

Complex Number

Problem 1

$$(a) \quad i^5 - i^7 + i^{10} = i - (-i) + (-1) = -2 + 2i.$$

$$(b) \quad \frac{1-i}{3+2i} = \frac{1-i}{3+2i} \left(\frac{3-2i}{3-2i} \right) = \frac{3-5i+2i^2}{9-4i^2} = \frac{3-5i-2}{9+4} = \frac{1}{13} - \frac{5}{13}i.$$

$$(c) \quad (2+i)^2(3-i) = \left(4+4i + \underbrace{i^2}_{=-1} \right) (3-i) = (3+4i)(3-i) = 9+9i-4i^2 = 13+9i.$$

$$(d) \quad (1-3i)^{-1} = \frac{1}{1-3i} = \frac{1}{1-3i} \left(\frac{1+3i}{1+3i} \right) = \frac{1+3i}{1-9i^2} = \frac{1+3i}{1+9} = \frac{1}{10} + \frac{3}{10}i.$$

Problem 2

Note that

$$\frac{1}{z} = 1+3i \Rightarrow z = \frac{1}{1+3i} = \frac{1}{1+3i} \left(\frac{1-3i}{1-3i} \right) = \frac{1-3i}{1-9i^2} = \frac{1}{10} - \frac{3}{10}i.$$

Problem 3

(a) The modulus and argument of $z_1 = 3 - 3i$ are given by

$$r = \sqrt{3^2 + (-3)^2} = \sqrt{18}, \quad \theta = -\tan^{-1} \frac{3}{3} = -\frac{\pi}{4}.$$

The Polar form and Euler form of z_1 are then given by

$$z_1 = \underbrace{\sqrt{18} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right)}_{\text{Polar Form}} = \underbrace{\sqrt{18} e^{-\frac{i\pi}{4}}}_{\text{Euler Form}}.$$

(b) The modulus and argument of $z_2 = \sqrt{6} + \sqrt{2}i$ are given by

$$r = \sqrt{(\sqrt{6})^2 + (\sqrt{2})^2} = \sqrt{8}, \quad \theta = \tan^{-1} \frac{\sqrt{2}}{\sqrt{6}} = \frac{\pi}{6}.$$

The Polar form and Euler form of z_2 are then given by

$$z_2 = \underbrace{\sqrt{8} \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)}_{\text{Polar Form}} = \underbrace{\sqrt{8} e^{\frac{i\pi}{6}}}_{\text{Euler Form}}.$$

(c) The modulus and argument of $z_3 = -2i$ is given by

$$r = \sqrt{0^2 + (-2)^2} = 2, \quad \theta = -\frac{\pi}{2}$$

The Polar form and Euler form of z_3 are then given by

$$z_3 = \underbrace{2 \left(\cos \left(-\frac{\pi}{2} \right) + i \sin \left(-\frac{\pi}{2} \right) \right)}_{\text{Polar Form}} = \underbrace{2e^{-\frac{i\pi}{2}}}_{\text{Euler Form}}.$$

(d) The modulus and argument of $z_4 = -4 - \sqrt{48}i$ are given by

$$r = \sqrt{(-4)^2 + (-\sqrt{48})^2} = 8, \quad \theta = -\left(\pi - \tan^{-1} \frac{\sqrt{48}}{4} \right) = -\left(\pi - \frac{\pi}{3} \right) = -\frac{2\pi}{3}.$$

The Polar form and Euler form of z_4 are then given by

$$z_4 = \underbrace{8 \left(\cos \left(-\frac{2\pi}{3} \right) + i \sin \left(-\frac{2\pi}{3} \right) \right)}_{\text{Polar Form}} = \underbrace{8e^{-i\frac{2\pi}{3}}}_{\text{Euler Form}}.$$

(e) The modulus and argument of $z_5 = -1 + 5i$ are given by

$$r = \sqrt{(-1)^2 + 5^2} = \sqrt{26}, \quad \theta = \pi - \tan^{-1} \frac{5}{1} = \pi - \tan^{-1} 5.$$

The Polar form and Euler form of z_5 are then given by

$$z_5 = \underbrace{\sqrt{26}(\cos(\pi - \tan^{-1} 5) + i \sin(\pi - \tan^{-1} 5))}_{\text{Polar Form}} = \underbrace{\sqrt{26}e^{i(\pi - \tan^{-1} 5)}}_{\text{Euler Form}}.$$

