

Version 1 (p.1-p.5)

Version 2 (p.6-p.10)

Version 3 (p.11-p.14)

①

**MA1201 Calculus and Basic Algebra II (Section: CC1)**

**Quiz 2: Vector, Complex Number and Matrix**

**Version 1**

Name:

SID:

Score:

Time allowed: 50 minutes

**Instruction:**

Write all your answer in the spaces provided. You need to show all your working steps and method used CLEARLY. Marks will be deducted for any unclear solution.

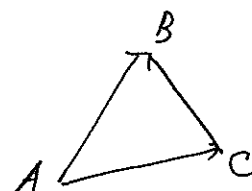
Problem 1

Let  $A = (1, 2, 3)$ ,  $B = (2, 1, 0)$ ,  $C = (1, 3, 3)$  and  $D = (x, 1, 0)$  be four points in 3D-space.

- (a) Find the area of  $\triangle ABC$ . (11 marks)  
 (b) If  $AB$  is perpendicular to  $BD$ , find the value of  $x$ . (11 marks)  
 (c) Find the projector vector of  $\vec{AB}$  onto  $\vec{AC}$  ( $\text{proj}_{\vec{AC}} \vec{AB}$ ). (11 marks)

(a) Note that

$$\vec{AB} = \vec{OB} - \vec{OA} = \vec{i} - \vec{j} - 3\vec{k}, \quad \vec{AC} = \vec{OC} - \vec{OA} = \vec{j}$$



The area of  $\triangle ABC$

$$= \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} |(\vec{i} - \vec{j} - 3\vec{k}) \times \vec{j}| = \frac{1}{2} |\vec{i} \times \vec{j} - \vec{j} \times \vec{j} - 3\vec{k} \times \vec{j}|$$

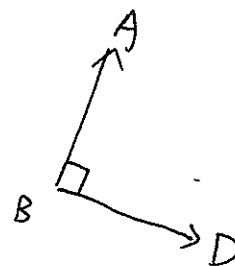
$$= \frac{1}{2} |3\vec{i} + \vec{k}| = \frac{1}{2} \sqrt{(3)^2 + 1^2} = \frac{\sqrt{10}}{2}$$

(b)  $\vec{BD} = \vec{OD} - \vec{OB} = (x-2)\vec{i}$

$$\because AB \perp BD \Rightarrow \vec{BA} \cdot \vec{BD} = |\vec{BA}| |\vec{BD}| \cos 90^\circ = 0$$

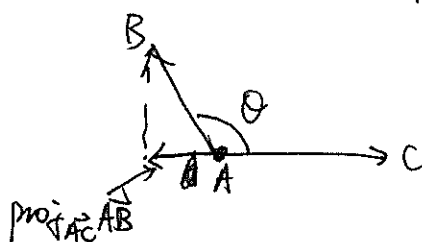
$$\Rightarrow (-\vec{i} + \vec{j} + 3\vec{k}) \cdot [(x-2)\vec{i}] = -(x-2) = 0$$

$$\Rightarrow x = 2$$



(c) Note that

$$\cos \theta = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| |\vec{AC}|} = \frac{1(0) + (-1)(1) + (-3)(0)}{\sqrt{1^2 + (-1)^2 + (-3)^2} \sqrt{0^2 + 1^2 + 0^2}} = \frac{-1}{\sqrt{11}} \Rightarrow \theta > 90^\circ$$



$$\text{Magnitude of } \text{proj}_{\vec{AC}} \vec{AB} = |\vec{AB}| \cos(180^\circ - \theta) = \sqrt{11} \left( \frac{1}{\sqrt{11}} \right) = 1$$

$$\text{Direction of } \text{proj}_{\vec{AC}} \vec{AB} = -\hat{AC} = -\frac{\vec{AC}}{|\vec{AC}|} = -\frac{\vec{j}}{1} = -\vec{j}$$

$$\therefore \text{proj}_{\vec{AC}} \vec{AB} = \text{magnitude} \times \text{direction} = -\vec{j}$$

## Problem 2

(a) Let  $z_1 = \sqrt{6} + \sqrt{2}i$  and  $z_2 = -\sqrt{2} + \sqrt{2}i$  be two complex numbers. Compute

(i)  $z_1^4$ , (ii)  $\sqrt[6]{\frac{z_1}{z_2}}$ , (iii)  $(z_2)^{\frac{3}{2}}$ .

