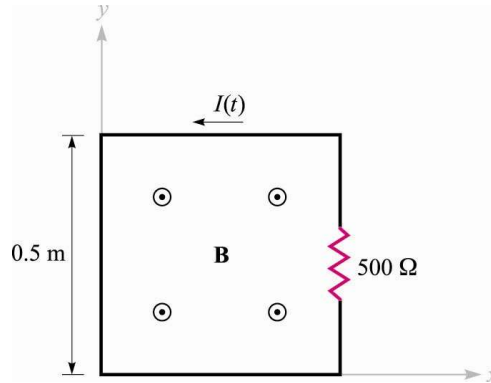


EE2104 Introduction to Electromagnetics
Assignment 4 Solution

Deadline (for uploading): 7:00 pm, 5 Dec 2020
Please put down your name and university ID.

1. A perfectly conducting filament containing a small $500\text{-}\Omega$ resistor is formed into a square, as illustrated below.



Find $I(t)$ if $\mathbf{B} = B_0 \sin(2\pi f_0 t + \phi_0) \hat{\mathbf{a}}_z$, where $B_0 = 1\text{ mT}$, $f_0 = 70\text{ Hz}$, and $\phi_0 = \frac{\pi}{4}$.

Solution:

The voltage generated from the varying $\mathbf{B}$ -field onto the square filament is

$$V(t) = -\frac{d\Lambda}{dt} = -\mathbf{S} \cdot \frac{d\mathbf{B}}{dt} = -S \cdot (\hat{\mathbf{a}}_z \cdot \hat{\mathbf{a}}_z) \cdot \frac{d[B_0 \sin(2\pi f_0 t + \phi_0)]}{dt}$$

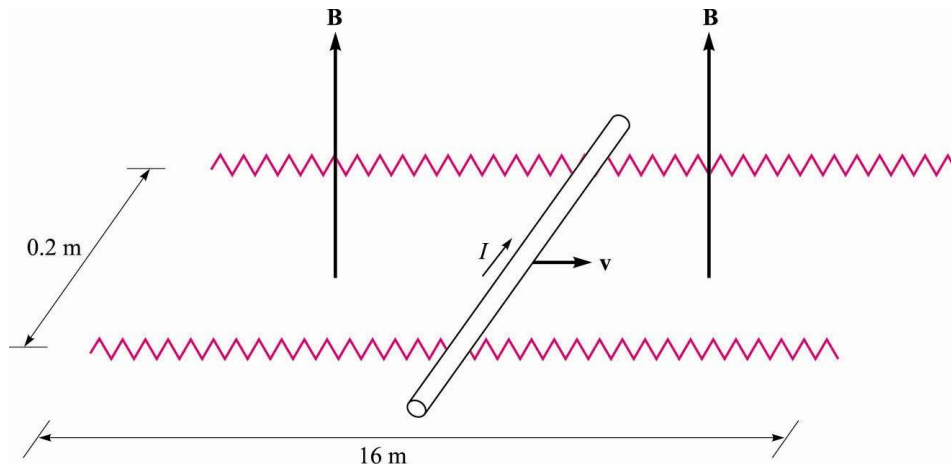
Thus, we have

$$V(t) = -0.25B_0 \frac{d\left[\sin\left(140\pi t + \frac{\pi}{4}\right)\right]}{dt} = -35\pi \cos\left(140\pi t + \frac{\pi}{4}\right) \text{ (mV)}$$

Therefore,

$$I(t) = \frac{V(t)}{R} = -0.07\pi \cos\left(140\pi t + \frac{\pi}{4}\right) \text{ (mA)}$$

2. The rails in the figure below each have a resistance of $5\ \Omega/\text{m}$. The bar with no resistance moves to the right at a constant speed of $9\ \text{m/s}$ in a uniform magnetic field of $80\ \text{mT}$. Find $I(t)$, $0 < t < 1\ \text{s}$, if the bar is at $x = 2\ \text{m}$ at $t = 0$ and a $3\text{-}\Omega$ resistor is present across the left end.