(f) The modulus and argument of $z_6 = 5 - 8i$ are given by

$$r = \sqrt{5^2 + (-8)^2} = \sqrt{89}, \quad \theta = -\tan^{-1} \frac{8}{5}.$$

The Polar form and Euler form of z_6 are then given by

$$z_6 = \underbrace{\sqrt{89} \left(\cos \left(-\tan^{-1} \frac{8}{5} \right) + i \sin \left(-\tan^{-1} \frac{8}{5} \right) \right)}_{\text{Polar Form}} = \underbrace{\sqrt{89}e^{i(-\tan^{-1} \frac{8}{5})}}_{\text{Euler Form}}.$$

$$(g) \quad ie^{\frac{i\pi}{4}} = e^{\frac{i\pi}{2}} e^{\frac{i\pi}{4}} = e^{i(\frac{\pi}{2} + \frac{\pi}{4})} = \underbrace{e^{\frac{i3\pi}{4}}}_{\text{Euler Form}} = \underbrace{\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}}_{\text{Polar Form}}.$$

$$(h) \quad -2e^{\frac{i\pi}{3}} = 2e^{i\pi} e^{\frac{i\pi}{3}} = 2e^{i(\pi + \frac{\pi}{3})} = 2e^{\frac{4i\pi}{3}} = 2e^{i(\frac{4\pi}{3} - 2\pi)} = \underbrace{2e^{-\frac{2i\pi}{3}}}_{\text{Euler Form}} = \underbrace{2 \cos \left(-\frac{2\pi}{3} \right) + i \sin \left(-\frac{2\pi}{3} \right)}_{\text{Polar Form}}.$$

$$(i) \quad e^{\frac{i\pi}{5}} + e^{-i\pi} = e^{i\left(\frac{\pi}{5} + (-\pi)\right)} \left[e^{\frac{i\pi}{5} - i\left(\frac{\pi}{5} + (-\pi)\right)} + e^{-i\pi - i\left(\frac{\pi}{5} + (-\pi)\right)} \right] = e^{-i\frac{2\pi}{5}} \left(e^{\frac{3i\pi}{5}} - e^{-\frac{3i\pi}{5}} \right) \\ = \underbrace{2 \cos \frac{3\pi}{5}}_{\text{negative!}} e^{-i\frac{2\pi}{5}} = \underbrace{\left(-2 \cos \frac{3\pi}{5} \right)}_{\text{positive}} (-1) e^{-i\frac{2\pi}{5}} = \left(-2 \cos \frac{3\pi}{5} \right) e^{i\pi} e^{-i\frac{2\pi}{5}} \\ = \underbrace{\left(-2 \cos \frac{3\pi}{5} \right) e^{\frac{3i\pi}{5}}}_{\text{Euler Form}} = \underbrace{\left(-2 \cos \frac{3\pi}{5} \right) \left(\cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5} \right)}_{\text{Polar Form}}.$$

$$(j) \quad 1 - e^{\frac{i\pi}{4}} = e^{i0} - e^{\frac{i\pi}{4}} = e^{i\left(\frac{0}{2} + \frac{\pi}{4}\right)} \left[e^{i0 - i\left(\frac{0}{2} + \frac{\pi}{4}\right)} - e^{-\frac{i\pi}{4} - i\left(\frac{0}{2} + \frac{\pi}{4}\right)} \right] = e^{\frac{i\pi}{8}} \left(e^{-\frac{i\pi}{8}} - e^{\frac{i\pi}{8}} \right) = e^{\frac{i\pi}{8}} \left(-2i \sin \frac{\pi}{8} \right) \\ = 2 \sin \frac{\pi}{8} (-i) e^{\frac{i\pi}{8}} = 2 \sin \frac{\pi}{8} e^{-\frac{i\pi}{2}} e^{\frac{i\pi}{8}} \\ = \underbrace{2 \sin \frac{\pi}{8} e^{-\frac{3i\pi}{8}}}_{\text{Euler Form}} = \underbrace{2 \sin \frac{\pi}{8} \left(\cos \left(-\frac{3\pi}{8} \right) + i \sin \left(-\frac{3\pi}{8} \right) \right)}_{\text{Polar Form}}.$$

Problem 4

Recall that $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$.

