L03: Wave Equations and Plane Waves

EE3109 Applied Electromagnetics

Lecture Notes

Prof. Alex M. H. Wong

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Review and Preview

- Last week we:
  - Reviewed electro/magneto/quasi- statics
  - Introduced Maxwell's equations
  - Examined electromagnetic material properties
  - Studied how EM fields behave at material boundaries

- This class we will look at:
  - The EM wave equation
  - The EM plane wave
  - Plane wave propagation in lossy media

Outline

- Course Matter
- Wave Equations
- Plane Waves
- Wave Propagation in Lossy Media

Wave Equations – Motivation

- We would like to write equations for (E, H, B, D) in a way that is **decoupled** from other quantities

- Without sources and in simple media, this can be done straightforwardly

Source-free wave equations $\vec{J} = 0, \rho = 0$

Wave is traveling in a simple non-conducting (σ=0) medium

$$\nabla \times \vec{\mathrm{E}} = -\mu \frac{\partial \vec{\mathrm{H}}}{\partial t} \qquad \nabla \times \vec{\mathrm{H}} = \varepsilon \frac{\partial \vec{\mathrm{E}}}{\partial t}$$

$$\nabla \cdot \vec{\mathrm{E}} = 0 \qquad \nabla \cdot \vec{\mathrm{H}} = 0$$

Consider

$$\nabla \times \nabla \times \vec{\mathrm{E}} = -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{\mathrm{H}}\right)$$

$$= -\mu\varepsilon \frac{\partial^2 \vec{\mathrm{E}}}{\partial t^2}$$

or

$$\nabla \times \nabla \times \vec{\mathrm{E}} + \mu\varepsilon \frac{\partial^2 \vec{\mathrm{E}}}{\partial t^2} = 0$$

$$\nabla \times \nabla \times \vec{E} = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

$$= -\nabla^2 \vec{E}$$

$$\therefore \nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

velocity of wave propagation : $u = \dfrac{1}{\sqrt{\mu\varepsilon}}$

Wave equation for E becomes : $\nabla^2 \vec{E} - \dfrac{1}{u^2}\dfrac{\partial^2 \vec{E}}{\partial t^2} = 0$ (1)

Similarly,

$$\nabla \times \nabla \times \vec{H} + \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

$$\Rightarrow \nabla^2 \vec{H} - \frac{1}{u^2}\frac{\partial^2 \vec{H}}{\partial t^2} = 0 \qquad (2)$$

Both (1) and (2) are called homogeneous vector wave equations

In rectangular coordinates, $-\nabla^2\vec{E} = -\nabla^2 E_x\hat{x} - \nabla^2 E_y\hat{y} - \nabla^2 E_z\hat{z}$

where $\nabla^2 \mathrm{E_x} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathrm{E_x}$

The wave equations become

$$\nabla^2 E_x - \frac{1}{u^2}\frac{\partial^2 E_x}{\partial \mathrm{t}^2} = 0 \qquad \nabla^2 H_x - \frac{1}{u^2}\frac{\partial^2 H_x}{\partial \mathrm{t}^2} = 0$$

$$\nabla^2 E_y - \frac{1}{u^2}\frac{\partial^2 E_y}{\partial \mathrm{t}^2} = 0 \qquad \nabla^2 H_y - \frac{1}{u^2}\frac{\partial^2 H_y}{\partial \mathrm{t}^2} = 0$$

$$\nabla^2 E_z - \frac{1}{u^2}\frac{\partial^2 E_z}{\partial \mathrm{t}^2} = 0 \qquad \nabla^2 H_z - \frac{1}{u^2}\frac{\partial^2 H_z}{\partial \mathrm{t}^2} = 0$$

All the 6 components are decoupled.

Review: The Helmholtz Equations

Applying the phasor transformation, we have:

Wave Equations		Helmholtz Equations
$\left(\nabla^2 - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = 0$	$\Rightarrow$	$(\nabla^2 + k^2)\mathbf{E} = 0$
$\left(\nabla^2 - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{H} = 0$		$(\nabla^2 + k^2)\mathbf{H} = 0$

where $k^2 = \frac{\omega^2}{u^2} = \omega^2\mu\varepsilon \Rightarrow k = \omega\sqrt{\mu\varepsilon}$

This means: $(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 + k^2)F_n = 0$
where $n = x, y$, or z, and $F = E$ or H.

