 8s + 15)Y(s) = \frac{2}{s^2}.$$

7.7.2 Example

■ Solution:

■ Solving:

$$Y(s) = \frac{2}{s^2(s^2 + 8s + 15)} = \frac{2}{s^2(s + 3)(s + 5)}.$$

■ Partial fraction expansion:

$$Y(s) = \frac{c_1 s + c_2}{s^2} + \frac{c_3}{s + 3} + \frac{c_4}{s + 5}.$$

$$■ \quad c_1 + c_2 + c_4 = 0, \quad 8c_1 + c_2 + 5c_3 + 3c_4 = 0, \quad 15c_1 + 8c_2 = 0, \quad 15c_2 = 2.$$

$$■ \quad c_1 = -\frac{16}{225}, \quad c_2 = \frac{2}{15}, \quad c_3 = \frac{1}{9}, \quad c_4 = -\frac{1}{25}.$$

■

$$Y(s) = -\frac{16}{225} \frac{1}{s} + \frac{2}{25} \frac{1}{s^2} + \frac{1}{9} \frac{1}{s + 3} - \frac{1}{25} \frac{1}{s + 5}.$$

7.7.2 Example

■ Solution:



$$y(t) = L^{-1}\{Y(s)\} = -\frac{16}{225} + \frac{2}{15}t + \frac{1}{9}e^{-3t} - \frac{1}{25}e^{-5t}.$$

7.7.3 Laplace Transform of A Convolution

■ **Theorem:** Let $H(s) = L\{h(t)\}$ and $G(s) = L\{g(t)\}$. Then

$$H(s)G(s) = L \left\{ \int_0^t h(t - \tau)g(\tau)d\tau \right\}$$

■ **Proof:**

■

$$\begin{aligned} L \left\{ \int_0^t h(t - \tau)g(\tau)d\tau \right\} &= \int_0^\infty e^{-st} \int_0^t h(t - \tau)g(\tau)d\tau dt \\ &= \int_0^\infty \int_0^t e^{-st} h(t - \tau)g(\tau)d\tau dt \end{aligned}$$

7.7.3 Laplace Transform of A Convolution

■ Proof:



$$L \left\{ \int_0^t h(t - \tau)g(\tau)d\tau \right\} = \int_0^\infty \int_0^t e^{-st}h(t - \tau)g(\tau)d\tau dt$$

■ Change the order of integration:

$$L \left\{ \int_0^t h(t - \tau)g(\tau)d\tau \right\} = \int_0^\infty \int_\tau^\infty e^{-st}h(t - \tau)g(\tau)dt d\tau,$$

■ Change variables $u = t - \tau$ and $du = dt$ for fixed τ .



$$\begin{aligned} L \left\{ \int_0^t h(t - \tau)g(\tau)d\tau \right\} &= \int_0^\infty \int_0^\infty e^{-s(u+\tau)}h(u)g(\tau)du d\tau \\ &= \int_0^\infty e^{-su}h(u)du \int_0^\infty e^{-s\tau}g(\tau)d\tau = H(s)G(s). \end{aligned}$$

7.7.3 Laplace Transform of A Convolution

■ **Corollary:** The kernel for the ODE

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \dots + a_n y = g(t)$$

is

$$h(t) = L^{-1}\{H(s)\} = L^{-1}\left\{\frac{1}{s^n + a_1 s^{(n-1)} + \dots + a_{n-1} s + a_n}\right\}.$$

■ **Proof:**

■

$$y(t) = \int_0^t h(t - \tau)g(\tau)d\tau.$$

7.7.3 Laplace Transform of A Convolution

■ Proof:

- Taking the Laplace transform

$$L\{y(t)\} = Y(s) = H(s)G(s).$$

- Assuming zero initial conditions

$$Y(s) = \frac{G(s)}{s^n + a_1 s^{(n-1)} + \dots + a_{n-1} s + a_n}$$

- Thus,

$$H(s) = \frac{1}{s^n + a_1 s^{(n-1)} + \dots + a_{n-1} s + a_n}.$$

and $h(t) = L^{-1}\{H(s)\}.$

- Engineers call $H(s)$ as the transfer function of the system.