Solution:

For $0 < t < 1\ \text{s}$, the voltage across the bar is

$$V(t) = -\frac{d\Lambda}{dt} = -\mathbf{B} \cdot \frac{d\mathbf{S}}{dt} = -B(\hat{\mathbf{a}}_z \cdot \hat{\mathbf{a}}_z) \cdot 0.2 \frac{d(2 + vt)}{dt} = -0.2Bv = -0.144\ (\text{V})$$

The resistance of the whole resistor is

$$R(t) = 3 + 2 \times 5 \times (2 + vt) = 23 + 10vt = 23 + 90t\ (\Omega)$$

Therefore, we have

$$I(t) = \frac{V(t)}{R(t)} = -\frac{0.144}{23 + 90t}\ (\text{A})$$

3. Consider a long coaxial cable of inner and outer radii a and b , respectively. The gap is filled with material of relative permittivity ϵ_r and relative permeability μ_r . The length of the cable is l . Using the formula for the fields associated with infinite line charge and current.

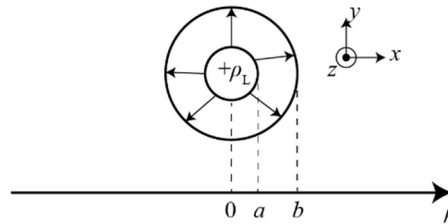
(a) Find the capacitance C .

(b) Find the inductance L .

In the next course on electromagnetics, the concept of characteristic impedance $Z = (L/C)^{1/2}$ will be introduced. For $\epsilon_r = 4$, $\mu_r = 1$, $b = 2\text{mm}$, and $a = 0.2\text{mm}$, evaluate Z and state its unit.

Solution:

Let the cable be placed on the z -axis.



- (a) For the capacitance, assume that the wire conductor is charged with $+\rho_L$ (c/m). Thus, we have

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\epsilon_r\rho} \hat{\mathbf{a}}_\rho$$

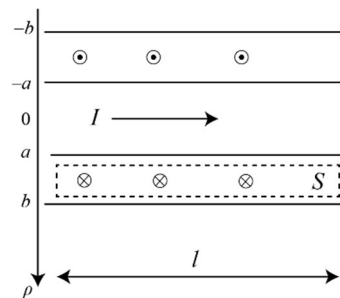
The voltage V in reference to the outer conductor is

$$V = -\int_b^a \mathbf{E} \cdot d\mathbf{L} = -\int_b^a \frac{\rho_L}{2\pi\epsilon_0\epsilon_r\rho} d\rho = \frac{\rho_L}{2\pi\epsilon_0\epsilon_r} \ln\left(\frac{b}{a}\right)$$

Therefore, the capacitance C is

$$C = \frac{\rho_L \cdot l}{V} = \frac{2\pi\epsilon_0\epsilon_r l}{\ln\left(\frac{b}{a}\right)}$$

- (b) For the inductance, assume that the current I flows along the wire conductor.



Then we have

$$\mathbf{B} = \frac{\mu_0\mu_r I}{2\pi\rho} \hat{\mathbf{a}}_\phi$$

Thus, the flux through the surface S is

$$\Phi = \int_S \frac{\mu_0\mu_r I}{2\pi\rho} dS = l \int_a^b \frac{\mu_0\mu_r I}{2\pi\rho} d\rho = l \frac{\mu_0\mu_r I}{2\pi} \ln\left(\frac{b}{a}\right)$$

The inductance L is

$$L = \frac{\Phi}{I} = l \frac{\mu_0\mu_r}{2\pi} \ln\left(\frac{b}{a}\right)$$

So the impedance is

$$Z = \sqrt{\frac{L}{C}} = \frac{1}{2\pi} \sqrt{\frac{\mu_0\mu_r}{\epsilon_0\epsilon_r}} \cdot \ln\left(\frac{b}{a}\right) = 69 \, \Omega$$

4. Sun light reaches the earth surface at intensity of 1 kW/m^2 . Although the radiation consists of a wide spectrum, the following calculation assumes a wavelength of $\lambda = 1 \text{ }\mu\text{m}$ in the infra-red range. We also assume, for simplicity, that the field is polarized along one direction so that the electromagnetic wave is described by:

$$\begin{aligned}\mathbf{E} &= E_0 \cos(kz - \omega t) \hat{\mathbf{a}}_x \\ \mathbf{H} &= H_0 \cos(kz - \omega t) \hat{\mathbf{a}}_y\end{aligned}$$

where the amplitudes E_0 and H_0 , the wavenumber k , and the angular frequency ω are constants. The propagation direction is denoted as the z -direction.