Uniform Plane Waves

- A Uniform Plane Wave is characterized by **E** and **H** that are **constant** at all points across an **infinite plane**.

- For simplicity, for now we consider a co-ordinate definition where this plane is the xy-plane

- In this case, both **E** and **H** are not functions of x and y (though they may point in the x- and y- directions)

- Hence, the Helmholtz equations become: (next page)

1. Uniform Plane Waves

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) E_x = 0 \qquad \left(\frac{\partial^2}{\partial z^2} + k^2\right) H_x = 0$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) E_y = 0 \qquad \left(\frac{\partial^2}{\partial z^2} + k^2\right) H_y = 0$$

The z-components of **E** and **H** are zero, because

$$E_z = \mathbf{E} \cdot \hat{\mathbf{z}} = \left(\frac{1}{j\omega\varepsilon} \nabla \times \mathbf{H}\right) \cdot \hat{\mathbf{z}} = \frac{1}{j\omega\varepsilon} (\nabla \times \mathbf{H}) \cdot \hat{\mathbf{z}}$$
$$= \frac{1}{j\omega\varepsilon}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}, \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}, \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \cdot \hat{\mathbf{z}}$$
$$= \frac{1}{j\omega\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = 0$$

(and the same goes for H_z)

Uniform Plane Waves

The E_x equation again: $\left(\frac{\partial^2}{\partial z^2} + k^2\right) E_x = 0$

The solution for E_x is:

<u>Phasor:</u> ($E_0^{\pm} = \left|E_0^{\pm}\right| \exp\left(j\phi_0^{\pm}\right)$)