Express your answer in Euler form.

(18 marks)

(b) Solve the equation for  $z$ :

$$z^{10} + 8z^5 - 9 = 0.$$

(15 marks)

(a) Note that  $z_1 = \sqrt{6} + \sqrt{2}i = \sqrt{8} \left[ \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right]$ .  
 $z_2 = -\sqrt{2} + \sqrt{2}i = 2 \left[ \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right]$

$\Rightarrow$  (i)  $z_1^4 = (\sqrt{8})^4 \left[ \cos \frac{4\pi}{6} + i \sin \frac{4\pi}{6} \right] = 64 e^{i(\frac{2\pi}{3})}$

(ii)  $\sqrt[6]{\frac{z_1}{z_2}} = \sqrt[6]{\frac{\sqrt{8}}{2} \left[ \cos \left( \frac{\pi}{6} - \frac{3\pi}{4} \right) + i \sin \left( \frac{\pi}{6} - \frac{3\pi}{4} \right) \right]}$   
 $= \sqrt[6]{\sqrt{2} \left( \cos \left( \frac{-14\pi}{24} \right) + i \sin \left( \frac{-14\pi}{24} \right) \right)} = 2^{\frac{1}{12}} \left[ \cos \frac{k\pi - \frac{7\pi}{12}}{6} + i \sin \frac{k\pi - \frac{7\pi}{12}}{6} \right]$   
 $= 2^{\frac{1}{12}} e^{i \left( \frac{k\pi}{3} - \frac{7\pi}{72} \right)}$  where  $k = 0, 1, 2, 3, 4, 5$ .

(iii)  $z_2^{\frac{3}{2}} = \left( 2 \left[ \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right] \right)^{\frac{3}{2}} = \left( 8 \left[ \cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4} \right] \right)^{\frac{1}{2}}$   
 $= \sqrt{8} \left( \cos \frac{k\pi + \frac{9\pi}{4}}{2} + i \sin \frac{k\pi + \frac{9\pi}{4}}{2} \right) = \sqrt{8} e^{i \left( \frac{k\pi}{2} + \frac{9\pi}{8} \right)}$  for  $k = 0, 1$ .

(4)

$$(b) \quad z^{10} + 8z^5 - 9 = 0$$

$$\Rightarrow y^2 + 8y - 9 = 0$$

$$y = z^5 \Rightarrow (y+9)(y-1) = 0$$

$$\Rightarrow y = z^5 = -9 \quad \text{or} \quad y = z^5 = 1$$

$$\Rightarrow z = \sqrt[5]{9(\cos \pi + i \sin \pi)} \quad \text{or} \quad z = \sqrt[5]{\cos 0 + i \sin 0}$$

$$= 9^{1/5} \left[ \cos \frac{k\pi + \pi}{5} + i \sin \frac{k\pi + \pi}{5} \right] = \cos \frac{k\pi}{5} + i \sin \frac{k\pi}{5}$$

where  $k=0, 1, 2, 3, 4$

## Problem 3

(a) Let  $A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 5 & x & 4 \end{pmatrix}$  and  $B = \begin{pmatrix} y & 0 & -1 \\ -2 & 1 & y \\ -1 & 5 & 4 \end{pmatrix}$  be two matrices.

(i) Find the value of  $\det A$ ,  $\det(A^T A)$ . Express your answer in terms of  $x$ . (Hint:  $\det(A^T) = \det A$ ) (11 marks)

(ii) If  $\det B = 0$ , find the value of  $y$ . (11 marks)

(b) Take  $y = 1$ , find  $B^{-1}$ , the inverse matrix of  $B$ . (12 marks)

expand along 1st row

$$(a)(i) \det A = 2 \times A_{11} + 1 \times A_{12} + 0 \times A_{13}$$

$$= 2 \times (-1)^{1+1} \det \begin{pmatrix} 0 & 1 \\ x & 4 \end{pmatrix} + 1 \times (-1)^{1+2} \det \begin{pmatrix} -1 & 1 \\ 5 & 4 \end{pmatrix}$$