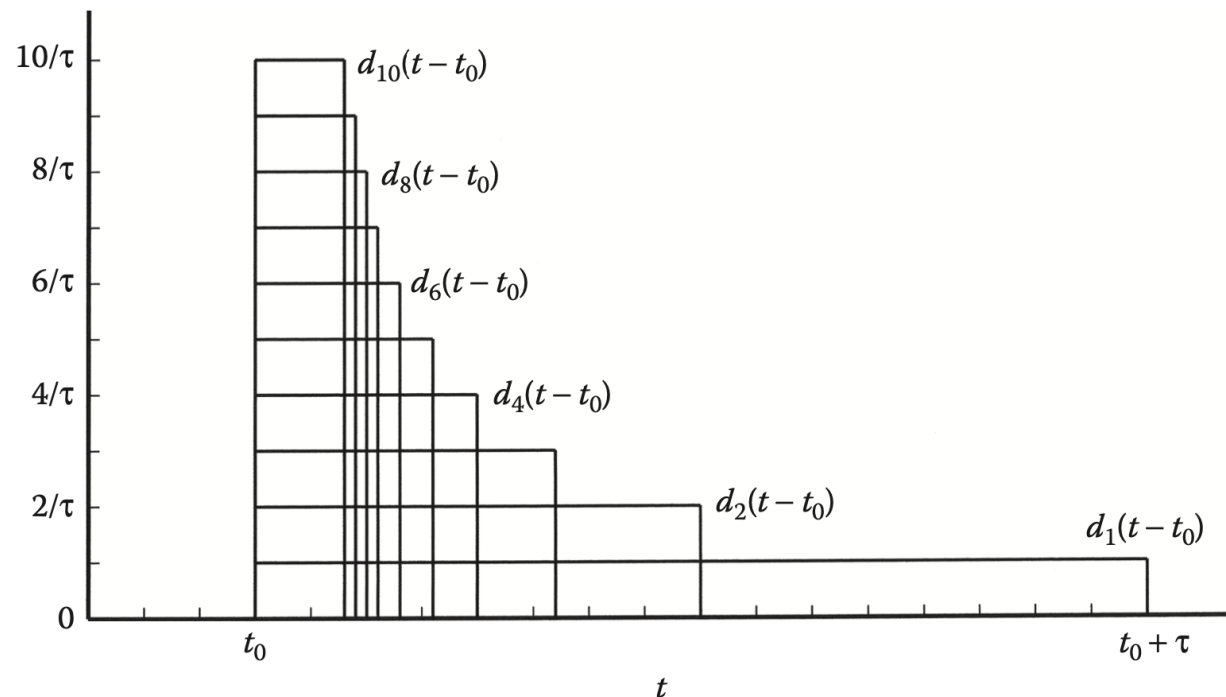
7.7.4 Differential Equations with Impulsive and Discontinuous Inputs

- The utility of the Laplace transform when the input (forcing function) in LTI ODEs is **impulsive** or **discontinuous**.

7.7.4.1 Defining the Dirac Delta Function

- Consider the sequence of functions defined for $t_0 \geq 0$

$$\begin{cases} d_n(t - t_0) = \frac{n}{\tau}, & 0 < t - t_0 < \frac{\tau}{n} \\ d_n(t - t_0) = 0, & \text{otherwise} \end{cases}$$



7.7.4.1 Defining the Dirac Delta Function

- Note the integral $I_n = \int_0^\infty d_n(t - t_0)dt = 1$ is independent of n .
- The sequence of integrals

$$J_n = \int_0^\infty d_n(t - t_0)f(t)dt.$$

■

$$J_n = \int_0^\infty d_n(t - t_0)f(t)dt = \frac{n}{\tau} \int_{t_0}^{t_0 + \tau/n} f(t)dt.$$

- If $f(t)$ is a continuous function, by the *mean value integral calculus theorem*, for some t^* such that $t_0 < t^* < t_0 + \tau/n$,

$$J_n = \frac{n}{\tau} \int_{t_0}^{t_0 + \tau/n} f(t)dt = \frac{n\tau}{\tau n} f(t^*) = f(t^*).$$

7.7.4.1 Defining the Dirac Delta Function

■ Hence,

$$\lim_{n \rightarrow \infty} J_n = \lim_{n \rightarrow \infty} \int_0^{\infty} d_n(t - t_0) f(t) dt = f(t_0).$$

■ The limit

$$\lim_{n \rightarrow \infty} d_n(t - t_0) = \delta(t - t_0)$$

is referred to as an **impulse** and carries the name Dirac delta function at t_0 .