$$E_x(z) = E_0^{+} e^{-jkz} + E_0^{-} e^{+jkz} = E_x^{+}(z) + E_x^{-}(z)$$

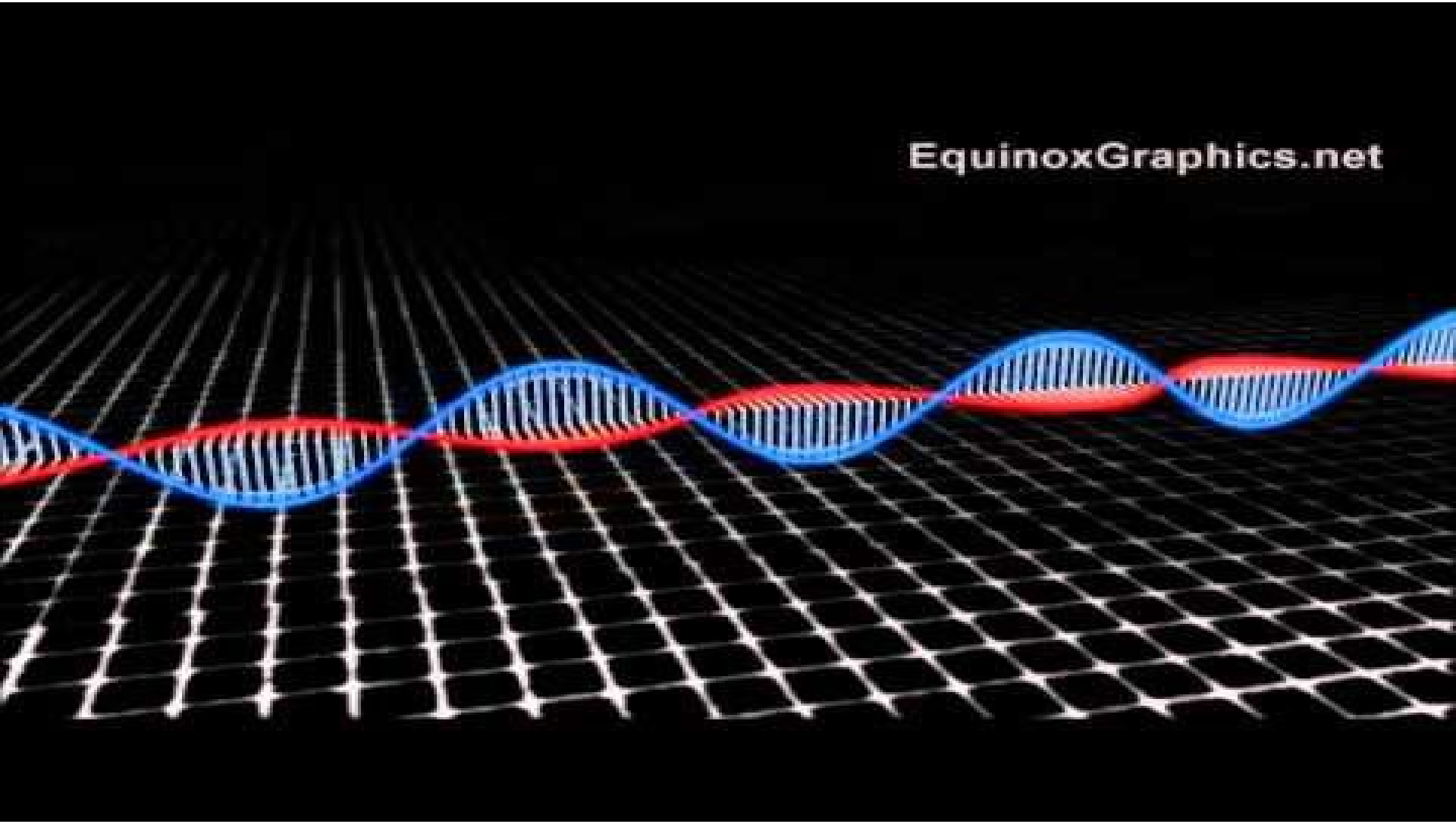
<u>Time domain:</u>

$$\begin{aligned} &E_x(z,t) \\ &= \left|E_0^{+}\right| \cos(\omega t - kz + \phi_0^{+}) + \left|E_0^{-}\right| \cos(\omega t + kz + \phi_0^{-}) \\ &= E_x^{+}(z,t) + E_x^{-}(z,t) \end{aligned}$$

A similar form applies for E_y, H_x and H_y.

Plane Waves….

Plane Wave Visualization



https://www.youtube.com/watch?v=aCTRjVEmeC0&feature=youtu.be

Travelling Plane Waves

Consider $E_x^+(z)$ (the same applies to E_y, H_x and H_y):

$$E_x^+(z) = E_0^+ e^{-jkz} \Rightarrow E_x^+(z,t) = |E_0^+| \cos(\omega t - kz + \phi_0^+)$$

Consider the phase of this travelling wave ($\varphi = \omega t - kz + \phi_0^+$):
The wavelength (λ) is the distance through which the wave changes in phase by 2π rad:

$$\boxed{\lambda \equiv |\Delta z| \text{ s.t. } \Delta\varphi = 2\pi \Rightarrow \lambda = \frac{2\pi}{k}}$$

The **phase velocity (v_p)** of the wave **is the rate at which the constant phase point travels**:

Rearrange the phase expression to get $z = \frac{\omega}{k} t + \underbrace{\frac{(\phi_0^+ - \varphi)}{k}}_{\text{Constant w.r.t. time}}$

$$\boxed{\Rightarrow v_p = \frac{dz}{dt} = \frac{\omega}{k} = f\lambda}$$

Exercise: Find the velocity of $E_x^-(z)$

Example 1: Relating E to H

Let us consider a **forward travelling plane wave**, with x denoting the direction of the E-field. i.e. $\mathbf{E} = E_x\hat{\mathbf{x}} = E_0 e^{-jkz}\hat{\mathbf{x}}$. What is the corresponding $\mathbf{H}$?