$$= 2 \times [0(4) - x(1)] - 1 \times [(-1)(4) - 5(1)] = -2x + 9$$

$$\det A^T A = (\det A^T)(\det A) = (\det A)(\det A) = (9 - 2x)^2.$$

$$(ii) \therefore \det B = y \times B_{11} + 0 \times B_{12} + (-1) \times B_{13}$$

$$= y \times (-1)^{1+1} \det \begin{pmatrix} 1 & y \\ 5 & 4 \end{pmatrix} + (-1) \times (-1)^{1+3} \det \begin{pmatrix} -2 & 1 \\ -1 & 5 \end{pmatrix}$$

$$= y(1 \times 4 - 5y) - ((-2)(5) - (-1)(1)) = -5y^2 + 4y + 9$$

$$\therefore \det B = 0 \Rightarrow -5y^2 + 4y + 9 = 0 \Rightarrow (-5y + 9)(y + 1) = 0$$

$$\Rightarrow y = \frac{9}{5} \text{ or } y = -1.$$

$$(iii) \text{ Take } y = 1, B = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 1 \\ -1 & 5 & 4 \end{pmatrix} \text{ and } \det B = -5 + 4 + 9 = 8.$$

$$B^{-1} = \frac{1}{\det B} \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}^T = \frac{1}{8} \begin{pmatrix} -1 & 7 & -9 \\ -5 & 3 & -5 \\ 1 & 1 & 1 \end{pmatrix}^T = \frac{1}{8} \begin{pmatrix} -1 & -5 & 1 \\ 7 & 3 & 1 \\ -9 & -5 & 1 \end{pmatrix}.$$

☺End of the Quiz☺

$$B_{11} = (-1)^{1+1} \det \begin{pmatrix} 1 & 1 \\ 5 & 4 \end{pmatrix} = -1$$

$$B_{12} = (-1)^{1+2} \det \begin{pmatrix} -2 & 1 \\ -1 & 4 \end{pmatrix} = 7$$

$$B_{13} = (-1)^{1+3} \det \begin{pmatrix} -2 & 1 \\ -1 & 5 \end{pmatrix} = -9$$

$$B_{21} = (-1)^{2+1} \det \begin{pmatrix} 0 & -1 \\ 5 & 4 \end{pmatrix} = -5$$

$$B_{22} = (-1)^{2+2} \det \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} = 3$$

$$B_{23} = (-1)^{2+3} \det \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix} = -5$$

$$B_{31} = (-1)^{3+1} \det \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = +1$$

$$B_{32} = (-1)^{3+2} \det \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} = 1$$

$$B_{33} = (-1)^{3+3} \det \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = 1$$

**MA1201 Calculus and Basic Algebra II (Section: CC1)**  
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**Instruction:**

**Write all your answer in the spaces provided. You need to show all your working steps and method used CLEARLY. Marks will be deducted for any unclear solution.**

## Problem 1

Let  $A = (0, 3, 1)$ ,  $B = (1, 0, 2)$ ,  $C = (-2, 1, 0)$  and  $D = (-1, x, 1)$  be three points on a 3D-plane.

- (a) Let  $\vec{a}$  be a vector with magnitude 5 and its direction is *opposite* to that of  $\overrightarrow{AD}$ , find this vector  $\vec{a}$ . (11 marks)
- (b) Find a vector which is perpendicular to both  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$ . (11 marks)
- (c) If the points  $A, B, C, D$  are coplanar, find the value of  $x$ . (11 marks)

(a) Note that  $\vec{AD} = \vec{OD} - \vec{OA} = -\vec{i} + (x-3)\vec{j}$ ,  $AD = \sqrt{(-1)^2 + (x-3)^2} = \sqrt{x^2 - 6x + 10}$

Direction of  $\vec{AD} = \hat{AD} = \frac{\vec{AD}}{|\vec{AD}|} = -\frac{1}{\sqrt{x^2 - 6x + 10}}\vec{i} + \frac{x-3}{\sqrt{x^2 - 6x + 10}}\vec{j}$ .

$$\vec{a} = 5 \times (-\hat{AD}) = -\frac{5}{\sqrt{x^2 - 6x + 10}}\vec{i} + \frac{5(x-3)}{\sqrt{x^2 - 6x + 10}}\vec{j}.$$