■

$$\int_0^{\infty} \delta(t - t_0) f(t) dt = f(t_0).$$

7.7.4.1 Defining the Dirac Delta Function

- Laplace transform

$$L\{\delta(t - t_0)\} = \int_0^{\infty} e^{-st} \delta(t - t_0) dt = e^{-st_0}.$$

- For $t_0 = 0$,

$$L\{\delta(t)\} = 1.$$

7.7.4.2 Ordinary Differential Equations with Impulsive Inputs

- Consider the n th order LTI ODE with no initial conditions and an input consisting of just the Dirac delta function at the origin:

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \dots + a_n y = \delta(t).$$

- By the time-domain methods,

$$y(t) = \int_0^t h(t - \tau) \delta(\tau) d\tau.$$

- Since $\int_0^\infty \delta(t - t_0) f(t) dt = f(t_0)$, it follows $y(t) = h(t)$.

7.7.4.2 Ordinary Differential Equations with Impulsive Inputs

- By the Laplace transform methods,

$$\begin{aligned} Y(s) &= \frac{G(s)}{s^n + a_1 s^{(n-1)} + \dots + a_{n-1} s + a_n} \\ &= \frac{1}{s^n + a_1 s^{(n-1)} + \dots + a_{n-1} s + a_n}. \end{aligned}$$

- Therefore,

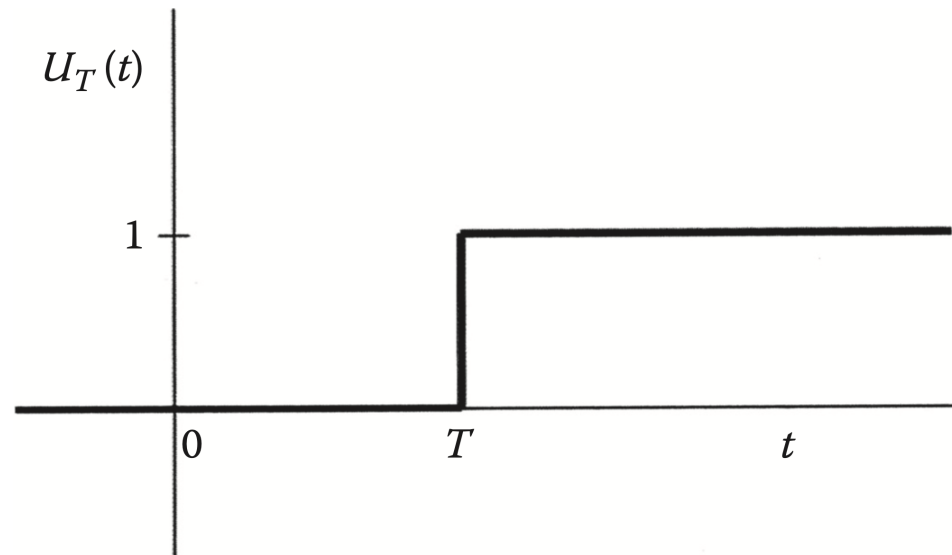
$$y(t) = L^{-1} \left\{ \frac{1}{s^n + a_1 s^{(n-1)} + \dots + a_{n-1} s + a_n} \right\}.$$

- Also, $y(t) = h(t)$. As the kernel of the ODE, $h(t)$ is often referred to by engineers as the **impulse response** of the system.

7.7.4.3 Ordinary Differential Equations with Discontinuous Inputs

- Define a **unit step** function beginning at time T as $u_T(t)$:

$$\begin{cases} u_T(t) = 0, & t \leq T \\ u_T(t) = 1, & t > T \end{cases}$$



7.7.4.3 Ordinary Differential Equations with Discontinuous Inputs

- **Theorem:** If $F(s)$ is the Laplace transform of a function $f(t)$, then the Laplace transform of the function delayed by time T is

$$L \{u_T(t)f(t - T)\} = \int_0^{\infty} e^{-st}u_T(t)f(t - T)dt = e^{-sT}F(s).$$

- **Proof:**

$$L \{u_T(t)f(t - T)\} = \int_0^{\infty} e^{-st}u_T(t)f(t - T)dt = \int_T^{\infty} e^{-st}f(t - T)dt.$$

Let $t' = t - T$, then

$$\int_T^{\infty} e^{-st}f(t - T)dt = \int_0^{\infty} e^{-s(t'+T)}f(t')dt' = e^{-sT} \int_0^{\infty} e^{-st'}f(t')dt'.$$

Thus $L \{u_T(t)f(t - T)\} = e^{-sT}F(s)$.