$$(a) \quad -\cos \theta - i \sin \theta = -\frac{e^{i\theta} + e^{-i\theta}}{2} - i \frac{e^{i\theta} - e^{-i\theta}}{2i} = -e^{i\theta} = e^{i\pi} e^{i\theta} = \underbrace{e^{i(\theta + \pi)}}_{\text{Euler Form}} \\ = \underbrace{\cos(\theta + \pi) + i \sin(\theta + \pi)}_{\text{Polar Form}}.$$

$$\begin{aligned}
 \text{(b)} \quad -\sin \theta + i \cos \theta &= -\frac{e^{i\theta} - e^{-i\theta}}{2i} + i \frac{e^{i\theta} + e^{-i\theta}}{2} = -\frac{i(e^{i\theta} - e^{-i\theta})}{-2} + i \frac{e^{i\theta} + e^{-i\theta}}{2} = ie^{i\theta} = e^{\frac{i\pi}{2}} e^{i\theta} \\
 &= \underbrace{e^{i(\theta + \frac{\pi}{2})}}_{\text{Euler Form}} = \underbrace{\cos\left(\theta + \frac{\pi}{2}\right) + i \sin\left(\theta + \frac{\pi}{2}\right)}_{\text{Polar Form}}.
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad 1 - \sin \theta + i \cos \theta &\stackrel{\text{from (b)}}{=} 1 + e^{i(\theta + \frac{\pi}{2})} = e^{i0} + e^{i(\theta + \frac{\pi}{2})} \\
 &= e^{i\left(\frac{0+\theta+\frac{\pi}{2}}{2}\right)} \left[e^{i0-i\left(\frac{0+\theta+\frac{\pi}{2}}{2}\right)} + e^{i(\theta+\frac{\pi}{2})-i\left(\frac{0+\theta+\frac{\pi}{2}}{2}\right)} \right] = e^{i\left(\frac{\theta}{2} + \frac{\pi}{4}\right)} \left[e^{-i\left(\frac{\theta}{2} + \frac{\pi}{4}\right)} + e^{i\left(\frac{\theta}{2} + \frac{\pi}{4}\right)} \right] \\
 &= \underbrace{2 \cos\left(\frac{\theta}{2} + \frac{\pi}{4}\right) e^{i\left(\frac{\theta}{2} + \frac{\pi}{4}\right)}}_{\text{Euler Form}} = \underbrace{2 \cos\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \left[\cos\left(\frac{\theta}{2} + \frac{\pi}{4}\right) + i \sin\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \right]}_{\text{Polar Form}}.
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad 1 + \cos \theta - i \sin \theta &= 1 + \frac{e^{i\theta} + e^{-i\theta}}{2} - i \frac{e^{i\theta} - e^{-i\theta}}{2i} = 1 + e^{-i\theta} = e^{i0} + e^{-i\theta} \\
 &= e^{i\left(\frac{0-\theta}{2}\right)} \left[e^{i0-i\left(\frac{0-\theta}{2}\right)} + e^{-i\theta-i\left(\frac{0-\theta}{2}\right)} \right] = e^{-\frac{i\theta}{2}} \left(e^{\frac{i\theta}{2}} + e^{-\frac{i\theta}{2}} \right) = \underbrace{2 \cos \frac{\theta}{2} e^{-\frac{i\theta}{2}}}_{\text{Euler Form}} \\
 &= \underbrace{2 \cos \frac{\theta}{2} \left[\cos\left(-\frac{\theta}{2}\right) + i \sin\left(-\frac{\theta}{2}\right) \right]}_{\text{Polar Form}}.
 \end{aligned}$$

Problem 5

$$\begin{aligned}
 \text{(a)} \quad \frac{1 + \sqrt{3}i}{2 - 2i} &= \frac{2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)}{\sqrt{8} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right)} \\
 &= \frac{1}{\sqrt{2}} \left(\cos \left(\frac{\pi}{3} - \left(-\frac{\pi}{4} \right) \right) + i \sin \left(\frac{\pi}{3} - \left(-\frac{\pi}{4} \right) \right) \right) = \frac{1}{\sqrt{2}} \left(\cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12} \right).
 \end{aligned}$$

$$\text{(b)} \quad (1 + i)^{-5} = \left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^{-5} = 2^{-\frac{5}{2}} \left(\cos \left(-\frac{5\pi}{4} \right) + i \sin \left(-\frac{5\pi}{4} \right) \right).$$