(b) Note that  $\begin{cases} \vec{AB} = \vec{OB} - \vec{OA} = \vec{i} - 3\vec{j} + \vec{k} \\ \vec{AC} = \vec{OC} - \vec{OA} = -2\vec{i} - 2\vec{j} - \vec{k} \end{cases}$

By the definition,  $\vec{AB} \times \vec{AC}$  is the vector perpendicular to both  $\vec{AB}$  and  $\vec{AC}$ , the required vector is

$$\begin{aligned} \vec{AB} \times \vec{AC} &= (\vec{i} - 3\vec{j} + \vec{k}) \times (-2\vec{i} - 2\vec{j} - \vec{k}) \\ &= -2(\vec{i} \times \vec{i}) - 2(\vec{i} \times \vec{j}) - (\vec{i} \times \vec{k}) + 6(\vec{j} \times \vec{i}) + 6(\vec{j} \times \vec{j}) \\ &\quad + 3(\vec{j} \times \vec{k}) - 2(\vec{k} \times \vec{i}) - 2(\vec{k} \times \vec{j}) - (\vec{k} \times \vec{k}) \\ &= 5\vec{i} - \vec{j} - 8\vec{k}. \end{aligned}$$

(c) If  $A, B, C, D$  are coplanar, then  $\vec{AD} \cdot (\vec{AB} \times \vec{AC}) = 0$

$$\Rightarrow (-\vec{i} + (x-3)\vec{j}) \cdot (5\vec{i} - \vec{j} - 8\vec{k}) = 0$$

$$\Rightarrow (-1)(5) + (x-3)(-1) + 0(-8) = 0$$

$$\Rightarrow -5 - (x-3) = 0 \Rightarrow x = -2.$$

## Problem 2

(a) Let  $z_1 = \sqrt{2} - \sqrt{2}i$  and  $z_2 = -\sqrt{3} + i$  be two complex numbers. Compute

(i)  $\left(\frac{z_2}{z_1}\right)^6$ , (ii)  $\sqrt[4]{z_1}$ , (iii)  $(z_2)^{\frac{4}{5}}$ .

Express your answer in Polar form.

(18 marks)

(b) Solve the equation

$$(z+3)^5 = -32(z-1)^5$$

(15 marks)

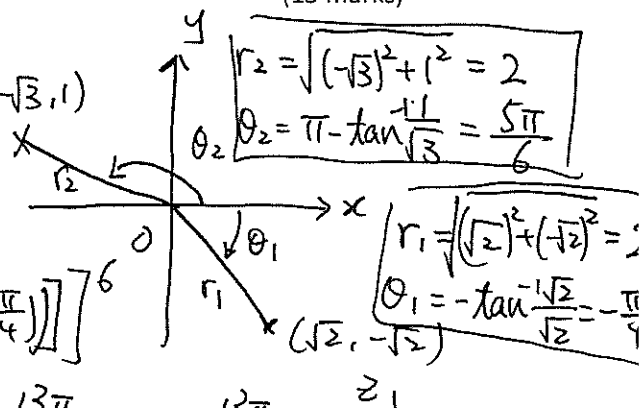
(a)  $z_2 = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$z_2 = (-\sqrt{3}, 1)$

$z_1 = 2\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)$

$\Rightarrow (i) \left(\frac{z_2}{z_1}\right)^6 = \left[\cos\left(\frac{5\pi}{6} - \left(-\frac{\pi}{4}\right)\right) + i\sin\left(\frac{5\pi}{6} - \left(-\frac{\pi}{4}\right)\right)\right]^6$

$$= \left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right)^6 = \cos\frac{13\pi}{2} + i\sin\frac{13\pi}{2}$$



(ii)  $\sqrt[4]{z_1} = \sqrt[4]{2\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)} = 2^{\frac{1}{4}}\left[\cos\frac{2k\pi - \frac{\pi}{4}}{4} + i\sin\frac{2k\pi - \frac{\pi}{4}}{4}\right]$

$$= 2^{\frac{1}{4}}\left(\cos\left(\frac{k\pi}{2} - \frac{\pi}{16}\right) + i\sin\left(\frac{k\pi}{2} - \frac{\pi}{16}\right)\right) \text{ for } k=0, 1, 2, 3$$