7.7.4.3 Ordinary Differential Equations with Discontinuous Inputs

- **Corollary:** The Laplace transform of $u_T(t)$ is the Laplace transform of 1 delayed by T :

$$L \{u_T(t)\} = L \{u_T(t) \cdot 1\} = e^{-sT} L \{1\} = \frac{e^{-sT}}{s}.$$

7.7.4.3 Ordinary Differential Equations with Discontinuous Inputs

- **Corollary:** The Laplace transform of $u_T(t)$ is the Laplace transform of 1 delayed by T :

$$L \{u_T(t)\} = L \{u_T(t) \cdot 1\} = e^{-sT} L \{1\} = \frac{e^{-sT}}{s}.$$

7.7.5 Short Table of Laplace Transforms (Final Version)

TABLE 7.2
Short Table of Laplace Transforms (Final Version)

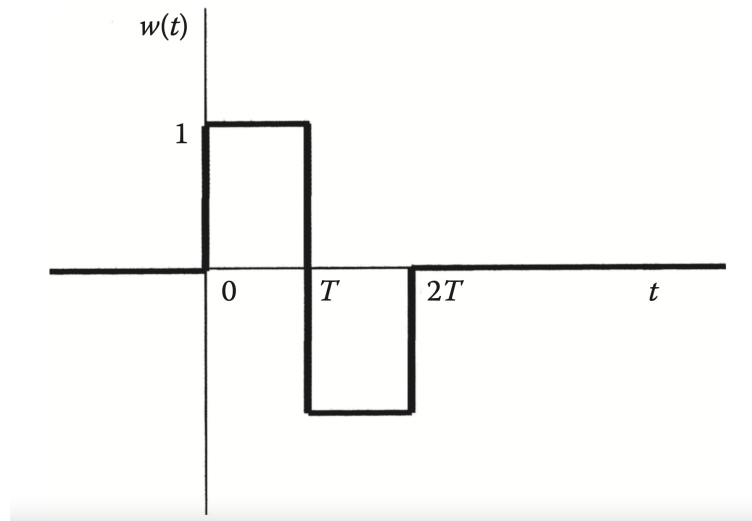
$f(t)$	$F(s)$
$\delta(t)$	1
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$e^{-\lambda t}$	$\frac{1}{s + \lambda}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$te^{-\lambda t}$	$\frac{1}{(s + \lambda)^2}$
$e^{-\lambda t} \sin \omega t$	$\frac{\omega}{(s + \lambda)^2 + \omega^2}$
$e^{-\lambda t} \cos \omega t$	$\frac{s + \lambda}{(s + \lambda)^2 + \omega^2}$
$u_T(t)f(t - T)$	$e^{-sT}F(s)$

7.7.6 Example

- **Problem:** Determine the output $y(t)$ of an underdamped second-order system

$$\frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + (\lambda^2 + \omega^2)y = w(t)$$

with the non-repeating square wave input $w(t)$ shown in Figure.



7.7.6 Example

■ Solution:

- Describe $w(t)$ as a combination of step functions occurring at different times:

$$w(t) = u_0(t) - 2u_T(t) + u_{2T}(t).$$

- Using Laplace transform:

$$L\{w(t)\} = \frac{1}{s} (1 - 2e^{-sT} + e^{-2sT}).$$

- Assuming zero initial conditions,

$$(s^2 + 2\lambda s + (\lambda^2 + \omega^2))Y(s) = \frac{1}{s} (1 - 2e^{-sT} + e^{-2sT}).$$

7.7.6 Example

■ Solution:

■ Solving,

$$Y(s) = \frac{1 - 2e^{-sT} + e^{-2sT}}{s(s^2 + 2\lambda s + (\lambda^2 + \omega^2))}.$$

■ Partial fraction expansion:

$$\begin{aligned} Y_0(s) &= \frac{1}{s(s^2 + 2\lambda s + (\lambda^2 + \omega^2))} \\ &= \frac{c_1}{s} + \frac{c_2 s + c_3}{s^2 + 2\lambda s + (\lambda^2 + \omega^2)} \\ &= \frac{s^2(c_1 + c_2) + s(2\lambda c_1 + c_3) + c_1(\lambda^2 + \omega^2)}{s(s^2 + 2\lambda s + (\lambda^2 + \omega^2))} \end{aligned}$$

■ Thus, $c_1 + c_2 = 0$, $2\lambda c_1 + c_3 = 0$, $c_1(\lambda^2 + \omega^2) = 1$.