Apply Faraday's Law:

$$H(z) = -\frac{1}{j\omega\mu}\nabla \times \mathbf{E} = -\frac{1}{j\omega\mu}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(E_0 e^{-jkz}, 0, 0\right)$$
$$= -\frac{1}{j\omega\mu}\frac{\partial}{\partial z}\left(E_0 e^{-jkz}\right)\hat{\mathbf{y}} = -E_0\frac{-jk}{j\omega\mu}e^{-jkz}\hat{\mathbf{y}} = \frac{k}{\omega\mu}E_x\hat{\mathbf{y}}$$

Recall: $k = \omega\sqrt{\mu\varepsilon}$. Define the **Intrinsic Impedance**: $\eta = \sqrt{\frac{\mu}{\varepsilon}}$

Rearrange the above to get: $H(z) = \frac{1}{\eta}E_x\hat{\mathbf{y}} = \frac{1}{\eta}E_0 e^{-jkz}\hat{\mathbf{y}}$

Remarks:

1. **Phase**: E and H travel in phase.

2. **Amplitude**: E and H differ in amplitude by the factor η. i.e.

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{|\mathbf{E}|}{|\mathbf{H}|} \equiv \text{Intrinsic Impedance } [\Omega]$$

3. **Direction**: Define $\mathbf{k} \equiv \text{Wave Vector } [\text{m}^{-1}]$

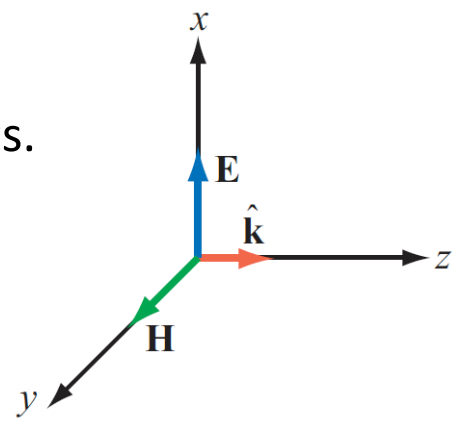
   Such that $|\mathbf{k}| = k$; $\hat{\mathbf{k}} = \frac{\mathbf{k}}{k}$ points in the direction of phase progression.

   (E, H, k) form a right-handed triplet: $\hat{\mathbf{E}} \times \hat{\mathbf{H}} = \hat{\mathbf{k}}$

These are true when μ and ε both have positive real parts.

Hence, for plane waves in this case:

$$\mathbf{E} = \eta \mathbf{H} \times \hat{\mathbf{k}}$$
$$\mathbf{H} = \frac{1}{\eta} \hat{\mathbf{k}} \times \mathbf{E}$$



Travelling Plane Waves (in general direction)

We have described mathematically a plane wave for which the E-field is in the x-direction, the H-field is in the y-direction and the direction of propagation (travelling) is in the z-direction.

One can transform this plane wave (e.g. using a co-ordinate transformation) to obtain the equation for a wave travelling in an arbitrary direction. The time-domain and phasor expressions (for the E-field) are (in the lossless case):

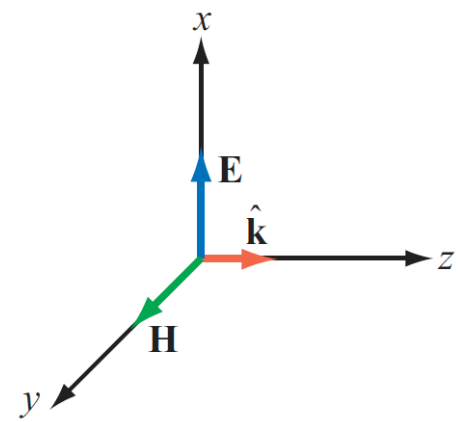
$$\mathbf{E}(\mathbf{r}, t) = |E_0| \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi_0)\ \hat{\mathbf{e}}$$
$$\mathbf{E}(\mathbf{r}) = |E_0| e^{j\phi_0} e^{-j\mathbf{k}\cdot\mathbf{r}}\ \hat{\mathbf{e}}$$

Where $\mathbf{k} \cdot \mathbf{r}$ is the inner product between $\mathbf{k}$ and $\mathbf{r}$. In Cartesian co-ordinates $\mathbf{r} = (x, y, z) = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, and

$$\mathbf{k} \cdot \mathbf{r} = \left(k_x, k_y, k_z\right) \cdot (x, y, z) = k_x x + k_y y + k_z z$$

The corresponding magnetic field is $\mathbf{H}(\mathbf{r}) = \frac{|E_0|}{\eta} e^{j\phi_0} e^{-j\mathbf{k}\cdot\mathbf{r}}\ \hat{\mathbf{h}}$, where $\hat{\mathbf{e}}, \hat{\mathbf{h}}$ and $\hat{\mathbf{k}}$ form a right-hand triplet.