(iii)  $(z_2)^{\frac{4}{5}} = \left\{ \left[ 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) \right]^4 \right\}^{\frac{1}{5}} = \left[ 16\left(\cos\frac{20\pi}{6} + i\sin\frac{20\pi}{6}\right) \right]^{\frac{1}{5}}$

$$= 16^{\frac{1}{5}}\left(\cos\frac{2k\pi + \frac{20\pi}{6}}{5} + i\sin\frac{2k\pi + \frac{20\pi}{6}}{5}\right)$$

$$= 16^{\frac{1}{5}}\left(\cos\left(\frac{2k\pi}{5} + \frac{2\pi}{3}\right) + i\sin\left(\frac{2k\pi}{5} + \frac{2\pi}{3}\right)\right) \text{ for } k=0, 1, 2, 3, 4.$$

(9)

$$(b) (z+3)^5 = -32(z-1)^5$$

$$\Rightarrow \left(\frac{z+3}{z-1}\right)^5 = -32 \Rightarrow \frac{z+3}{z-1} = \sqrt[5]{-32} = \sqrt[5]{32(\cos \pi + i \sin \pi)}$$

$$\Rightarrow \frac{z+3}{z-1} = 2 \underbrace{\left( \cos \frac{2k\pi + \pi}{5} + i \sin \frac{2k\pi + \pi}{5} \right)}_{\substack{\uparrow \\ \text{call it } \omega_k}}, \quad k=0,1,2,3,4.$$

$$\Rightarrow \frac{z+3}{z-1} = 2\omega_k \Rightarrow z+3 = 2\omega_k z - 2\omega_k$$

$$\Rightarrow (2\omega_k - 1)z = 3 + 2\omega_k$$

$$\Rightarrow z = \frac{3 + 2\omega_k}{2\omega_k - 1} \quad \text{where } \omega_k = \cos \frac{(2k+1)\pi}{5} + i \sin \frac{(2k+1)\pi}{5} \text{ and } k=0,1,2,3,4.$$

$$= \frac{-2\cos \frac{(2k+1)\pi}{5} + 3 + i \sin \frac{(2k+1)\pi}{5}}{(2\cos \frac{(2k+1)\pi}{5} - 1 + i \sin \frac{(2k+1)\pi}{5})}$$

$$= \frac{(2\cos \frac{(2k+1)\pi}{5} + 3 + i \sin \frac{(2k+1)\pi}{5})(2\cos \frac{(2k+1)\pi}{5} - 1 - i \sin \frac{(2k+1)\pi}{5})}{(2\cos \frac{(2k+1)\pi}{5} - 1 + i \sin \frac{(2k+1)\pi}{5})(2\cos \frac{(2k+1)\pi}{5} - 1 - i \sin \frac{(2k+1)\pi}{5})}$$

$$C = \cos \frac{(2k+1)\pi}{5}$$

$$S = \sin \frac{(2k+1)\pi}{5}$$

$$= \frac{[(2C+3) + iS][(2C-1) - iS]}{[(2C-1) + iS][(2C-1) - iS]}$$

$$= \frac{(2C+3)(2C-1) + S^2}{(2C-1)^2 + S^2} + \frac{S(2C-1) - (2C+3)S}{(2C-1)^2 + S^2} i$$

$$= \dots$$

(They are optional)

## Problem 3

Let  $A = \begin{pmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 4y & 0 & 5y \end{pmatrix}$  be a  $3 \times 3$  matrix

(a) Find the value of  $\det B$ ,  $\det(B^4)$ . Express your answer in terms of  $y$ . (11 marks)

(b) If  $\det A = 0$ , find the value of  $\lambda$ . (11 marks)

(c) Take  $\lambda = 0$ , find the value of  $A^{-1}$ , the inverse of  $A$ . (12 marks)

expand along 2nd row

$$(a) \det B = \cancel{b_{21}}^0 B_{21} + \cancel{b_{22}}^3 B_{22} + \cancel{b_{23}}^0 B_{23} = 3 \times (-1)^{2+2} \det \begin{pmatrix} 1 & 2 \\ 4y & 5y \end{pmatrix}$$

$$= 3((1)(5y) - (4y)(2)) = -9y.$$

$$\det(B^4) = (\det B)^4 = (-9y)^4 = 6561y^4 \quad (\text{since } \det(CD) = (\det C)(\det D))$$

