

EE 3121 Summary

Chap. 1 Modeling of physical systems via ODE: Electric Circuits, Mechanical systems, etc.

Chap. 2 First-order linear ODE

$$\frac{dy}{dt} + p(t)y = g(t), \quad y(t_0) = y_0$$

- Method of separating variables
- Integrating factor
- General solution:

$$y(t) = y_0 e^{-\int_{t_0}^t p(\tau) d\tau} + \int_{t_0}^t e^{-\int_{\tau}^t p(\tau) d\tau} g(\tau) d\tau$$

Chap. 3 First-order nonlinear ODE

- Separation of variables

$$\frac{dy}{dt} = f(t)g(y), \quad y(t_0) = y_0$$

$$\Rightarrow \frac{dy}{g(y)} = f(t) dt$$

Example :

$$\frac{dy}{dt} + 8y + y^2 + 15 = 0, \quad y(0) = 0$$

$$\Rightarrow \frac{dy}{dt} = -(y^2 + 8y + 15)$$

$$\Rightarrow \frac{dy}{y^2 + 8y + 15} = -dt$$

$$\Rightarrow \frac{dy}{(y+3)(y+5)} = -dt$$

Partial fraction expansion of $\frac{1}{(y+3)(y+5)}$.

- Successive Approximation

Chap. 4 Existence and Uniqueness

- Lipschitz condition

$$|f(t, y_1) - f(t, y_2)| \leq K |y_1 - y_2|$$

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

- Estimation of time for existence

$$|t - t_0| \leq \min\left\{a, \frac{b}{M}\right\}$$

$$|t - t_0| \leq a, \quad |y - y_0| \leq b, \quad |f(t, y)| \leq M$$

- Successive approximation

$$y_n(t) = y_0 + \int_{t_0}^t f(\tau, y_{n-1}(\tau)) d\tau$$

- Vector field (direction field)
- Equilibrium and stability

Chap. 5 Second-Order Linear ODE

$$\frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + r(t) y = g(t)$$

$$y'(t_0) = y_1, \quad y(t_0) = y_0$$

- Fundamental solutions of homogeneous equation
- Wronskian
- Kernel function
- Variation of parameters method

$$y(t) = \int_{t_0}^t K(t, \tau) g(\tau) d\tau$$

- Undetermined coefficients method

e^{-kt}	Ce^{-kt}
t	$Ct + D$
$\sin \omega t$	$C \sin \omega t + D \cos \omega t$

- Linear Time-Invariant Second-order ODE:
Solve characteristic roots.

Chap. 6 High-Order Linear ODE

$$\frac{d^n y}{dt^n} + a_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_{n-1}(t) \frac{dy}{dt} + a_n(t) y = g(t)$$

- Extension of all concepts and methods developed for second-order linear ODE.

Chap. 7 Laplace Transform

- Elementary transform pairs
- Properties
 - Differentiation
 - Integration
 - Convolution
- Application of Laplace transform for Solving LTI ODE.
- Partial Fraction expansion
- Inverse Laplace transform

chap. 8 State Space Representation

$$\dot{x} = A(t)x + B(t)u$$

$$x(t_0) = x_0$$

- Fundamental matrix
- Transition matrix

$$\begin{cases} \frac{\partial \Phi(t, t_0)}{\partial t} = A(t) \Phi(t, t_0) \\ \Phi(t_0, t_0) = I \end{cases}$$

- Solution

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^t \Phi(t, \tau) B(\tau) u(\tau) d\tau$$

- LTI state-space equation

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$

— solve eigenvalues/eigenvectors of A

— Calculate $\Phi(t) = e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$

$$\Rightarrow x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}B u(\tau) d\tau$$

chap. 9 PDE

- Parabolic PDE
- Hyperbolic PDE
- Elliptic PDE
- Solutions in separating variables and infinite series.
- Nonlinear ODE: solved in the form of an infinite series.
- Bessel functions.

EE 3121 Final Examination

- 6 problems, 100 points
- Coverage: Cumulative up to parabolic PDE.
- Open book.
- Calculator may be helpful but not necessary.

Problem 1: Modeling of physical systems using differential equations (both scalar equation and state-space equation).

Problem 2: First-order ODE (nonlinear, nonhomogeneous): 1. Solution. 2. Qualitative analysis such as equilibrium points and stability, vector field map, estimation of time for existence of a solution.

Problem 3: Laplace transform for solving differential equations: 1. Properties of Laplace transform; 2. Partial fraction; 3. Elementary transform pairs.

Problem 4: State-space representation
1. Solution form; 2. state transition matrix and its properties.

Problem 5: Computation of transition matrix.

1. General
$$\begin{cases} \frac{\partial \Phi(t, t_0)}{\partial t} = A(t) \Phi(t, t_0) \\ \Phi(t_0, t_0) = I \end{cases}$$

2. Laplace transform

$$\Phi(t) = \mathcal{L}^{-1} \{ (sI - A)^{-1} \} = e^{At}$$

Problem 6 Parabolic PDE

1. Separation
2. Infinite series