

1. Boundaries : $\begin{cases} x=0 \\ y=0 \\ y+z=2 \\ x=4-y^2 \end{cases}$

$$\iiint_V f(x, y, z) dv = \int_0^2 \int_0^{2-z} \int_0^{4-y^2} f(x, y, z) dx dy dz$$

$$= \int_0^2 \int_0^{4-y^2} \int_0^{2-y} f(x, y, z) dz dx dy$$

$$= \int_0^1 \int_0^{4-(2-z)^2} \int_0^{2-z} f(x, y, z) dy dx dz +$$

$$\int_0^1 \int_{4-(2-z)^2}^4 \int_0^{\sqrt{4-x}} f(x, y, z) dy dx dz$$

2. (a) $u + v = 3x$

15 $v - 2u = 3y$

$$\Rightarrow x = \frac{1}{3}(u+v)$$

$$y = \frac{1}{3}(v-2u)$$

$$\frac{\partial x}{\partial u} = \frac{1}{3}$$

$$\frac{\partial x}{\partial v} = \frac{1}{3}$$

$$\frac{\partial y}{\partial u} = -\frac{2}{3}$$

$$\frac{\partial y}{\partial v} = \frac{1}{3}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} = \frac{1}{3} \times \frac{1}{3} - \left(-\frac{2}{3}\right) \times \frac{1}{3} = \frac{1}{3}$$

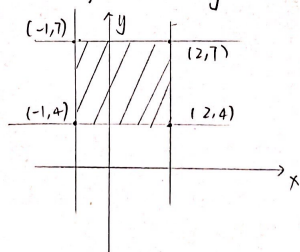
(b) $y = -2x + 4 \Rightarrow \frac{1}{3}(v-2u) = -\frac{2}{3}(u+v) + 4 \rightarrow v = 4$

$y = -2x + 7 \Rightarrow \frac{1}{3}(v-2u) = -\frac{2}{3}(u+v) + 7 \rightarrow v = 7$

$y = x - 2 \Rightarrow \frac{1}{3}(v-2u) = \frac{1}{3}(u+v) - 2 \rightarrow u = 2$

$y = x + 1 \Rightarrow \frac{1}{3}(v-2u) = \frac{1}{3}(u+v) + 1 \rightarrow u = -1$

Transformed region:



$$(c) \iint_R (2x^2 - xy - y^2) dx dy = \iint_R (2x+y)(x-y) dx dy$$

$$= \int_4^7 \int_{-1}^2 \frac{1}{3} uv du dv$$

$$= \frac{1}{3} \int_4^7 \left(\frac{1}{2} v u^2 \right) \Big|_{-1}^2 dv$$

$$= \frac{1}{3} \int_4^7 \frac{3}{2} v dv$$

$$= \frac{1}{3} \cdot \frac{3}{2} v^2 \Big|_4^7$$

$$= \frac{33}{4}$$

3. (a).

$$x = r \cos \theta$$

$$y = r \sin \theta$$

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The region R is

$$\begin{cases} r \leq 2 \\ -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3} \\ r \cos \theta \geq 1 \end{cases}$$

$$(b): J = \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = r$$

$$\iint_R \frac{1}{\sqrt{x^2 + y^2}} dx dy = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{\frac{1}{\cos \theta}}^2 \frac{1}{r} \cdot r dr d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (2 - \sec \theta) d\theta$$

$$= (2\theta - \ln |\sec \theta + \tan \theta|) \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \frac{4\pi}{3} - \ln(\sqrt{3} + 2) + \ln(2 - \sqrt{3})$$

4.

Boundaries:

$$\begin{cases} z = x \\ z = 0 \\ x = \sqrt{1-y^2} \\ x = 0 \\ y = 1 \\ y = -1 \end{cases}$$

We set

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$$

Then boundaries can be transferred into

$$\begin{cases} z = r \cos \theta \\ z = 0 \\ r = 1 \\ r \cos \theta = 0 \\ r \sin \theta = 1 \\ r \sin \theta = -1 \end{cases}$$

The region is $0 \leq z \leq r \cos \theta$

$$0 \leq x \leq \sqrt{1-y^2} \Rightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad 0 \leq r \leq 1$$

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Hence
$$\int_{-1}^1 \int_0^{\sqrt{1-y^2}} \int_0^x (x^2 + y^2) dz dx dy$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \int_0^{r \cos \theta} r^2 r dz dr d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r^4 \cos \theta dr d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left. \frac{1}{5} r^5 \cos \theta \right|_0^1 d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5} \cos \theta d\theta = \left. \frac{1}{5} \sin \theta \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{5}$$

5.