$$(b) \det A = \cancel{a_{11}}^{(2-\lambda)} A_{11} + \cancel{a_{12}}^{=0} A_{12} + \cancel{a_{13}}^{=0} A_{13} = (2-\lambda) \times (-1)^{1+1} \det \begin{pmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix}$$

$$= (2-\lambda)[(1-\lambda)(-1-\lambda) - 3(1)] = (2-\lambda)(\lambda^2 - 4) = -(\lambda-2)^2(\lambda+2)$$

$$\det A = 0 \Rightarrow -(\lambda-2)^2(\lambda+2) = 0 \Rightarrow \lambda = 2 \text{ or } \lambda = -2.$$

$$(c) \text{ Take } \lambda = 0, A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix} \text{ and } \det A = -8.$$

$$\Rightarrow A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = \frac{1}{-8} \begin{pmatrix} -4 & 2 & 2 \\ 0 & -2 & -6 \\ 0 & -2 & 2 \end{pmatrix}^T = -\frac{1}{8} \begin{pmatrix} -4 & 0 & 0 \\ 2 & -2 & -2 \\ 2 & -6 & 2 \end{pmatrix}$$

$$A_{11} = (-1)^{1+1} \det \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} = -4 \quad A_{21} = (-1)^{1+2} \det \begin{pmatrix} 0 & 0 \\ 3 & -1 \end{pmatrix} = 0 \quad \text{or } \begin{pmatrix} 1/2 & 0 & 0 \\ -1/4 & 1/4 & 1/4 \\ -1/4 & 3/4 & -1/4 \end{pmatrix}$$

$$A_{12} = (-1)^{1+2} \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2 \quad A_{22} = (-1)^{2+2} \det \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} = -2$$

$$A_{13} = (-1)^{1+3} \det \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = 2 \quad A_{23} = (-1)^{2+3} \det \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = -6$$

☺ End of the Quiz ☺

$$A_{31} = (-1)^{3+1} \det \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = 0$$

$$A_{32} = (-1)^{3+2} \det \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = -2$$

$$A_{33} = (-1)^{3+3} \det \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = 2$$

**MA1201 Calculus and Basic Algebra II (Section: CC1)**  
**Quiz 2: Vector, Complex Number and Matrix**  
**Version 3**

Name:

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Time allowed: 50 minutes

**Instruction:**

Write all your answer in the spaces provided. You need to show all your working steps and method used CLEARLY. Marks will be deducted for any unclear solution.

**Problem 1**

Let  $A = (-1, 0, 3)$ ,  $B = (0, 2, -1)$ ,  $C = (2, 1, 0)$  and  $D = (-1, 2, 1)$  be three points on a 3D-plane.

(a) Find the angle between the vectors  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$

(11 marks)

(b) Determine if the points  $B, C$  and  $D$  are collinear

(11 marks)

(c) Find the volume of parallelepiped with  $A, B, C, D$  as adj. vertices.

(11 marks)

(a) Note that 
$$\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{i} + 2\vec{j} - 4\vec{k} \Rightarrow |\overrightarrow{AB}| = \sqrt{1^2 + 2^2 + (-4)^2} = \sqrt{21} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 3\vec{i} + \vec{j} - 3\vec{k} \Rightarrow |\overrightarrow{AC}| = \sqrt{3^2 + 1^2 + (-3)^2} = \sqrt{19} \end{cases}$$

$$\cos \theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}| |\overrightarrow{AC}|} = \frac{1 \times 3 + 2 \times 1 + (-4) \times (-3)}{\sqrt{21} \sqrt{19}} \Rightarrow \cos \theta = \frac{17}{\sqrt{21} \sqrt{19}} \Rightarrow \theta =$$

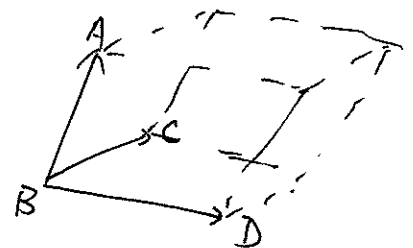
(b) Note that 
$$\begin{cases} \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 2\vec{i} - \vec{j} + \vec{k} \\ \overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = -\vec{i} + 2\vec{k} \end{cases}$$

