

EE2104 Introduction to Electromagnetics
Assignment 2 Solution

Deadline (for uploading): 11:59 pm, 30 Sep 2021
Please put down your name and university ID.

1. Consider a line at $x = -L/3$ to $2L/3$. It contains electric charges of density $\rho_L = \rho_{L0} (x/L)^2$ for some constant ρ_{L0} in C/m. Find the total charge.

Solution:

The total charge is

$$\begin{aligned} Q &= \int_{-L/3}^{2L/3} \rho_L dx \\ &= \int_{-L/3}^{2L/3} \rho_{L0} (x/L)^2 dx \\ &= \int_{-L/3}^{2L/3} \rho_{L0} (x^2/L^2) dx \\ &= \frac{\rho_{L0}}{L^2} \int_{-L/3}^{2L/3} x^2 dx \\ &= \frac{\rho_{L0}}{L^2} \cdot \left(\frac{x^3}{3} \right) \Big|_{x=-L/3}^{2L/3} \\ &= \frac{\rho_{L0}}{L^2} \cdot \frac{L^3}{9} \\ &= \frac{\rho_{L0} L}{9} \end{aligned}$$

2. A sphere centered at origin has radius b . It contains electric charge of density $\rho_v(r, \theta, \phi)$ in C/m^3 .
- (a) Find the total charge when the density $\rho_v = \rho_0$ for some constant ρ_0 .
- (b) Find the total charge when the density $\rho_v = \rho_0 \exp(-r^3/b^3)$ for some constant ρ_0 .

Solution:

(a) The total charge is

$$\begin{aligned}
 Q &= \int_{\text{sphere } r < b} \rho_v(r, \theta, \phi) dV \\
 &= \int_{\text{sphere } r < b} \rho_0 dV \\
 &= \frac{4\rho_0\pi b^3}{3}
 \end{aligned}$$

(b) The total charge is

$$\begin{aligned}
 Q &= \int_{\text{sphere } r < b} \rho_v(r, \theta, \phi) dV \\
 &= \int_{\text{sphere } r < b} \rho_0 \exp(-r^3/b^3) dV \\
 &= \rho_0 \int_{\text{sphere } r < b} e^{-r^3/b^3} r^2 \sin\theta d\theta d\phi dr \\
 &= \rho_0 \int_0^{2\pi} d\phi \cdot \int_0^\pi \sin\theta d\theta \cdot \int_0^b e^{-r^3/b^3} r^2 dr \\
 &= 4\pi\rho_0 \frac{b^3}{3} \int_0^b e^{-\frac{r^3}{b^3}} d\left(\frac{r^3}{b^3}\right) \\
 &= -\frac{4\pi\rho_0 b^3}{3} \left(e^{-\frac{r^3}{b^3}} \right) \Big|_0^b \\
 &= \frac{4\pi\rho_0 b^3}{3} (1 - e^{-1})
 \end{aligned}$$

3. The magnetic field from a current along the z -axis is proportional to

$$\mathbf{H}(x, y, z) = \frac{y\hat{x} - x\hat{y}}{2\pi(x^2 + y^2)}$$

- (a) Express $\mathbf{H}$ in terms of the cylindrical coordinate system. Then, briefly sketch the field.
 (b) Evaluate curl $\mathbf{H}$.
 (c) Evaluate div $\mathbf{H}$.

Solution:

- (a) The transformation from the rectangular to the cylindrical coordinate system is given by

$$\begin{cases} x = \rho \cos \phi \\ y = \rho \sin \phi \\ \hat{x} = \cos \phi \hat{\rho} - \sin \phi \hat{\phi} \\ \hat{y} = \sin \phi \hat{\rho} + \cos \phi \hat{\phi} \end{cases}$$

