

1. Evaluate the limit $\lim_{x \rightarrow 0} \frac{\sin(2x)}{\sin\left(\frac{x}{2}\right)}$.
2. Evaluate the limit $\lim_{x \rightarrow 0} \frac{3\sin 2x \tan x}{(x^3 + 5)x^2}$.
3. Evaluate the limit $\lim_{x \rightarrow 0} \frac{1 - \cos^3 x}{x^2}$.
4. Evaluate the limit $\lim_{x \rightarrow \infty} \frac{3x^2 - 4x + 2}{x^2 + 5}$.
5. Let $f(x) = \begin{cases} 3x - 2 & , \text{ if } x < 2 \\ 1 & , \text{ if } x = 2 \\ x^2 & , \text{ if } x > 2 \end{cases}$. Determine whether $f(x)$ is continuous at $x = 2$. Give your reason.
6. Let $g(x) = \begin{cases} \cos(\pi x) & , \text{ if } x < 1 \\ x + 2 & , \text{ if } x \geq 1 \end{cases}$. Determine whether $g(x)$ is continuous at $x = 1$. Give your reason.
7. Let $f(x) = \begin{cases} \frac{8x^2 + 2x - 3}{2x - 1} & , \text{ if } x \neq \frac{1}{2} \\ 5 & , \text{ if } x = \frac{1}{2} \end{cases}$. Is $f(x)$ continuous at $x = \frac{1}{2}$? Give your reason.
8. Let $f(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & , \text{ if } x \neq 1 \\ c & , \text{ if } x = 1 \end{cases}$. Find the value of c for which $f(x)$ is continuous at $x = 1$. Give your reason.
9. Let $f(x) = \begin{cases} \sqrt{x} & , \text{ if } x \geq 1 \\ x^2 & , \text{ if } x < 1 \end{cases}$. Evaluate $\lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1}$, if the limit exists.
10. Let $g(x) = |\tan x|$, for $x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$. Determine whether $g(x)$ is differentiable at $x = 0$. Give your reason.
11. Let $f(x) = |\cos x|$, $x \in \mathbf{R}$. Is $f(x)$ differentiable at $x = 0$? Give your reason.

12. (a) Evaluate the limit $\lim_{x \rightarrow 0} \left(x \sin \frac{1}{x} \right)$.

(Hint: Since $-1 \leq \sin \theta \leq 1$ for all values of θ , it follows that $-|x| \leq x \sin \frac{1}{x} \leq |x|$ for all values of $x (\neq 0)$.)

- (b) Let $g(x) = \begin{cases} x^2 \sin \frac{1}{x} & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases}$. Determine whether $g(x)$ is differentiable at $x = 0$, if so, find $g'(0)$.

13. Show from first principles that $\frac{d}{dx}(x^{\frac{1}{3}}) = \frac{1}{3}x^{-\frac{2}{3}}$, for $x \neq 0$.

(Hint: If p is a negative integer or a fraction index, then

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots,$$

the expansion is valid only if $|x| < 1$.)

14. Without the use of De L'Hôpital rule, prove that, for all integers n ,

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}.$$

(Hint: Consider three cases.

1. For $n = 1$.

2. For n be a positive integer and $n > 1$, we have

$$x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1}).$$

3. For n be a negative integer, we let $y = x^{-1}$ and $b = a^{-1}$.)

Answers:

1. 4
2. $\frac{6}{5}$
3. $\frac{3}{2}$
4. 3
5. $\lim_{x \rightarrow 2} f(x) = 4 \neq 1 = f(2)$, so $f(x)$ is not continuous at $x = 2$.
6. $\lim_{x \rightarrow 1^-} g(x) = -1 \neq 3 = \lim_{x \rightarrow 1^+} g(x)$, so $\lim_{x \rightarrow 1} g(x)$ does not exist.
 $\therefore g(x)$ is not continuous at $x = 1$.
7. $\lim_{x \rightarrow \frac{1}{2}} f(x) = 5 = f\left(\frac{1}{2}\right)$, so $f(x)$ is continuous at $x = \frac{1}{2}$.
8. $\lim_{x \rightarrow 1} f(x) = 2$. Since $f(x)$ is continuous at $x = 1$, we have $\lim_{x \rightarrow 1} f(x) = f(1)$, i.e. $c = 2$.
9. $\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = 2 \neq \frac{1}{2} = \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1}$, so the limit $\lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1}$ does not exist.

$$10. \lim_{x \rightarrow 0^-} \frac{g(x) - g(0)}{x - 0} = -1 \text{ but } \lim_{x \rightarrow 0^+} \frac{g(x) - g(0)}{x - 0} = 1.$$

Since $\lim_{x \rightarrow 0^-} \frac{g(x) - g(0)}{x - 0} \neq \lim_{x \rightarrow 0^+} \frac{g(x) - g(0)}{x - 0}$, the limit $\lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0}$ does not exist.