$$\begin{aligned} \therefore \text{Area of } \triangle BCD &= \frac{1}{2} |\overrightarrow{BC} \times \overrightarrow{BD}| = \frac{1}{2} \left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ -1 & 0 & 2 \end{vmatrix} \right| \\ &= \frac{1}{2} \left| -2(\vec{i} \times \vec{j}) + (\vec{j} \times \vec{i}) - (\vec{k} \times \vec{i}) + 4(\vec{i} \times \vec{k}) - 2(\vec{j} \times \vec{k}) + 2(\vec{k} \times \vec{j}) \right| \\ &= \frac{1}{2} \left| -2\vec{i} - 5\vec{j} - \vec{k} \right| = \frac{1}{2} \sqrt{(-2)^2 + (-5)^2 + (-1)^2} = \frac{\sqrt{30}}{2} \neq 0 \end{aligned}$$

$\Rightarrow B, C, D$  are NOT collinear.

(c) The volume of parallelepiped

$$\begin{aligned} &= \left| \overrightarrow{BA} \cdot (\overrightarrow{BC} \times \overrightarrow{BD}) \right| \\ &= \left| (-\vec{i} - 2\vec{j} + 4\vec{k}) \cdot (-2\vec{i} - 5\vec{j} - \vec{k}) \right| \\ &= \left| (-1)(-2) + (-2)(-5) + 4(-1) \right| = 8. \end{aligned}$$



## Problem 2

(a) Let  $z_1 = 3\sqrt{2} + \sqrt{6}i$  and  $z_2 = -\sqrt{3} - i$  be two complex numbers. Compute

(i)  $(z_1)^{-10}$ , (ii)  $\sqrt[4]{z_1 z_2}$ , (iii)  $(z_2)^{\frac{2}{3}}$ .

Express your answer in Polar form.

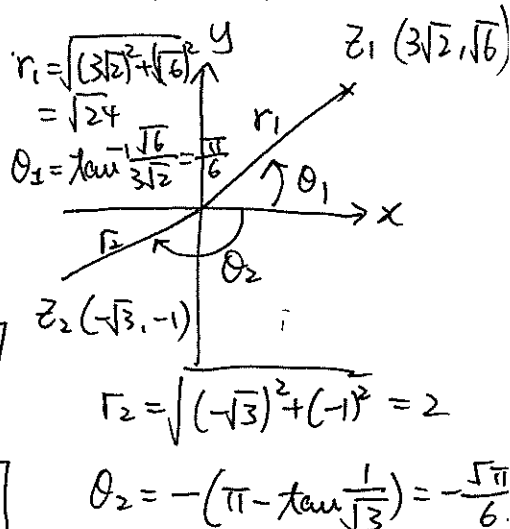
(18 marks)

(b) Solve the equation

$$\frac{z^5}{3i + z^5} = 2$$

(15 marks)

(a)  $z_1 = 3\sqrt{2} + \sqrt{6}i = \sqrt{24} \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$   
 $z_2 = -\sqrt{3} - i = 2 \left( \cos \left( -\frac{5\pi}{6} \right) + i \sin \left( -\frac{5\pi}{6} \right) \right)$



$\Rightarrow$  (i)  $(z_1)^{-10} = (\sqrt{24})^{-10} \left[ \cos \left( -\frac{10\pi}{6} \right) + i \sin \left( -\frac{10\pi}{6} \right) \right]$   
 $= \frac{1}{24^5} \left[ \cos \left( -\frac{5\pi}{3} \right) + i \sin \left( -\frac{5\pi}{3} \right) \right]$

(ii)  $\sqrt[4]{z_1 z_2} = \sqrt[4]{2\sqrt{24} \left( \cos \left( \frac{\pi}{6} - \frac{5\pi}{6} \right) + i \sin \left( \frac{\pi}{6} - \frac{5\pi}{6} \right) \right)}$   
 $2\sqrt{24} = \sqrt{96} = 96^{\frac{1}{2}} = 96^{\frac{1}{2}} \left( \cos \frac{2k\pi - 2\pi}{4} + i \sin \frac{2k\pi - 2\pi}{4} \right)$  for  $k=0, 1, 2, 3$ .