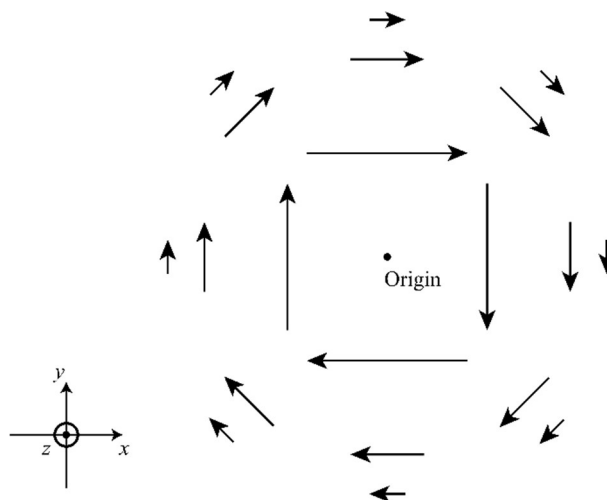
Thus, the field is

$$\begin{aligned} \mathbf{H}(x, y, z) &= \frac{y\hat{x} - x\hat{y}}{2\pi(x^2 + y^2)} \\ &= \frac{\rho \sin \phi (\cos \phi \hat{\rho} - \sin \phi \hat{\phi}) - \rho \cos \phi (\sin \phi \hat{\rho} + \cos \phi \hat{\phi})}{2\pi \rho^2} \\ &= \frac{-\hat{\phi}}{2\pi \rho} \end{aligned}$$

Therefore, we have

$$\mathbf{H}(\rho, \phi) = \frac{-\hat{\phi}}{2\pi \rho}$$

The field is plotted in the figure below.



(b) $\text{curl } \mathbf{H}$ is

$$\nabla \times \mathbf{H} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & \frac{-\rho}{2\pi\rho} & 0 \end{vmatrix} = 0$$

(c) $\text{div } \mathbf{H}$ is

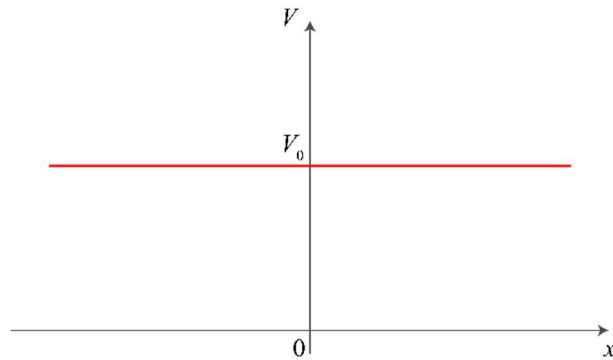
$$\nabla \cdot \mathbf{H} = \frac{1}{\rho} \left[\frac{\partial}{\partial \phi} \left(-\frac{1}{2\pi\rho} \right) \right] = 0$$

4. Consider a voltage $V(x, y) = V_0 \exp(-k^2 y^2)$, where V_0 is a constant in volts and k is a positive constant in m^{-1} .
- Is $V(x, y)$ a scalar field or a vector field?
 - Plot V versus x at $y = 0$. Label the axes accurately.
 - Plot V versus y at $x = 0$. Label the axes accurately.
 - Plot the contour lines on the xy -plane for $V = 0.2V_0, 0.4V_0, 0.6V_0, 0.8V_0$, and V_0 .
 - Find $\mathbf{E} = -\text{grad}(V)$.

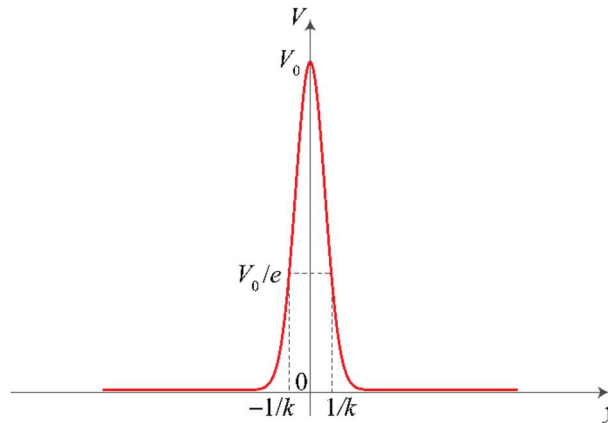
Solution:

(a) $V(x, y)$ is a scalar field.