(c) We first note that

$$\begin{aligned}
 \frac{(1 - i)(\sqrt{3} + i)}{2i} &= \frac{\left[\sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right) \right] \left[2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \right]}{2 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)} \\
 &= \sqrt{2} \left(\cos \left(-\frac{\pi}{4} + \frac{\pi}{6} - \frac{\pi}{2} \right) + i \sin \left(-\frac{\pi}{4} + \frac{\pi}{6} - \frac{\pi}{2} \right) \right) \\
 &= \sqrt{2} \left[\cos \left(-\frac{7\pi}{12} \right) + i \sin \left(-\frac{7\pi}{12} \right) \right].
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 \left(\frac{(1 - i)(\sqrt{3} + i)}{2i} \right)^{12} &= \left[\sqrt{2} \left[\cos \left(-\frac{7\pi}{12} \right) + i \sin \left(-\frac{7\pi}{12} \right) \right] \right]^{12} = 64 (\cos(-7\pi) + i \sin(-7\pi)) \\
 &= -64.
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \sqrt[6]{-\sqrt{48} + 4i} &= \sqrt[6]{8 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)} = 8^{\frac{1}{6}} \left(\cos \frac{2k\pi + \frac{5\pi}{6}}{6} + i \sin \frac{2k\pi + \frac{5\pi}{6}}{6} \right) \\
 &= 8^{\frac{1}{6}} \left(\cos \left(\frac{k\pi}{3} + \frac{5\pi}{36} \right) + i \sin \left(\frac{k\pi}{3} + \frac{5\pi}{36} \right) \right), \quad k = 0, 1, 2, \dots, 5.
 \end{aligned}$$

(e) We first note that

$$\frac{4+4i}{(-2+\sqrt{12}i)^3} = \frac{\sqrt{32} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)}{\left[4 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \right]^3} = \frac{\sqrt{32} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)}{64 \underbrace{\left(\cos 2\pi + i \sin 2\pi \right)}_{=1}} = \frac{1}{2\sqrt{32}} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right).$$

Thus, we have

$$\begin{aligned}\sqrt{\frac{4+4i}{(-2+\sqrt{12}i)^3}} &= \sqrt{\frac{1}{2\sqrt{32}}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)} = 2^{-\frac{7}{4}}\left(\cos\frac{2k\pi+\frac{\pi}{4}}{2}+i\sin\frac{2k\pi+\frac{\pi}{4}}{2}\right) \\ &= 2^{-\frac{7}{4}}\left(\cos\left(k\pi+\frac{\pi}{8}\right)+i\sin\left(k\pi+\frac{\pi}{8}\right)\right), \quad k=0,1.\end{aligned}$$

(f)

$$\begin{aligned} (-2 - 2i)^{\frac{3}{4}} &= \sqrt[4]{\left[\sqrt{8}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)\right]^3} = \sqrt[4]{8^{\frac{3}{2}}\left(\cos\frac{15\pi}{4} + i\sin\frac{15\pi}{4}\right)} \\ &= 8^{\frac{3}{8}}\left(\cos\frac{2k\pi + \frac{15\pi}{4}}{4} + i\sin\frac{2k\pi + \frac{15\pi}{4}}{4}\right), \quad \text{for } k = 0, 1, 2, 3. \end{aligned}$$

(g)

$$\begin{aligned}\sin \theta - i \cos \theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} - i \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{i(e^{i\theta} - e^{-i\theta})}{-2} - i \frac{e^{i\theta} + e^{-i\theta}}{2} = -ie^{i\theta} = e^{-\frac{i\pi}{2}} e^{i\theta} \\ &= e^{i(\theta - \frac{\pi}{2})}.\end{aligned}$$

Thus

$$(\sin \theta - i \cos \theta)^5 = \left[e^{i(\theta - \frac{\pi}{2})} \right]^5 = e^{i(5\theta - \frac{5\pi}{2})}.$$

(h)

$$1 + e^{\frac{i\pi}{4}} = e^{i0} + e^{\frac{i\pi}{4}} = e^{\frac{i\pi}{8}} \left(e^{-\frac{i\pi}{8}} + e^{\frac{i\pi}{8}} \right) = 2 \cos \frac{\pi}{8} e^{\frac{i\pi}{8}} = 2 \cos \frac{\pi}{8} \left(\cos \frac{\pi}{8} + i \sin \frac{\pi}{8} \right).$$