7.7.6 Example

■ Solution:

- The solutions are $c_1 = \frac{1}{\lambda^2 + \omega^2}$, $c_2 = \frac{-1}{\lambda^2 + \omega^2}$, $c_3 = \frac{-2\lambda}{\lambda^2 + \omega^2}$.
- Therefore,

$$\begin{aligned} Y_0(s) &= \frac{1}{\lambda^2 + \omega^2} \left[\frac{1}{s} - \frac{s + 2\lambda}{s^2 + 2\lambda s + (\lambda^2 + \omega^2)} \right] \\ &= \frac{1}{\lambda^2 + \omega^2} \left[\frac{1}{s} - \frac{s + \lambda}{(s + \lambda)^2 + \omega^2} - \frac{\lambda}{(s + \lambda)^2 + \omega^2} \right] \end{aligned}$$

- We see that

$$y_0(t) = L^{-1} \{Y_0(s)\} = \frac{1}{\lambda^2 + \omega^2} \left[1 - e^{-\lambda t} \cos(\omega t) - \frac{\lambda}{\omega} e^{-\lambda t} \sin(\omega t) \right]$$

7.7.6 Example

■ Solution:



$$y_0(t) = L^{-1} \{Y_0(s)\} = \frac{1}{\lambda^2 + \omega^2} \left[1 - e^{-\lambda t} \cos(\omega t) - \frac{\lambda}{\omega} e^{-\lambda t} \sin(\omega t) \right]$$



$$\begin{aligned} y(t) &= L^{-1} \{Y(s)\} = L^{-1} \left\{ \frac{1 - 2e^{-sT} + e^{-2sT}}{s(s^2 + 2\lambda s + (\lambda^2 + \omega^2))} \right\} \\ &= y_0(t) - 2u_T(t)y_0(t - T) + u_{2T}(t)y_0(t - 2T). \end{aligned}$$

7.7.7 Initial and Final Value Theorems

■ Initial Value Theorem

Let $F(s)$ be the Laplace transform of a **continuous** function $f(t)$ and let $df/dt \leq M$ for all t . Then

$$f(0) = \lim_{s \rightarrow \infty} sF(s).$$

■ Proof:

■ Start by $L\{\frac{df}{dt}\} = sL\{f(t)\} - f(0)$.

■ Also,

$$L\{\frac{df}{dt}\} = \int_0^{\infty} e^{-st} \frac{df}{dt} dt.$$

■ So,

$$sF(s) = f(0) + \int_0^{\infty} e^{-st} \frac{df}{dt} dt.$$

7.7.7 Initial and Final Value Theorems

■ Proof:

■ Since

$$\left| \int_0^\infty e^{-st} \frac{df}{dt} dt \right| \leq \int_0^\infty \left| \frac{df}{dt} \right| e^{-st} dt \leq M \int_0^\infty e^{-st} dt = \frac{M}{s},$$

we have

$$\lim_{s \rightarrow \infty} \left| \int_0^\infty e^{-st} \frac{df}{dt} dt \right| \leq \lim_{s \rightarrow \infty} \frac{M}{s} = 0.$$

■ Hence,

$$\lim_{s \rightarrow \infty} sF(s) = f(0).$$

- The theorem also holds for functions **continuous everywhere except at the origin**, provided that the theorem is amended to

$$\lim_{s \rightarrow \infty} sF(s) = f(0^+).$$

where $f(0^+)$ is understood to be the value of the function just after $t = 0$. This is useful for systems with an impulsive input at $t = 0$.

7.7.7 Initial and Final Value Theorems

■ Final Value Theorem

Let $F(s)$ be the Laplace transform of a **continuous** function $f(t)$ that **has a limit** as $t \rightarrow \infty$. Then

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

■ Proof:

■ Start by

$$L\left\{\frac{df}{dt}\right\} = sL\{f(t)\} - f(0).$$

■ Then

$$sL\{f(t)\} = f(0) + \int_0^{\infty} e^{-st} \frac{df}{dt} dt.$$

7.7.7 Initial and Final Value Theorems

■ Proof:

■ Now

$$\int_0^{\infty} e^{-st} \frac{df}{dt} dt = \lim_{T \rightarrow \infty} \int_0^T e^{-st} \frac{df}{dt} dt.$$