(b) When $y = 0$, we have $V(x, 0) = V(x, y) = V_0$.



(c) When $x = 0$, we have $V(0, y) = V_0 \exp(-k^2 y^2)$.



(d) When $V(x, y) = 0.2V_0$, we have

$$V_0 \exp(-k^2 y^2) = 0.2V_0$$

$$e^{-k^2 y^2} = 0.2$$

$$y^2 = \frac{1.6}{k^2}$$

So the contour lines on the xy -plane are two lines for $V = 0.2V_0$.

Similarly, we have another contour lines and a point on the xy -plane

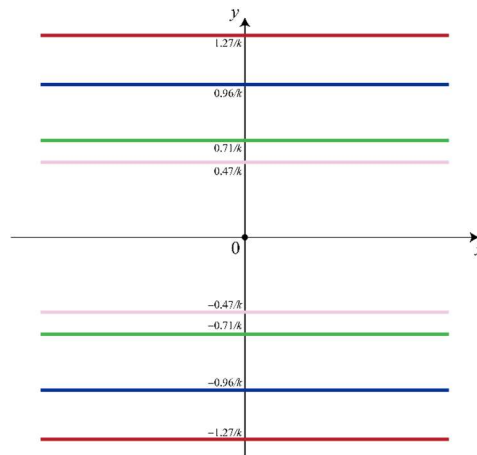
$$y^2 = \frac{0.92}{k^2}$$

$$y^2 = \frac{0.51}{k^2}$$

$$y^2 = \frac{0.22}{k^2}$$

$$y^2 = 0$$

The contour lines on the xy -plane are plotted in the figure below.



(e) $\mathbf{E} = -\text{grad}(V)$

$$= -\left(\frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}}\right)$$

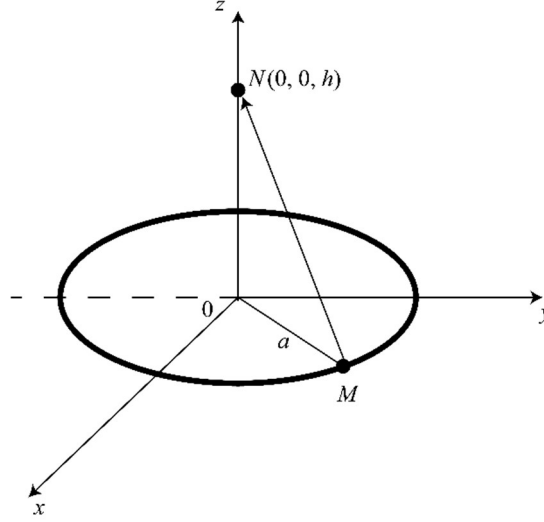
$$= 2V_0k^2ye^{-k^2y^2}\hat{\mathbf{y}}$$

5. From the Coulomb's law, derive the electric field measured at $(x, y, z) = (0, 0, h)$ from a charged circle centered at the origin. The circle is oriented along the xy -plane with radius a . The line charge density is given by

$$\rho_L(\phi) = \rho_{L0} \sin^2 2\phi$$

where ϕ is measured from the x -axis and ρ_{L0} (C/m) is a constant.