- Express k and ω in terms of λ .
- Using the Maxwell's equations, express H_0 in terms of E_0 .
- By calculating the energy density carried by the sun light, find the value of E_0 .

Solution:

- By definition, the phase of the field increases by 2π when z increases by λ . Then we have

$$k(z + \lambda) = kz + 2\pi$$

Thus,

$$k = 2\pi/\lambda$$

From the wave equation, we have

$$\nabla^2 \cos(kz - \omega t) = \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \cos(kz - \omega t)$$

Therefore,

$$k = \sqrt{\mu_0 \epsilon_0} \omega = \omega/c$$

So

$$k = ck = 2\pi c/\lambda$$

where $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/s}$ is the speed of light.

- From the Maxwell's equations,

$$\nabla \times [E_0 \cos(kz - \omega t) \hat{\mathbf{a}}_x] = -\mu_0 \frac{\partial}{\partial t} [H_0 \cos(kz - \omega t) \hat{\mathbf{a}}_y]$$

Thus,

$$E_0 \frac{\partial}{\partial z} [\cos(kz - \omega t)] \hat{\mathbf{a}}_y = -\mu_0 H_0 \omega \sin(kz - \omega t) \hat{\mathbf{a}}_y$$

Therefore, we have

$$H_0 = \frac{kE_0}{\mu_0 \omega}$$

- The energy density is

$$W = \frac{I}{c} = (1 \text{ kW/m}^2)/(3 \times 10^8 \text{ m/s}) = 3.33 \times 10^{-6} \text{ J/m}^3$$

Also,

$$W = \frac{1}{2} \mu_0 \frac{H_0^2}{2} + \frac{1}{2} \epsilon_0 \frac{E_0^2}{2} = \frac{\epsilon_0 E_0^2}{2} = 3.33 \times 10^{-6} \text{ J/m}^3$$

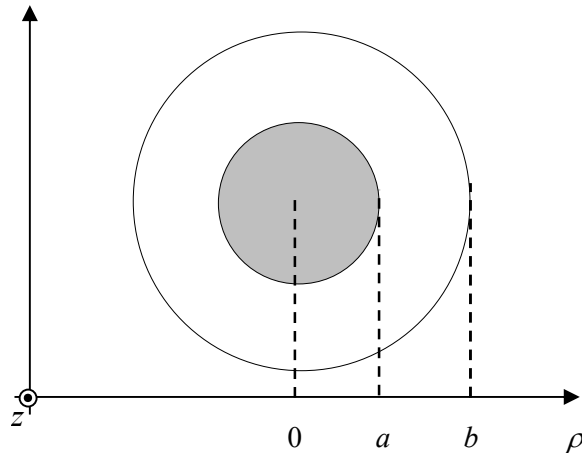
So

$$E_0 = 867 \text{ V/m}$$

5. Consider the following DC generator where a rotating disc is put under a magnetic field. The field is uniformly pointing perpendicular to the disc as

$$\mathbf{B} = B_0 \hat{\mathbf{a}}_z$$

where B_0 is a constant. The disc is rotating in the counter-clockwise sense at frequency f . Using the Faraday's law of electromagnetic induction, find the e.m.f. maintained between radii a and b .



Solution:

Consider a charge q at position ρ .

We have

$$\text{Velocity} = \rho \omega \hat{\mathbf{a}}_y$$

$$\text{Current element} = q \rho \omega \hat{\mathbf{a}}_y$$

$$\text{Force } \mathbf{F} = q \rho \omega \hat{\mathbf{a}}_y \times B_0 \hat{\mathbf{a}}_z = q \rho \omega B_0 \hat{\mathbf{a}}_x$$

The energy acquired when the charge moves from radius $\rho = a$ to $\rho = b$ is

$$W = \int_{\rho=a}^b q \rho \omega B_0 d\rho = q \omega B_0 \frac{b^2 - a^2}{2}$$

Therefore, the e.m.f. maintained between radius $\rho = a$ to $\rho = b$ is

$$V = \frac{W}{q} = \frac{b^2 - a^2}{2} \omega B_0 = (b^2 - a^2) \pi f B_0$$