Thus, we conclude that

$$\sqrt[4]{1 + e^{\frac{i\pi}{4}}} = \sqrt[4]{2 \cos \frac{\pi}{8} \left(\cos \frac{\pi}{8} + i \sin \frac{\pi}{8} \right)} = \left(2 \cos \frac{\pi}{8} \right)^{\frac{1}{4}} \left(\cos \frac{2k\pi + \frac{\pi}{8}}{4} + i \sin \frac{2k\pi + \frac{\pi}{8}}{4} \right),$$

$$k = 0, 1, 2, 3.$$

Problem 6

$$\begin{aligned} \left(\frac{1+i \tan \theta}{1-i \tan \theta}\right)^5 &= \left(\frac{1+i \frac{\sin \theta}{\cos \theta}}{1-i \frac{\sin \theta}{\cos \theta}}\right)^5 = \left(\frac{\cos \theta+i \sin \theta}{\cos \theta-i \sin \theta}\right)^5 = \frac{(\cos \theta+i \sin \theta)^5}{(\cos (-\theta)+i \sin (-\theta))^5} \\ &= \frac{\cos 5 \theta+i \sin 5 \theta}{\cos (-5 \theta)+i \sin (-5 \theta)} = \frac{\cos 5 \theta+i \sin 5 \theta}{\cos 5 \theta-i \sin 5 \theta} = \frac{1+i \frac{\sin 5 \theta}{\cos 5 \theta}}{1-i \frac{\sin 5 \theta}{\cos 5 \theta}} = \frac{1+i \tan 5 \theta}{1-i \tan 5 \theta} . \end{aligned}$$

Problem 7

(a) Since $|z| = r = 1$, then the polar form of z is seen to be $z = \cos \theta + i \sin \theta$. Then

$$z_0 = \frac{1+z}{1-z} = \frac{(1+\cos\theta) + i\sin\theta}{(1-\cos\theta) - i\sin\theta} = \frac{(1+\cos\theta) + i\sin\theta}{(1-\cos\theta) - i\sin\theta} \left(\frac{(1-\cos\theta) + i\sin\theta}{(1-\cos\theta) + i\sin\theta} \right) \\ = \frac{(1-\cos^2\theta) + [\sin\theta(1-\cos\theta) + \sin\theta(1+\cos\theta)]i + i^2\sin^2\theta}{(1-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{\sin^2 \theta - \sin^2 \theta + 2 \sin \theta i}{2 - 2 \cos \theta} = \frac{\sin \theta}{1 - \cos \theta} i \quad \text{Take } b = \frac{\sin \theta}{1 - \cos \theta} \quad \equiv bi.$$

Thus z_0 is purely imaginary.

- (b) Since $z = \cos \theta + i \sin \theta$, then $\bar{z} = \cos \theta - i \sin \theta$. The remaining argument is similar to that of (a). We omit the detail here.

Problem 8

- (a) $\frac{3z}{1+\bar{z}} = \frac{3(1+3i)}{1+(1-3i)} = \frac{3+9i}{2-3i} = \frac{3+9i}{2-3i} \left(\frac{2+3i}{2+3i} \right) = \frac{6+24i+27i^2}{4-9i^2} = \frac{-21+24i}{13}.$
- (b) $(z+3\bar{z})^2 = [(1+3i)+3(1-3i)]^2 = (4-6i)^2 = 16-48i+36i^2 = -20-48i.$
- (c) $\sqrt[4]{z+\bar{z}} = \sqrt[4]{(1+3i)+(1-3i)} = \sqrt[4]{2} = \sqrt[4]{2(\cos 0 + i \sin 0)} = 2^{\frac{1}{4}} \left(\cos \frac{2k\pi}{4} + i \sin \frac{2k\pi}{4} \right), \text{ for } k = 0, 1, 2, 3.$