Right-Hand Triplet



- (x, y, z) and $(\mathbf{E}, \mathbf{H}, \mathbf{k})$ both form right-hand triplets (can apply the right-hand rule to show their directional relationship)

<u>Directions</u>

$$\hat{\mathbf{E}} \times \hat{\mathbf{H}} = \hat{\mathbf{k}} \qquad \hat{\mathbf{H}} \times \hat{\mathbf{E}} = -\hat{\mathbf{k}}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{E}} = \hat{\mathbf{H}} \qquad \hat{\mathbf{E}} \times \hat{\mathbf{k}} = -\hat{\mathbf{H}}$$

$$\hat{\mathbf{H}} \times \hat{\mathbf{k}} = \hat{\mathbf{E}} \qquad \hat{\mathbf{k}} \times \hat{\mathbf{H}} = -\hat{\mathbf{E}}$$

<u>Amplitude:</u>

$$\eta = \frac{|\mathbf{E}|}{|\mathbf{H}|}$$

<u>Combined</u>

$$\Rightarrow \mathbf{H} = \frac{1}{\eta}\hat{\mathbf{k}} \times \mathbf{E} = -\frac{1}{\eta}\mathbf{E} \times \hat{\mathbf{k}}; \quad \boldsymbol{E} = \eta\mathbf{H} \times \hat{\mathbf{k}} = -\eta\hat{\mathbf{k}} \times \mathbf{H}$$

Example 2 (Ulaby Example 7-1)

The electric field of a 1-MHz plane wave travelling in the +z-direction in air points along the x-direction. If this field reaches a peak value of 1.2π [mV/m] at $t = 0$ and $z = 50$ [m], obtain an expression for E(z,t) and H(z,t), then plot them as a function of z at t=0.

Example 7-1 cont.

Outline

- Course Matter
- Wave Equations
- Plane Waves
- Wave Propagation in Lossy Media

Revisit: Uniform Plane Waves

Helmholtz equation for E_x: $\left(\frac{\partial^2}{\partial z^2} + k^2\right) E_x = 0$

A similar form applies for E_y, H_x and H_y .

<u>Phasor Solution:</u>

$$E_x(z) = E_0^+ e^{-jkz} + E_0^- e^{+jkz} = E_x^+(z) + E_x^-(z)$$

What happens if the medium is lossy?

$$k \rightarrow k_c = \omega\sqrt{\mu_0 \varepsilon_c}$$

The <u>wave number</u> becomes complex!

Generalize the formulation:

$$\text{Propagation Constant} \equiv$$
$$\gamma = \alpha + j\beta = jk_c = j\omega\sqrt{\mu_0 \varepsilon_c}$$

What is α and β in terms of the material parameters?

$$(\alpha + j\beta)^2 = (j\omega\sqrt{\mu\varepsilon_c})^2 = -\omega^2\mu\varepsilon_c$$

$$(\alpha^2 - \beta^2) + j2\alpha\beta = -\omega^2\mu\varepsilon' + j\omega^2\mu\varepsilon''$$

Equating the real and imaginary parts gives:

$$(\alpha^2 - \beta^2) = -\omega^2\mu\varepsilon';\ 2\alpha\beta = \omega^2\mu\varepsilon''$$

One can easily write α in terms of β (or vice versa) with the second equation. Substitution into the first one allows one to find the following:

$$\text{Attenuation Constant} \equiv \alpha = \omega\left\{\frac{\mu\varepsilon'}{2}\left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1\right]\right\}^{1/2} \quad [\text{Np/m}]$$