■ So,

$$\begin{aligned} \lim_{s \rightarrow 0} \int_0^{\infty} e^{-st} \frac{df}{dt} dt &= \lim_{s \rightarrow 0} \lim_{T \rightarrow \infty} \int_0^T e^{-st} \frac{df}{dt} dt \\ &= \lim_{T \rightarrow \infty} \lim_{s \rightarrow 0} \int_0^T e^{-st} \frac{df}{dt} dt = \lim_{T \rightarrow \infty} \int_0^T \frac{df}{dt} dt \\ &= \lim_{T \rightarrow \infty} f(T) - f(0). \end{aligned}$$

■

$$\lim_{s \rightarrow 0} sF(s) = f(0) + \lim_{T \rightarrow \infty} f(T) - f(0) = \lim_{T \rightarrow \infty} f(T).$$

7.7.8 Ordinary Differential Equations with Periodic Inputs

- Consider the response of n th order LTI ODEs to a periodic input:
 $g(t + T) = g(t)$ for all $t > 0$.
- Using Laplace transform:

$$\begin{aligned} G(s) &= L \{g(t)\} = \int_0^{\infty} e^{-st} g(t) dt \\ &= \int_0^T e^{-st} g(t) dt + \int_T^{2T} e^{-st} g(t) dt + \dots + \int_{nT}^{(n+1)T} e^{-st} g(t) dt + \dots \\ &= \sum_{n=0}^{\infty} G_n(s). \end{aligned}$$

7.7.8 Ordinary Differential Equations with Periodic Inputs

- Consider the n th term

$$G_n(s) = \int_{nT}^{(n+1)T} e^{-st} g(t) dt.$$

- Let $t' = t - nT$.

$$\begin{aligned} G_n(s) &= \int_0^T e^{-s(t'+nT)} g(t' + nT) dt = \int_0^T e^{-s(t'+nT)} g(t') dt \\ &= (e^{-sT})^n \int_0^T e^{-st'} g(t') dt = (e^{-sT})^n G_0(s). \end{aligned}$$

■

$$G(s) = G_0(s) \sum_{n=0}^{\infty} (e^{-sT})^n.$$

7.7.8 Ordinary Differential Equations with Periodic Inputs



$$G(s) = G_0(s) \sum_{n=0}^{\infty} (e^{-sT})^n.$$

■ Since

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{for } |x| < 1,$$



$$G(s) = G_0(s) \sum_{n=0}^{\infty} (e^{-sT})^n = \frac{G_0(s)}{1 - e^{-sT}}.$$

7.7.8.1 Response to a Periodic Input

- Assuming zero initial conditions, the Laplace transform $Y(s)$ of the response of an n th order LTI ODE to a periodic input is

$$Y(s) = \frac{G_0(s)}{(1 - e^{-sT})(s^n + a_1 s^{(n-1)} + \dots + a_{n-1} s + a_n)}.$$

- **Theorem:** The response of a stable, damped n th order LTI ODE to a bounded periodic input is **eventually periodic**.

- **Proof:**



$$Y(s)(1 - e^{-sT}) = \frac{G_0(s)}{s^n + a_1 s^{(n-1)} + \dots + a_{n-1} s + a_n}.$$

7.7.8.1 Response to a Periodic Input

■ Proof:

■ Now,

$$\begin{aligned} L^{-1} \{Y(s)(1 - e^{-sT})\} &= L^{-1} \{Y(s)\} - L^{-1} \{e^{-sT}Y(s)\} \\ &= y(t) - u_T(t)y(t - T). \end{aligned}$$

■ Hence,

$$y(t) - u_T(t)y(t - T) = L^{-1} \left\{ \frac{G_0(s)}{s^n + a_1s^{(n-1)} + \dots + a_{n-1}s + a_n} \right\}.$$

■ For $t > T$, that is

$$y(t) - y(t - T) = L^{-1} \left\{ \frac{G_0(s)}{s^n + a_1s^{(n-1)} + \dots + a_{n-1}s + a_n} \right\}.$$

7.7.8.1 Response to a Periodic Input

■ Proof:

- To show that the response to a periodic input eventually becomes periodic, we must show that

$$\lim_{t \rightarrow \infty} (y(t) - y(t - T)) = 0.$$

- We must show that

$$\lim_{t \rightarrow \infty} L^{-1} \left\{ \frac{G_0(s)}{s^n + a_1 s^{(n-1)} + \dots + a_{n-1} s + a_n} \right\} = 0.$$

- Since the LTI ODE is stable and damped, the limit $\lim_{t \rightarrow \infty} (y(t) - y(t - T))$ exists.