Problem 9

Note that

$$\begin{aligned} \operatorname{Re} \left(\frac{z-1}{z+1} \right) &= \frac{1}{2} \left(\frac{z-1}{z+1} + \overline{\frac{z-1}{z+1}} \right) \stackrel{\bar{z}=1/\bar{z}}{=} \frac{1}{2} \left(\frac{z-1}{z+1} + \frac{\bar{z}-1}{\bar{z}+1} \right) = \frac{1}{2} \frac{(z-1)(\bar{z}+1) + (\bar{z}-1)(z+1)}{(z+1)(\bar{z}+1)} \\ &= \frac{1}{2} \frac{2z\bar{z}-2}{(z+1)(\bar{z}+1)} \stackrel{z\bar{z}=|z|^2}{=} \frac{|z|^2-1}{(z+1)(\bar{z}+1)} \stackrel{|z|=1}{=} \frac{1-1}{(z+1)(\bar{z}+1)} = 0. \\ \operatorname{Im} \left(\frac{z-1}{z+1} \right) &= \frac{1}{2i} \left(\frac{z-1}{z+1} - \overline{\frac{z-1}{z+1}} \right) \stackrel{\bar{z}=1/\bar{z}}{=} \frac{1}{2i} \left(\frac{z-1}{z+1} - \frac{\bar{z}-1}{\bar{z}+1} \right) = \frac{1}{2i} \frac{(z-1)(\bar{z}+1) - (\bar{z}-1)(z+1)}{(z+1)(\bar{z}+1)} \\ &= \frac{1}{2i} \frac{2z-2\bar{z}}{(z+1)(\bar{z}+1)} = \frac{1}{i} \frac{z-\bar{z}}{z\bar{z}+z+\bar{z}+1} = \frac{1}{i} \frac{2i \operatorname{Im}(z)}{|z|^2+2\operatorname{Re}(z)+1} = \frac{\operatorname{Im}(z)}{1+\operatorname{Re}(z)}. \end{aligned}$$

Problem 10

$$\begin{aligned} |z_1+z_2|^2 - |z_1-z_2|^2 &= (z_1+z_2)\overline{(z_1+z_2)} - (z_1-z_2)\overline{(z_1-z_2)} \\ &= (z_1+z_2)(\bar{z}_1+\bar{z}_2) - (z_1-z_2)(\bar{z}_1-\bar{z}_2) \\ &= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 - z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 - z_2\bar{z}_2 = 2(z_1\bar{z}_2 + z_2\bar{z}_1) \\ &= 2(z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) = 4\operatorname{Re}(z_1\bar{z}_2). \end{aligned}$$

Remark: The last equality follows from the fact that $z + \bar{z} = 2\operatorname{Re}(z)$.

Problem 11

(a) $z^6 = -3 + \sqrt{3}i \Rightarrow z = \sqrt[6]{-3 + \sqrt{3}i} = \sqrt[6]{\sqrt{12} \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)}$

$$= 12^{\frac{1}{12}} \left(\cos \frac{2k\pi + \frac{5\pi}{6}}{6} + i \sin \frac{2k\pi + \frac{5\pi}{6}}{6} \right)$$

$$= 12^{\frac{1}{12}} \left(\cos \left(\frac{k\pi}{3} + \frac{5\pi}{36} \right) + i \sin \left(\frac{k\pi}{3} + \frac{5\pi}{36} \right) \right), \text{ for } k = 0, 1, 2, \dots, 5.$$

(b) $(1-z)^7 + (1+z)^7 = 0 \Rightarrow \left(\frac{1-z}{1+z} \right)^7 = -1 \Rightarrow \frac{1-z}{1+z} = (\cos \pi + i \sin \pi)^{\frac{1}{7}}$

$$\Rightarrow \frac{1-z}{1+z} = \underbrace{\cos \frac{2k\pi + \pi}{7} + i \sin \frac{2k\pi + \pi}{7}}_{\omega_k}, \quad k = 0, 1, 2, \dots, 6.$$

$$\Rightarrow z = \frac{1 - \omega_k}{1 + \omega_k}$$

$$\begin{aligned}
(c) \quad z^{10} - 5z^5 - 6 &= 0 \Rightarrow (z^5 - 6)(z^5 + 1) = 0 \\
&\Rightarrow z^5 = 6 \text{ or } z^5 = -1 \\
&\Rightarrow z = \sqrt[5]{6(\cos 0 + i \sin 0)} = 6^{\frac{1}{5}} \left(\cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5} \right) \text{ or} \\
&\quad z = \sqrt[5]{(\cos \pi + i \sin \pi)} = \cos \frac{2k\pi + \pi}{5} + i \sin \frac{2k\pi + \pi}{5} \\
&\text{where } k = 0, 1, 2, 3, 4.
\end{aligned}$$