Hence, $g(x)$ is not differentiable at $x = 0$.

$$11. \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = 0, \text{ so the limit } \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} \text{ exists.}$$

Hence, $f(x)$ is differentiable at $x = 0$.

$$12. (a) \quad -|x| \leq x \sin \frac{1}{x} \leq |x| \text{ for all } x \neq 0.$$

Since $\lim_{x \rightarrow 0} (-|x|) = 0$ and $\lim_{x \rightarrow 0} |x| = 0$, by the Sandwich Theorem,

$$\lim_{x \rightarrow 0} \left(x \sin \frac{1}{x} \right) = 0.$$

$$(b) \quad \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \left(x \sin \frac{1}{x} \right) = 0, \text{ from (a). } \therefore \text{ The limit exists.}$$

Hence, $g(x)$ is differentiable at $x = 0$ and $g'(0) = 0$.

$$\begin{aligned} 13. \quad \frac{d}{dx}(x^{\frac{1}{3}}) &= \lim_{h \rightarrow 0} \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^{\frac{1}{3}} \left(1 + \frac{h}{x} \right)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^{\frac{1}{3}} \left(1 + \frac{1}{3} \left(\frac{h}{x} \right) + \frac{1}{3} \left(-\frac{2}{3} \right) \frac{\left(\frac{h}{x} \right)^2}{2!} + \frac{1}{3} \left(-\frac{2}{3} \right) \left(-\frac{5}{3} \right) \frac{\left(\frac{h}{x} \right)^3}{3!} + \dots \right) - x^{\frac{1}{3}}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{3} \left(\frac{h}{x} \right) x^{\frac{1}{3}} + \frac{1}{3} \left(-\frac{2}{3} \right) \frac{\left(\frac{h}{x} \right)^2}{2!} x^{\frac{1}{3}} + \frac{1}{3} \left(-\frac{2}{3} \right) \left(-\frac{5}{3} \right) \frac{\left(\frac{h}{x} \right)^3}{3!} x^{\frac{1}{3}} + \dots}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{1}{3} x^{-\frac{2}{3}} + \frac{1}{3} \left(-\frac{2}{3} \right) \frac{h x^{-\frac{5}{3}}}{2!} + \frac{1}{3} \left(-\frac{2}{3} \right) \left(-\frac{5}{3} \right) \frac{h^2 x^{-\frac{8}{3}}}{3!} + \dots \right] \\ &= \frac{1}{3} x^{-\frac{2}{3}} \end{aligned}$$

$$14. \text{ Case 1: For } n=1, \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = \dots = na^{n-1}$$

$$\text{Case 2: For } n \text{ be a positive integer and } n > 1, \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = \dots = na^{n-1}$$

Case 3: For n be a negative integer, let $y = x^{-1}$ and $b = a^{-1}$, then

$$\begin{aligned}
 \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= \lim_{y \rightarrow b} \frac{y^{-n} - b^{-n}}{y^{-1} - b^{-1}} = \lim_{y \rightarrow b} \frac{\frac{y^{-n} - b^{-n}}{y - b}}{\frac{y^{-1} - b^{-1}}{y - b}} = \frac{\lim_{y \rightarrow b} \frac{y^{-n} - b^{-n}}{y - b}}{\lim_{y \rightarrow b} \frac{b - y}{yb}} \\
 &= \frac{(-n)b^{(-n)-1}}{\lim_{y \rightarrow b} \left(-\frac{1}{yb} \right)} = \frac{(-n)b^{(-n)-1}}{-\frac{1}{b^2}} = nb^{-n+1} = n(a^{-1})^{-n+1} \\
 &= na^{n-1}
 \end{aligned}$$