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$$(a) V = \int_0^1 \int_{-1}^1 \int_0^{y^2} 1 \, dz \, dy \, dx$$

$$(b) \quad x^2 + y^2 = 1 \quad z = -y \quad z = 0$$

$$\text{Let } x = r \cos \theta, \quad y = r \sin \theta \quad x^2 + y^2 = r^2 \quad 0 \leq r \leq 1 \quad 0 \leq \theta \leq \pi$$

$$V = \int_0^\pi \int_0^1 \int_{-r \sin \theta}^0 r \, dz \, dr \, d\theta$$

$$(c) \quad \text{Let } x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

$$1 - x^2 - y^2 = -\sqrt{1 - x^2 - y^2} \Rightarrow z = 1 - r^2 \quad z = -\sqrt{1 - r^2} \quad 0 \leq r \leq 1 \quad 0 \leq \theta \leq 2\pi$$

$$V = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{1-r^2} r \, dz \, dr \, d\theta$$

$$(d) \quad z \geq 0 \Rightarrow \rho \cos \phi \geq 0 \quad -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$$

$$\text{Transformed region: } \omega > \phi \leq \rho \leq 2$$

$$-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$$

$$0 \leq \theta \leq 2\pi$$

$$V = \iiint V = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_{\omega}^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

6.

$$(a) \text{curl}(\vec{F}) = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{pmatrix} \quad \text{where } \begin{aligned} f_1 &= y+z \\ f_2 &= z+x \\ f_3 &= x+y \end{aligned}$$

$$= (\partial_y f_3 - \partial_z f_2) \vec{i} + (\partial_z f_1 - \partial_x f_3) \vec{j} + (\partial_x f_2 - \partial_y f_1) \vec{k}$$

$$= (1-1) \vec{i} + (1-1) \vec{j} + (1-1) \vec{k} = \vec{0}$$

Hence $\vec{F}$ is conservative. Suppose $\phi(x, y, z)$ is the potential function.

$$\therefore \phi_x = y+z \Rightarrow \phi = x(y+z) + f(y, z)$$

$$\phi_y = x + f_y = x+z \Rightarrow f_y = z \quad f(y, z) = yz + g(z)$$

$$\therefore \phi = x(y+z) + yz + g(z)$$

$$\phi_z = x+y + g'(z) = x+y \quad \therefore g(z) = C \quad \text{where } C \text{ is constant}$$

$$\therefore \phi(x, y, z) = xy + xz + yz + C, \text{ where } C \text{ is arbitrary constant}$$

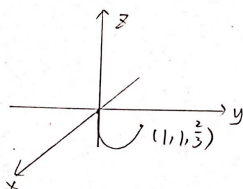
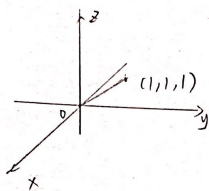
$$(b) \text{div}(\vec{F}) = \frac{\partial(y+z)}{\partial x} + \frac{\partial(z+x)}{\partial y} + \frac{\partial(x+y)}{\partial z} = 0$$

Hence $\text{div} \vec{F}$ at $(1, 2, 3)$ is 0

(c) From (a), we get $\text{curl}(\vec{F}) = \vec{0}$

7. (a)

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$$(b) \vec{r}(t) = \vec{i} + 2t\vec{j} + 2t^2\vec{k}$$

$$|\vec{r}(t)| = \sqrt{1^2 + (2t)^2 + (2t^2)^2} = 2t^2 + 1$$

$$\begin{aligned} \int_C xz \, dC &= \int_0^1 t \cdot \frac{2}{3} t^2 \cdot (2t^2 + 1) \, dt \\ &= \left(\frac{2}{9} t^6 + \frac{1}{6} t^4 \right) \Big|_0^1 = \frac{7}{18} \end{aligned}$$

$$\begin{aligned} (c) \quad \vec{F} &= (y+z)\vec{i} + (z+x)\vec{j} + (x+y)\vec{k} \\ &= 2t\vec{i} + 2t\vec{j} + 2t\vec{k} \end{aligned}$$

$$\vec{r}'(t) = \vec{i} + \vec{j} + \vec{k}$$

$$\begin{aligned} \int_C \vec{F} \, dC &= \int_0^1 (2t, 2t, 2t) \cdot (1, 1, 1) \, dt \\ &= 3t^2 \Big|_0^1 = 3 \end{aligned}$$

8. (a) Set $\begin{cases} x=x \\ y=y \\ z=4-y^2 \end{cases} \quad \begin{matrix} 0 \leq x \leq 2 \\ -2 \leq y \leq 2 \end{matrix}$

