

MA2001 Quiz II (CB)

First Name: _____ Last Name: Soluto ID: _____

Question 1 (10 marks) Let $\vec{F}(x, y, z) = by \vec{i} + (x + z \cos y) \vec{j} + \sin y \vec{k}$, where b is a real constant.

(i) Find a value of b such that $\vec{F}$ is conservative.

(ii) With the value of b in part (i) such that $\vec{F}$ is conservative, determine a potential function $\varphi(x, y, z)$ such that $\nabla\varphi = \vec{F}$.

Soluto (i) $\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ by & x+z\cos y & \sin y \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ by & \sin y \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ by & \sin y \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x+z\cos y & \sin y \end{vmatrix} \vec{k}$

$$= (\cos y - \cos y) \vec{i} - 0 \vec{j} + (1 - b) \vec{k} = 0$$

$$\Rightarrow b = 1$$

(ii) $\vec{F} = y \vec{i} + (x + z \cos y) \vec{j} + \sin y \vec{k}$

$\nabla\varphi = \vec{F} \Leftrightarrow \begin{cases} \varphi_x = y & \text{--- ①} \\ \varphi_y = x + z \cos y & \text{--- ②} \\ \varphi_z = \sin y & \text{--- ③} \end{cases}$

① $\Rightarrow \varphi = yx + g(y, z)$

$\varphi_y = x + \frac{\partial g}{\partial y}$

② $\Rightarrow x + \frac{\partial g}{\partial y} = x + z \cos y \Rightarrow \frac{\partial g}{\partial y} = z \cos y$

$\Rightarrow g = z \sin y + h(z)$

$\Rightarrow \varphi = xy + z \sin y + h(z)$

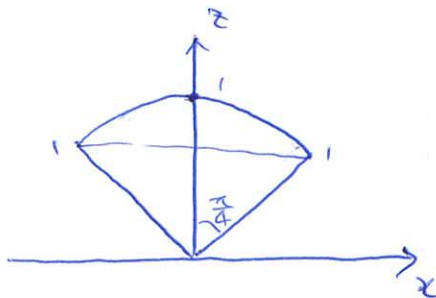
$\Rightarrow \varphi_z = \sin y + h'(z)$

③ $\Rightarrow \sin y + h'(z) = \sin y \Rightarrow h'(z) = 0 \Rightarrow h(z) = c$

$\Rightarrow \varphi(x, y, z) = xy + z \sin y + c$

Question 2 (20 marks)

- a) Evaluate $\iiint_V z dx dy dz$, where V is the region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the upper half sphere $x^2 + y^2 + z^2 = 1$.
- b) Use a) or otherwise, to evaluate $\iiint_V (3 + z) dx dy dz$.



Soln: a) $\iiint_V z dx dy dz$

$$= \iiint_{V_{\rho\phi\theta}} (\rho \cos\phi) \cdot \rho^2 \sin\phi d\rho d\phi d\theta$$

$$= \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} \rho^3 \sin\phi \cos\phi d\theta d\phi d\rho$$

$$= \int_0^1 \rho^3 d\rho \times \int_0^{\pi/4} \sin\phi \cos\phi d\phi \times \int_0^{2\pi} d\theta$$

$$= \left. \frac{\rho^4}{4} \right|_0^1 \times \left. \frac{\sin^2\phi}{2} \right|_0^{\pi/4} \times 2\pi$$

$$= \frac{1}{4} \times \frac{1}{4} \times 2\pi = \frac{\pi}{8}$$

b) $\iiint_V dx dy dz = \int_0^1 \int_0^{\pi/4} \int_0^{2\pi} \rho^2 \sin\phi d\theta d\phi d\rho$

$$= \int_0^1 \rho^2 d\rho \times \int_0^{\pi/4} \sin\phi d\phi \times \int_0^{2\pi} d\theta$$

$$= \left. \frac{\rho^3}{3} \right|_0^1 \times (-\cos\phi) \Big|_0^{\pi/4} \times 2\pi$$

$$= \frac{1}{3} \times \left(1 - \frac{\sqrt{2}}{2}\right) \times 2\pi = \frac{(2-\sqrt{2})}{3} \pi$$

$$\Rightarrow (*) \iiint_V (3+z) dx dy dz = 3 \iiint_V dx dy dz + \iiint_V z dx dy dz$$

$$= 3 \times \frac{(2-\sqrt{2})}{3} \pi + \frac{\pi}{8}$$

$$= \left(2 - \sqrt{2} + \frac{1}{8}\right) \pi$$

$$= \left(\frac{17}{8} - \sqrt{2}\right) \pi$$

for $(**)$ $\iiint_V (1+z) dx dy dz = \iiint_V dx dy dz + \iiint_V z dx dy dz = \frac{(2-\sqrt{2})}{3} \pi + 3 \times \frac{\pi}{8} = \frac{(25-8\sqrt{2})\pi}{24}$