(iii)  $z_2^{\frac{2}{3}} = \left[ 2 \left( \cos \left( -\frac{5\pi}{6} \right) + i \sin \left( -\frac{5\pi}{6} \right) \right) \right]^{\frac{2}{3}} = \left[ 4 \left( \cos \left( -\frac{5\pi}{3} \right) + i \sin \left( -\frac{5\pi}{3} \right) \right) \right]^{\frac{1}{3}}$   
 $= 4^{\frac{1}{3}} \left( \cos \frac{2k\pi - \frac{5\pi}{3}}{3} + i \sin \frac{2k\pi - \frac{5\pi}{3}}{3} \right)$  for  $k=0, 1, 2$ .

(b)  $\frac{z^5}{3i + z^5} = 2 \Rightarrow z^5 = 6i + 2z^5 \Rightarrow z^5 = -6i = z = \sqrt[5]{-6i}$   
 $\Rightarrow z = \sqrt[5]{6 \left[ \cos \left( -\frac{\pi}{2} \right) + i \sin \left( -\frac{\pi}{2} \right) \right]} = 6^{\frac{1}{5}} \left[ \cos \frac{2k\pi - \frac{\pi}{2}}{5} + i \sin \frac{2k\pi - \frac{\pi}{2}}{5} \right]$  for  $k=0, 1, 2, 3, 4$ .

## Problem 3

Let  $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -2 & 0 \\ x & 2x & 1 \end{pmatrix}$  and  $B = \begin{pmatrix} y & 1 & 0 \\ 2 & y & 3 \\ 0 & 4 & y \end{pmatrix}$  be a  $3 \times 3$  matrix.

- (a) Find the  $\det A$  and  $\det(A^4)$ . Express your answer in term of  $x$ . (11 marks)  
 (b) Find the value of  $y$  such that the matrix  $B$  is NOT invertible. (11 marks)  
 (c) Take  $y = 1$ , find the inverse of matrix  $B$ . (12 marks)

expand along first row

$$(a) \det A = \overset{=0}{a_{11}}A_{11} + \overset{=0}{a_{12}}A_{12} + \overset{=1}{a_{13}}A_{13} = 1 \times (-1)^{1+3} \det \begin{pmatrix} 1 & -2 \\ x & 2x \end{pmatrix}$$

$$= (1)(2x) - (x)(-2) = 4x.$$

$$\det(A^4) = (\det A)^4 = (4x)^4 = 256x^4 \leftarrow \text{Note that } \det(CD) = (\det C)(\det D)$$

(b)  $B$  is not invertible  $\Leftrightarrow \det B = 0$

$$\therefore \det B = \overset{y}{b_{11}}B_{11} + \overset{1}{b_{12}}B_{12} + \overset{0}{b_{13}}B_{13} = y[(-1)^{1+1} \det \begin{pmatrix} 2 & 3 \\ 0 & y \end{pmatrix}] + (-1)^{1+2} \det \begin{pmatrix} 2 & 3 \\ 0 & y \end{pmatrix}$$

$$= y(y^2 - 12) - (2y - (0)(3)) = y^3 - 14y.$$

$$\Rightarrow \det B = 0 \Rightarrow y^3 - 14y = 0 \Rightarrow y(y^2 - 14) = 0 \Rightarrow y = 0 \text{ or } y = \pm\sqrt{14}.$$

(c) Take  $y = 1$ ,  $B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix}$  and  $\det B = -13$ .

$$\text{Then } B^{-1} = \frac{1}{\det B} \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}^T \Rightarrow -\frac{1}{13} \begin{pmatrix} -11 & -2 & 8 \\ -1 & 1 & -4 \\ 3 & -3 & -1 \end{pmatrix}^T = -\frac{1}{13} \begin{pmatrix} -11 & -1 & 3 \\ -2 & 1 & -3 \\ 8 & -4 & -1 \end{pmatrix}$$

$B_{ij} = \text{cofactor}$

$$B_{11} = (-1)^{1+1} \det \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = -11$$

$$B_{21} = (-1)^{2+1} \det \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = -1$$

$$B_{12} = (-1)^{1+2} \det \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = -2$$

$$B_{22} = (-1)^{2+2} \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$B_{13} = (-1)^{1+3} \det \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix} = 8$$

$$B_{23} = (-1)^{2+3} \det \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix} = -4$$

$$B_{31} = (-1)^{3+1} \det \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = 3$$

$$B_{32} = (-1)^{3+2} \det \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = -3$$

$$B_{33} = (-1)^{3+3} \det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = -1$$