

Name: _____ SID: _____

MA2001 (CB1)

Quiz 1

Semester A, 2023-24

Answer all questions. Write your answers in the space provided.

1. State whether the following statements are true (T) or false (F). [4 marks]

(a) If all the eigenvalues of matrix B are distinct, then B is diagonalizable. Answer: T(b) If matrix C is diagonalizable, then C is invertible. Answer: F(c) If $\det(M) = 1$, then one of the eigenvalues of M is 1. Answer: F(d) The quadratic form $q = -4x_1^2 - x_2^2 - 2x_3^2 + 4x_1x_2$ is negative definite. Answer: F

$$= -(2x_1 - x_2)^2 - 2x_3^2$$

2. If A is a 2×2 matrix with $A \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \end{pmatrix}$ and $A \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \end{pmatrix}$.(a) Find the eigenvalues and the corresponding eigenvectors of A . [4 marks]

$$A \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\therefore \lambda_1 = 5, \quad \vec{x}_1 = k \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad k \neq 0$$

$$A \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \end{pmatrix} = -3 \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\therefore \lambda_2 = -3, \quad \vec{x}_2 = k \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \quad k \neq 0$$

(b) Write down an invertible matrix P and a diagonal matrix D such that $AP = PD$. [4 marks]

$$P = \begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}$$

(c) Find the matrix A .

[4 marks]

$$AP = PD$$

$$\Rightarrow A = PDP^{-1}$$

$$= \begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix} \frac{1}{4} \begin{pmatrix} -3 & -2 \\ -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -10 & -6 \\ 5 & 9 \end{pmatrix} \frac{1}{4} \begin{pmatrix} -3 & -2 \\ -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 36 & 32 \\ -24 & -28 \end{pmatrix} \frac{1}{4}$$

$$= \begin{pmatrix} 9 & 8 \\ -6 & -7 \end{pmatrix}$$

(d) Find the eigenvalues of $A^4 - 3I$, where I is the 2×2 identity matrix.

[2 marks]

$$\lambda_1 = 5^4 - 3 = 622$$

$$\lambda_2 = (-3)^4 - 3 = 78$$

(e) Is the quadratic form $q = [x_1 \ x_2] A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ positive definite, negative definite, positive semidefinite, negative semidefinite or indefinite? Show how you obtain your answer.

[2 marks]

$$\begin{aligned} q &= 9x_1^2 + 2x_1x_2 - 7x_2^2 \\ &= 9(x_1^2 + \frac{2}{9}x_1x_2) - 7x_2^2 \\ &= \underbrace{9}_{>0} \left(x_1 + \frac{1}{9}x_2\right)^2 - \underbrace{\frac{64}{9}}_{<0} x_2^2 \end{aligned}$$

$\therefore$ indefinite

- End -