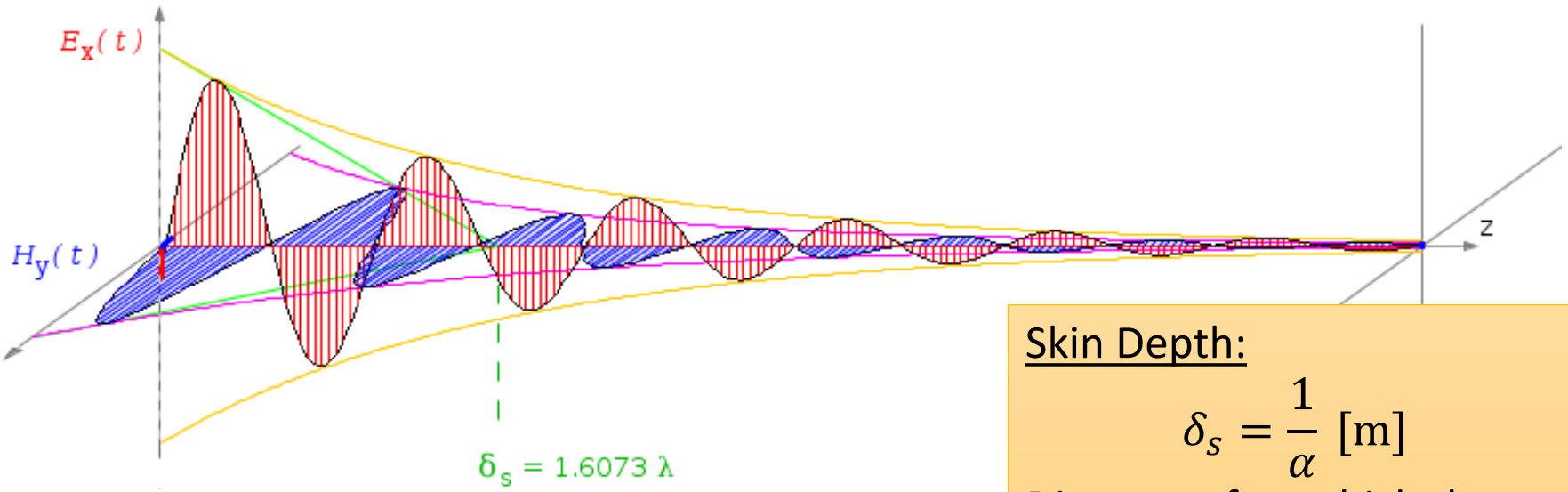
$$\text{Phase Constant} \equiv \beta = \omega\left\{\frac{\mu\varepsilon'}{2}\left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1\right]\right\}^{1/2} \quad [\text{rad/m}]$$

Hence the (positively travelling) electric field phasor can be written as

$$E_x = E_0^+ e^{-(jk_c)z} = E_0^+ e^{-(\alpha + j\beta)z} = E_0^+ e^{-\alpha z} e^{-j\beta z}$$

and the corresponding time-domain E-field is:

$$E_x(z,t) = |E_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \angle E_0^+)$$



<u>Skin Depth:</u>

$$\delta_s = \frac{1}{\alpha}\ [\mathrm{m}]$$

Distance after which the wave decays to $e^{-1} \cong 0.37$ of its original value.

Complex Characteristic Impedance

The complex characteristic impedance can also be written in terms of material parameters:

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon' - j\varepsilon''}} = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - \frac{j\varepsilon''}{\varepsilon'}\right)^{-\frac{1}{2}}$$

With this definition, the corresponding magnetic field phasor (for E_x^+) is written as:

$$\mathbf{H} = -\frac{1}{\eta_c}\mathbf{E} \times \hat{\mathbf{k}} = -H_0^+ e^{-\alpha z} e^{-j\beta z}\,\hat{\mathbf{y}}, \text{where } H_0^+ = \frac{E_0^+}{\eta_c}$$

And the corresponding time domain H field is:

$$\mathbf{H}(z,t) = |H_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \angle H_0^+ + \pi)\,\hat{\mathbf{y}}$$

Note: **H** is no longer in phase with **E**, because of η_c

Simplified Equations for Lossless, Low-loss and Conductive Media

Table 7-1: Expressions for α, β, η_c, u_p, and λ for various types of media.