7.7.8.1 Response to a Periodic Input

■ Proof:

■ Using the final value theorem,

$$\begin{aligned} & \lim_{t \rightarrow \infty} L^{-1} \left\{ \frac{G_0(s)}{s^n + a_1 s^{(n-1)} + \dots + a_{n-1} s + a_n} \right\} \\ &= \lim_{s \rightarrow 0} \frac{s G_0(s)}{s^n + a_1 s^{(n-1)} + \dots + a_{n-1} s + a_n} \\ &= \lim_{s \rightarrow 0} \frac{s G_0(s)}{a_n}. \end{aligned}$$

■ Since $g(t)$ is bounded,

$$|G_0(s)| \leq \int_0^T |g(t)| e^{-st} dt \leq M \int_0^T e^{-st} dt = \frac{M}{s} (1 - e^{-sT}).$$

■

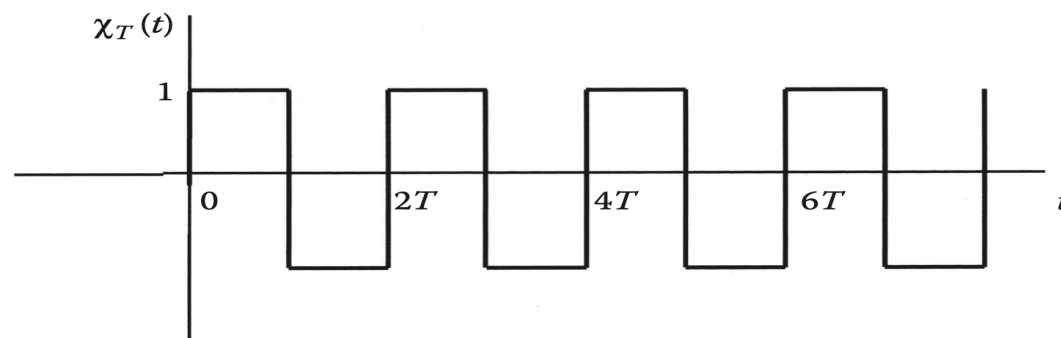
$$\lim_{s \rightarrow 0} \left| \frac{s G_0(s)}{a_n} \right| \leq \lim_{s \rightarrow 0} \frac{M}{a_n} (1 - e^{-sT}) = 0 \Rightarrow \lim_{t \rightarrow \infty} (y(t) - y(t - T)) = 0.$$

7.7.8.2 Resonance with a Nonsinusoidal Periodic Input

- Consider a second-order underdamped system excited by a periodic square wave $\chi_T(t)$:

$$\begin{cases} \chi_T(t) = +1, & 2nT < t < (2n+1)T \\ \chi_T(t) = -1, & (2n+1)T < t < (2n+2)T, \quad n = 0, 1, 2, \dots \end{cases}$$

- Note that $\chi_T(t)$ has period $2T$.



7.7.8.2 Resonance with a Nonsinusoidal Periodic Input

- The ODE for a second-order underdamped system with input $\chi_T(t)$ is

$$\frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + (\lambda^2 + \omega^2)y = \chi_T(t).$$

- Using Laplace transform

$$\begin{aligned} L\{\chi_T(t)\} &= \frac{\int_0^{2T} e^{-st}(u_0(t) - 2u_T(t) + u_{2T}(t))dt}{1 - e^{-2sT}} \\ &= \frac{1}{1 - e^{-2sT}} \frac{(1 - e^{-sT})^2}{s} = \frac{1 - e^{-sT}}{s(1 + e^{-sT})}. \end{aligned}$$

7.7.8.2 Resonance with a Nonsinusoidal Periodic Input

- Assuming zero initial conditions,

$$Y(s) = L\{y(t)\} = \frac{1 - e^{-sT}}{s(1 + e^{-sT})(s^2 + 2\lambda s + (\lambda^2 + \omega^2))}.$$

- The problem is the transcendental function $1 + e^{-sT}$ in the denominator. It is not in the short table and not amenable to the techniques we used, such as a partial fraction expansion.

7.7.8.2 Resonance with a Nonsinusoidal Periodic Input

- Extending the results in 7.7.6 Example,

$$y(t) = \frac{1}{\lambda^2 + \omega^2} \left(1 - e^{-\lambda t} \cos(\omega t) - \frac{\lambda}{\omega} e^{-\lambda t} \sin(\omega t) \right) \\ + \frac{2}{\lambda^2 + \omega^2} \sum_{n=1}^N (-1)^n u_{nT}(t) \times \\ \left(1 - e^{-\lambda(t-nT)} \cos(\omega(t-nT)) - \frac{\lambda}{\omega} e^{-\lambda(t-nT)} \sin(\omega(t-nT)) \right),$$

where N begins at zero and advances by one with each switch of the square wave.