$$\begin{aligned}
(d) \quad z^8 - 2\sqrt{3}z^4 + 4 &= 0 \xrightarrow{y=z^4} y^2 - 2\sqrt{3}y + 4 = 0 \\
&\Rightarrow y = \frac{2\sqrt{3} \pm \sqrt{(-2\sqrt{3})^2 - 4(1)(4)}}{2} = \sqrt{3} \pm i \\
&\Rightarrow z^4 = \sqrt{3} + i \text{ or } z^4 = \sqrt{3} - i \\
&\Rightarrow z = \sqrt[4]{2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)} \text{ or } z = \sqrt[4]{2 \left(\cos \left(-\frac{\pi}{6} \right) + i \sin \left(-\frac{\pi}{6} \right) \right)} \\
&\Rightarrow z = 2^{\frac{1}{4}} \left(\cos \frac{2k\pi + \frac{\pi}{6}}{4} + i \sin \frac{2k\pi + \frac{\pi}{6}}{4} \right) \text{ or } z = 2^{\frac{1}{4}} \left(\cos \frac{2k\pi - \frac{\pi}{6}}{4} + i \sin \frac{2k\pi - \frac{\pi}{6}}{4} \right). \\
&\text{where } k = 0, 1, 2, 3.
\end{aligned}$$

$$\begin{aligned}
(e) \quad \frac{z^5}{1+z^5} &= \sqrt{3}i \Rightarrow z^5 = \sqrt{3}i + \sqrt{3}iz^5 \\
&\Rightarrow z^5 = \frac{\sqrt{3}i}{1 - \sqrt{3}i} = \frac{\sqrt{3} \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)}{2 \left(\cos \left(-\frac{\pi}{3} \right) + i \sin \left(-\frac{\pi}{3} \right) \right)} = \frac{\sqrt{3}}{2} \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) \\
&\Rightarrow z = \sqrt[5]{\frac{\sqrt{3}}{2} \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)} = \frac{3^{\frac{1}{10}}}{2^{\frac{1}{5}}} \left(\cos \frac{2k\pi + \frac{5\pi}{6}}{5} + i \sin \frac{2k\pi + \frac{5\pi}{6}}{5} \right) \\
&\text{where } k = 0, 1, 2, 3, 4.
\end{aligned}$$

Problem 12

Since $3 + i$ is one of the root, it follows that $\overline{3 + i} = 3 - i$ is also the root of the same equation. By factor theorem, $[z - (3 + i)]$ and $[z - (3 - i)]$ are the factors of the polynomial on L.H.S. This implies that the product $[z - (3 + i)][z - (3 - i)] = z^2 - 6z + 10$ is also the factor of the same polynomial.

Using long division, the equation can be solved as

$$\begin{aligned}
z^4 - 8z^3 + 27z^2 - 50z + 50 &= 0 \\
&\Rightarrow (z^2 - 6z + 10)(z^2 - 2z + 5) = 0 \\
&\Rightarrow z = 3 \pm i \text{ or } z = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2} = 1 \pm 2i.
\end{aligned}$$

Problem 13

(a) We first note that

$$(\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta \dots (1)$$

On the other hand, one cause Binomial theorem and obtain

$$\begin{aligned}
&(\cos \theta + i \sin \theta)^5 \\
&= \cos^5 \theta + 5 \cos^4 \theta \sin \theta i + 10 \cos^3 \theta \sin^2 \theta i^2 + 10 \cos^2 \theta \sin^3 \theta i^3 + 5 \cos \theta \sin^4 \theta i^4 \\
&\quad + \sin^5 \theta i^5 \\
&= (\cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta) \\
&\quad + i(5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta) \dots (2).
\end{aligned}$$

By comparing the *real part* between the equations (1) and (2), we get

$$\begin{aligned}\cos 5\theta &= \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta \\ &= \cos^5 \theta - 10 \cos^3 \theta (1 - \cos^2 \theta) + 5 \cos \theta (1 - \cos^2 \theta)^2 \\ &= 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta.\end{aligned}$$

- (b) We shall consider the expression $(\cos \theta + i \sin \theta)^3$. We first note that
 $(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta \dots (1)$