	Any Medium	Lossless Medium ($\sigma = 0$)	Low-loss Medium ($\varepsilon''/\varepsilon' \ll 1$)	Good Conductor ($\varepsilon''/\varepsilon' \gg 1$)	Units
$\alpha =$	$\omega\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}-1\right]\right]^{1/2}$	0	$\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\pi f \mu\sigma}$	(Np/m)
$\beta =$	$\omega\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}+1\right]\right]^{1/2}$	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}$	$\sqrt{\pi f \mu\sigma}$	(rad/m)
$\eta_c =$	$\sqrt{\frac{\mu}{\varepsilon'}}\left(1-j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$(1+j)\frac{\alpha}{\sigma}$	(Ω)
$u_p =$	ω/β	$1/\sqrt{\mu\varepsilon}$	$1/\sqrt{\mu\varepsilon}$	$\sqrt{4\pi f/\mu\sigma}$	(m/s)
$\lambda =$	$2\pi/\beta = u_p/f$	u_p/f	u_p/f	u_p/f	(m)

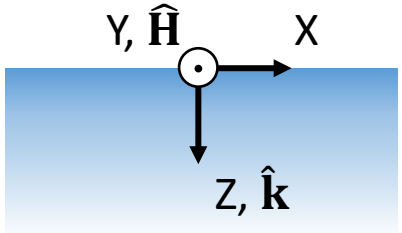
Notes: $\varepsilon' = \varepsilon$; $\varepsilon'' = \sigma/\omega$; in free space, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$; in practice, a material is considered a low-loss medium if $\varepsilon''/\varepsilon' = \sigma/\omega\varepsilon < 0.01$ and a good conducting medium if $\varepsilon''/\varepsilon' > 100$.

Example: Plane Wave in Seawater

A uniform plane wave is traveling in seawater. Assume that the $x-y$ plane resides just below the sea surface and the wave travels in the $+z$-direction into the water. The constitutive parameters of seawater are $\varepsilon_r = 80$, $\mu_r = 1$, and $\sigma = 4$ S/m. If the magnetic field at $z = 0$ is $\mathbf{H}(0, t) = \hat{\mathbf{y}}\, 100 \cos(2\pi \times 10^3 t + 15°)$ (mA/m),

(a) obtain expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$, and

(b) determine the depth at which the magnitude of $\mathbf{E}$ is 1% of its value at $z = 0$.



Example Cont'd

Example Cont'd

Example Cont'd

Example Cont'd

On Nepers and dB

- α is usually [Np/m], where Np, or Neper, is unitless
  - Eg. 1 [Np/m] means the **E/H-field amplitude decays to e^{-1}** of its original value after 1 [m].
- It is useful to also quote the decay in [dB/m]
  - 1 [Bell/m] means the **plane wave's power decays to $1/10$** of its original value after **1 [m]**.
  - 1 [dB/m] means the **plane wave's power decays to $1/10$** of its original value after **10 [m]**.

- (Will be shown in Lect. 6: A plane wave's power density is proportional to the square of the E-field.)

Converting Nepers <---> dB

-1 Neper is the logarithmic measure when the wave amplitude decays to e^{-1} of its original value; **-10 dB** is logarithmic measure when the wave power decays to 1/10 of its original value.

Converting -1 Np into dB: Consider $E(z_1) = E(z_0)e^{-1}$ (i.e. the E-field decays by 1 Np from z_0 to z_1). The decay in dB is:

$$20\log\left(\frac{E(z_1)}{E(z_0)}\right) = 20\log(e^{-1}) = -\frac{20\ln e}{\ln 10} = -\frac{20}{\ln 10}\ [\mathrm{dB}]$$
$$\cong -8.686\ [\mathrm{dB}]$$

Hence 1 Np = 8.686 dB or 1 dB = 0.1151 Np

Summary

- In this class we looked at:
  - The EM wave equation
  - The EM plane wave
  - Relationship between E and H fields in a plane wave
  - Wave decay and propagation in lossy media