Solution:



Considering a differential angle $d\phi$, a differential arc length on the charged circle is expressed as

$$dl = a \cdot d\phi$$

Therefore, at position $M(a\cos\phi, a\sin\phi, 0)$ on the circle, the differential arc dl carries a charge of

$$dQ = \rho_L(\phi) \cdot dl = \rho_{L0} \sin^2 2\phi \cdot dl = \rho_{L0} \sin^2 2\phi \cdot (a \cdot d\phi) = a\rho_{L0} \sin^2 2\phi d\phi$$

By Coulomb's law, the differential electric field at $N(0, 0, h)$ is

$$d\mathbf{E} = \frac{dQ}{4\pi\epsilon_0 |\mathbf{R}_{MN}|^2} \frac{\mathbf{R}_{MN}}{|\mathbf{R}_{MN}|} = \frac{dQ \mathbf{R}_{MN}}{4\pi\epsilon_0 |\mathbf{R}_{MN}|^3} = \frac{a\rho_{L0} \sin^2 2\phi d\phi \cdot (-a\cos\phi \hat{x} - a\sin\phi \hat{y} + h\hat{z})}{4\pi\epsilon_0 (\sqrt{a^2 + h^2})^3}$$

where $\mathbf{R}_{MN}$ is the vector pointing to N from M .

(i) The x -component of the total electric field at position N is

$$E_x = \int_{\phi=0}^{2\pi} dE_x = \int_{\phi=0}^{2\pi} \frac{a\rho_{L0} \sin^2 2\phi \cdot (-a\cos\phi)}{4\pi\epsilon_0 (\sqrt{a^2 + h^2})^3} d\phi = \frac{-a^2 \rho_{L0}}{4\pi\epsilon_0 (a^2 + h^2)^{\frac{3}{2}}} \int_{\phi=0}^{2\pi} \sin^2 2\phi \cos\phi d\phi$$

where

$$\sin^2 2\phi \cos\phi = \frac{1}{2} \cos\phi - \frac{1}{4} (\cos 5\phi + \cos 3\phi)$$

Then we have

$$\int_{\phi=0}^{2\pi} \sin^2 2\phi \cos\phi d\phi = \int_{\phi=0}^{2\pi} \left(\frac{1}{2} \cos\phi - \frac{1}{4} (\cos 5\phi + \cos 3\phi) \right) d\phi = 0$$

Thus, we have

$$E_x = 0$$

(ii) Similarly, the y -component of the total electric field at position N is

$$E_y = \int_{\phi=0}^{2\pi} dE_y = \int_{\phi=0}^{2\pi} \frac{a\rho_{L0}\sin^2 2\phi \cdot (-a\sin\phi)}{4\pi\epsilon_0(\sqrt{(a^2 + h^2)})^3} d\phi = \frac{-a^2\rho_{L0}}{4\pi\epsilon_0(a^2 + h^2)^{\frac{3}{2}}} \int_{\phi=0}^{2\pi} \sin^2 2\phi \sin\phi d\phi$$

where

$$\sin^2 2\phi \sin\phi = \frac{1}{2}\sin\phi - \frac{1}{4}(\sin 5\phi - \sin 3\phi)$$

Then we have

$$\int_{\phi=0}^{2\pi} \sin^2 2\phi \sin\phi d\phi = \int_{\phi=0}^{2\pi} \left(\frac{1}{2}\sin\phi - \frac{1}{4}(\sin 5\phi - \sin 3\phi) \right) d\phi = 0$$

Thus,

$$E_y = 0$$

(iii) The z -component of the total electric field at position N is

$$E_z = \int_{\phi=0}^{2\pi} dE_z = \int_{\phi=0}^{2\pi} \frac{a\rho_{L0}\sin^2 2\phi \cdot h}{4\pi\epsilon_0(\sqrt{(a^2 + h^2)})^3} d\phi = \frac{ah\rho_{L0}}{4\pi\epsilon_0(a^2 + h^2)^{\frac{3}{2}}} \int_{\phi=0}^{2\pi} \sin^2 2\phi d\phi$$

Therefore, we have

$$E_z = \frac{ah\rho_{L0}}{4\epsilon_0(a^2 + h^2)^{\frac{3}{2}}}$$

So the electric field at $(x, y, z) = (0, 0, h)$ is

$$\mathbf{E} = E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}} + E_z \hat{\mathbf{z}} = \frac{ah\rho_{L0}}{4\epsilon_0(a^2 + h^2)^{\frac{3}{2}}} \hat{\mathbf{z}}$$