(b) $\vec{r}(x,y) = x\vec{i} + y\vec{j} + (4-y^2)\vec{k}$

$\vec{r}_x = \vec{i} \quad \vec{r}_y = \vec{j} - 2y\vec{k}$

$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = 2y\vec{j} + \vec{k}$

$|\vec{r}_x \times \vec{r}_y| = \sqrt{4y^2 + 1}$

$\iint_S xz \, dS = \int_{-2}^2 \int_0^2 x(4-y^2) \sqrt{4y^2+1} \, dx \, dy$

$= \int_{-2}^2 (4-y^2) \sqrt{4y^2+1} \, dy \cdot \int_0^2 x \, dx$

We consider $\int (4-y^2) \sqrt{4y^2+1} \, dy$ part. Let $y = \frac{1}{2} \tan \theta$

$$\begin{aligned} \int (4-y^2) \sqrt{4y^2+1} \, dy &= \int (4 - \frac{1}{4} \tan^2 \theta) \sec \theta d(\frac{1}{2} \tan \theta) \\ &= \int [4 - \frac{1}{4} (\sec^2 \theta - 1)] \frac{1}{2} \sec^3 \theta \, d\theta \\ &= \frac{17}{8} \int \sec^3 \theta \, d\theta - \frac{1}{8} \int \sec^5 \theta \, d\theta \end{aligned}$$

$$\begin{aligned} \int \sec^3 \theta \, d\theta &= \int \sec \theta d(\tan \theta) = \sec \theta \times \tan \theta - \int \tan \theta \sec \theta \, d\theta \\ &= \sec \theta \tan \theta - \int (\sec^3 \theta - \sec \theta) \, d\theta \\ &= \sec \theta \tan \theta - \int \sec^3 \theta \, d\theta + \ln |\tan \theta + \sec \theta| + C \\ \therefore \int \sec^3 \theta \, d\theta &= \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\tan \theta + \sec \theta| + C \end{aligned}$$

Similarly $\int \sec^5 \theta \, d\theta = \frac{1}{4} \sec^3 \theta \tan \theta + \frac{3}{8} \sec \theta \tan \theta + \frac{3}{8} \ln |\sec \theta + \tan \theta| + C$

$$\therefore \int (4-y^2) \sqrt{4y+1} dy = \frac{17}{8} \left(\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right) - \frac{1}{8} \times \left(\frac{1}{4} \sec^3 \theta \tan \theta + \frac{3}{8} \sec \theta \tan \theta + \frac{3}{8} \ln |\sec \theta + \tan \theta| \right) + C$$

$$= \frac{65}{64} \sec \theta \tan \theta + \frac{65}{64} \ln |\sec \theta + \tan \theta| - \frac{1}{32} \sec^3 \theta \tan \theta + C$$

$$\text{Hence } \int_{-2}^2 (4-y^2) \sqrt{4y+1} dy = \int_{\tan^{-1}(-4)}^{\tan^{-1}(4)} \left(4 - \frac{1}{4} (\sec^2 \theta - 1) \right) \frac{1}{2} \sec^3 \theta d\theta$$

$$= \left(\frac{65}{64} \sec \theta \tan \theta + \frac{65}{64} \ln |\sec \theta + \tan \theta| - \frac{1}{32} \sec^3 \theta \tan \theta \right) \Bigg|_{\tan^{-1}(-4)}^{\tan^{-1}(4)}$$

$$= \frac{65}{8} \sqrt{17} - \frac{17}{4} \sqrt{17} + \frac{65}{32} \ln (\sqrt{17} + 4)$$

$$\therefore \int_0^2 \int_{-2}^2 (4-y^2) \sqrt{4y+1} dy dx = \frac{31}{4} \sqrt{17} + \frac{65}{16} \ln (\sqrt{17} + 4)$$

$$(c) \iint_S \vec{F} \cdot d\vec{S} = \int_{-2}^2 \int_0^2 (4+y-y^2, 4+x-y^2, x+y) \cdot (0, 2y, 1) dx dy$$

$$= \int_{-2}^2 \int_0^2 (9y + x + 2xy - 2y^3) dx dy$$

$$= \int_{-2}^2 \left((9y - 2y^3)x + (2y+1) \frac{x^2}{2} \Big|_0^2 \right) dy$$

$$= 2 \int_{-2}^2 (1 + 11y - 2y^3) dy$$

$$= 2 \left(y + \frac{11}{2} y^2 - \frac{2}{4} y^4 \Big|_{-2}^2 \right)$$

$$= 2 \times 4 = 8$$