On the other hand, one can use Binomial theorem and obtain

$$\begin{aligned}(\cos \theta + i \sin \theta)^3 &= \cos^3 \theta + 3 \cos^2 \theta \sin \theta i + 3 \cos \theta \sin^2 \theta i^2 + \sin^3 \theta i^3 \\ &= (\cos^3 \theta - 3 \cos \theta \sin^2 \theta) + i(3 \cos^2 \theta \sin \theta - \sin^3 \theta) \dots (2).\end{aligned}$$

By comparing the *imaginary part* between the equation (1) and (2), we obtain

$$\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta = 3(1 - \sin^2 \theta) \sin \theta - \sin^3 \theta = 3 \sin \theta - 4 \sin^3 \theta.$$

Problem 14

- (a) Using the fact that $z - \frac{1}{z} = 2i \sin \theta$, we obtain

$$\left(z - \frac{1}{z}\right)^5 = (2i \sin \theta)^5 = 32 \sin^5 \theta i^5 = 32 \sin^5 \theta i \dots (1).$$

On the other hand, one can use Binomial theorem and obtain

$$\begin{aligned}\left(z - \frac{1}{z}\right)^5 &= z^5 - 5z^3 + 10z - 10\left(\frac{1}{z}\right) + 5\left(\frac{1}{z^3}\right) - \frac{1}{z^5} = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right) \\ &= 2i \sin 5\theta - 10i \sin 3\theta + 20i \sin \theta \dots (2).\end{aligned}$$

By comparing (1) and (2), we obtain

$$\begin{aligned}32 \sin^5 \theta i &= 2i \sin 5\theta - 10i \sin 3\theta + 20i \sin \theta \\ \Rightarrow \sin^5 \theta &= \frac{1}{16} (\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta).\end{aligned}$$

- (b) Using the fact that $z - \frac{1}{z} = 2i \sin \theta$ and $z + \frac{1}{z} = 2 \cos \theta$, we get

$$\left(z - \frac{1}{z}\right)^3 \left(z + \frac{1}{z}\right)^4 = (2i \sin \theta)^3 (2 \cos^4 \theta) = -128 \sin^3 \theta \cos^4 \theta i \dots (1)$$

One can use Binomial theorem and obtain

$$\begin{aligned}\left(z - \frac{1}{z}\right)^3 \left(z + \frac{1}{z}\right)^4 &= \left[\left(z - \frac{1}{z}\right)\left(z + \frac{1}{z}\right)\right]^3 \left(z + \frac{1}{z}\right) = \left(z^2 - \frac{1}{z^2}\right)^3 \left(z + \frac{1}{z}\right) \\ &= \left(z^6 - 3z^2 + \frac{3}{z^2} - \frac{1}{z^6}\right) \left(z + \frac{1}{z}\right) = z^7 - 3z^3 + \frac{3}{z} - \frac{1}{z^5} + z^5 - 3z + \frac{3}{z^3} - \frac{1}{z^7} \\ &= \left(z^7 - \frac{1}{z^7}\right) + \left(z^5 - \frac{1}{z^5}\right) - 3\left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right) \\ &= (2 \sin 7\theta + 2 \sin 5\theta - 6 \sin 3\theta - 6 \sin \theta)i \dots (2)\end{aligned}$$

By comparing (1) and (2), we obtain

$$\begin{aligned}-128 \sin^3 \theta \cos^4 \theta i &= (2 \sin 7\theta + 2 \sin 5\theta - 6 \sin 3\theta - 6 \sin \theta)i \\ \Rightarrow \sin^3 \theta \cos^4 \theta &= -\frac{1}{64} (\sin 7\theta + \sin 5\theta - 3 \sin 3\theta - 3 \sin \theta).\end{aligned}$$

- (c) Using the result obtained in (b), the integral can be computed as

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^4 \theta d\theta &= -\frac{1}{64} \int_0^{\frac{\pi}{2}} (\sin 7\theta + \sin 5\theta - 3 \sin 3\theta - 3 \sin \theta) d\theta \\ &= -\frac{1}{64} \left[-\frac{1}{7} \cos 7\theta - \frac{1}{5} \cos 5\theta + \cos 3\theta + 3 \cos \theta \right]_0^{\frac{\pi}{2}} = -\frac{2}{35}.\end{aligned}$$