7.7.8.2 Resonance with a Nonsinusoidal Periodic Input

- Only consider the steady-state solution:

$$y_{SS}(t) = \frac{(-1)^N}{c^2} \left(1 - \frac{2c/\omega}{D} q(\delta t) \right),$$

where δt is the time elapsed since the last switch, $c = \sqrt{\lambda^2 + \omega^2}$, $q(\delta t) = e^{-\lambda \delta t} (\cos(\omega \delta t - \theta) + e^{-\lambda T} \cos(\omega(\delta t - T) + \theta))$, $\theta = \tan^{-1}(\lambda/\omega)$, and, most importantly,

$$D = 1 + 2e^{-\lambda T} \cos(\omega T) + e^{-2\lambda T}$$

.

- The value D determines the peak magnitude of the steady-state response: the smaller is D , the larger is the response.

7.7.8.2 Resonance with a Nonsinusoidal Periodic Input

- How D , and hence the peak response, varies with the period of the square wave, $2T$?
- D reaches a local minimum when $\omega T = (2n + 1)\pi$.
- There are an infinite number of solutions to this. Call them T_n . At these points,

$$D = 1 - 2e^{-\lambda T_n} + e^{-2\lambda T_n} = (1 - e^{-\lambda T_n})^2.$$

- $T_0 = \pi/\omega$ provides the global minimum of D such that

$$D = (1 - e^{-\pi \frac{\lambda}{\omega}})^2.$$

7.7.8.2 Resonance with a Nonsinusoidal Periodic Input

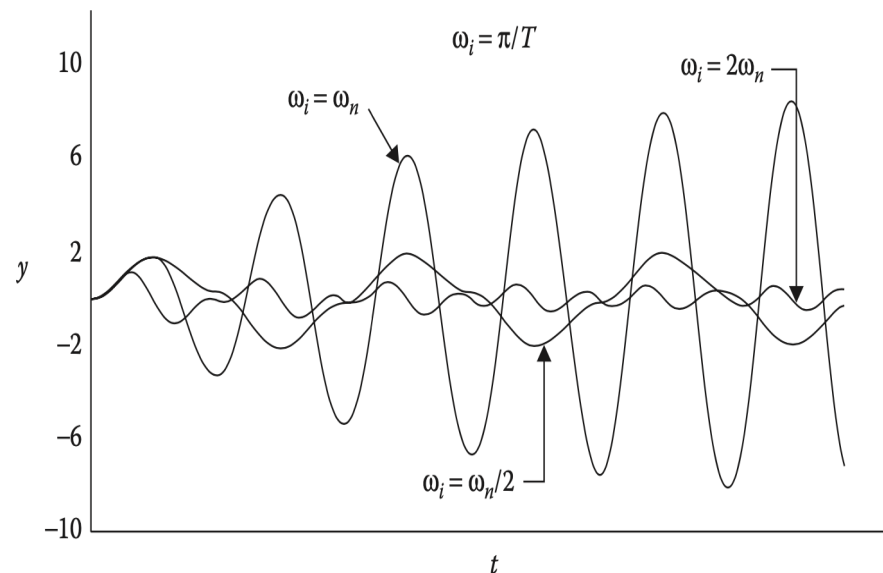
- The system has very low damping; that is, $\lambda/\omega \ll 1$, D can be very small, and hence the peak steady-state response quite large.
- When $T = T_0$ the *radian frequency* of the square-wave input is

$$\omega_{\text{input}} = \frac{2\pi}{2T_0} = \frac{\pi}{\pi/\omega} = \omega.$$

Hence resonance can occur when the frequency of the periodic input equals or is nearly equal to the natural frequency of the system, independent of the inputs waveform.

7.7.8.2 Resonance with a Nonsinusoidal Periodic Input

- The figure graphs the result of numerically integrating the equation $\frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + (\lambda^2 + \omega^2)y = \chi_T(t)$ for three different square-wave input frequencies for the same underdamped second-order system.



- Output of an underdamped second-order system of natural frequency ω when excited by square-wave inputs of unit amplitude but different frequencies ω_i .

END