

## Suggested Solution

MA1201 Calculus and Basic Linear Algebra II  
Midterm Test

Name:

Student ID:

Score:

Time allowed: 100 minutes

1. Evaluate the following integrals.

(a)  $\int x^2(6-x^3)^{10} dx$

(6 points)

Let  $y = 6 - x^3$ , then  $\frac{dy}{dx} = -3x^2 \Rightarrow dx = -\frac{1}{3x^2} dy$

$$\int x^2(6-x^3)^{10} dx = -\frac{1}{3} \int y^{10} dy$$

$$= -\frac{1}{3} \frac{y^{11}}{11} + C$$

$$= -\frac{1}{33} (6-x^3)^{11} + C.$$

(b)  $\int \sin^2 x \cos x dx$

(6 points)

(Method 1)

Let  $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$   
 $\Rightarrow dy = \cos x dx$

$$\int \sin^2 x \cos x dx$$

$$= \int y^2 dy$$

$$= \frac{y^3}{3} + C$$

$$= \frac{(\sin x)^3}{3} + C$$

(Method 2) Use product-to-sum formula

$$\int \sin^2 x \cos x dx$$

$$= \frac{1}{2} \int (1 - \cos 2x) \cos x dx$$

$$= \frac{1}{2} \int \cos x dx - \frac{1}{2} \int \cos 2x \cos x dx$$

$$= \frac{1}{2} \sin x - \frac{1}{4} \int (\cos 3x + \cos x) dx$$

$$= \frac{1}{2} \sin x - \frac{1}{12} \sin 3x - \frac{1}{4} \sin x + C$$

$$= \frac{1}{4} \sin x - \frac{1}{12} \sin 3x + C.$$

(c)  $\int \frac{2x-6}{(x-1)(x+1)(x^2+1)} dx$

(12 points)

We consider

$$\frac{2x-6}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$\Rightarrow 2x-6 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$$

Put  $x=-1$ ,  $-8 = -4B \Rightarrow B=2$

Put  $x=1$ ,  $-4 = 4A \Rightarrow A=-1$

Put  $x=0$ ,  $-6 = A-B-D \Rightarrow D=3$

Put  $x=2$ ,  $-2 = 15A+5B+3(2C+D) \Rightarrow C=-1$

$$\int \frac{2x-6}{(x-1)(x+1)(x^2+1)} dx = -\int \frac{1}{x-1} dx + \int \frac{2}{x+1} dx - \int \frac{x}{x^2+1} dx + \int \frac{3}{x^2+1} dx$$

(d)  $\int x^2 e^{-3x} dx$

$y = x^2 \Rightarrow \frac{dy}{dx} = 2x$

$$= -\ln|x-1| + 2\ln|x+1| - \frac{1}{2} \int \frac{1}{y} dy + 3 \tan^{-1} x$$

$$= -\ln|x-1| + 2\ln|x+1| - \frac{1}{2} \ln|x^2+1| + 3 \tan^{-1} x + C.$$

(8 points)

Using integration by parts with  $u=x^2$ ,  $dv=e^{-3x} dx$

$$\int x^2 e^{-3x} dx$$

$$\Rightarrow v = \int e^{-3x} dx = -\frac{1}{3} e^{-3x}$$

$$= -\frac{1}{3} x^2 e^{-3x} + \int \frac{1}{3} e^{-3x} d(x^2)$$

$$= -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \int x e^{-3x} dx \quad \leftarrow \begin{matrix} u=x \\ dv=e^{-3x} dx \Rightarrow v=-\frac{1}{3} e^{-3x} \end{matrix}$$

$$= -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \left[ -\frac{1}{3} x e^{-3x} + \int \frac{1}{3} e^{-3x} dx \right]$$

$$= -\frac{1}{3} x^2 e^{-3x} - \frac{2}{9} x e^{-3x} - \frac{1}{27} e^{-3x} + C.$$

(e)  $\int \frac{1}{(x^2+9)^{3/2}} dx$

(8 points)

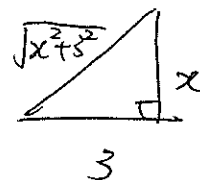
Let  $x = 3 \tan \theta$  (so that  $x^2 + 9 = 9 \tan^2 \theta + 9 = 9 \sec^2 \theta$ )

$$\Rightarrow \frac{dx}{d\theta} = 3 \sec^2 \theta \Rightarrow dx = 3 \sec^2 \theta d\theta$$

$$\tan \theta = \frac{x}{3}$$

$$\int \frac{1}{(x^2+9)^{3/2}} dx = \int \frac{1}{(9 \sec^2 \theta)^{3/2}} (3 \sec^2 \theta) d\theta$$

$$= \int \frac{1}{27 \sec^3 \theta} (3 \sec^2 \theta) d\theta$$



$$= \frac{1}{9} \int \frac{1}{\sec \theta} d\theta$$

$$= \frac{1}{9} \int \cos \theta d\theta$$

$$= \frac{1}{9} \sin \theta + C = \frac{1}{9} \frac{x}{\sqrt{x^2+9}} + C.$$

(f)  $\int_{-1}^2 |x^2 - 1| dx$

(8 points)

Note that  $|x^2 - 1| = \begin{cases} x^2 - 1 & \text{if } x^2 - 1 \geq 0 \\ -(x^2 - 1) & \text{if } x^2 - 1 < 0 \end{cases} = \begin{cases} x^2 - 1 & \text{if } x \geq 1 \text{ or } x \leq -1 \\ 1 - x^2 & \text{if } -1 < x < 1 \end{cases}$

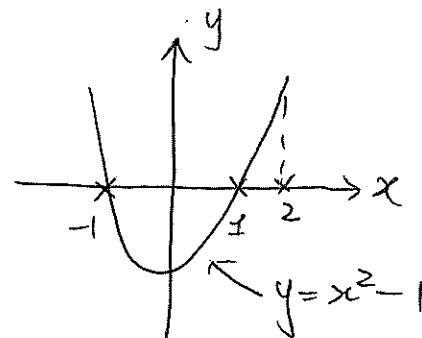
$$\int_{-1}^2 |x^2 - 1| dx$$

$$= \int_{-1}^{-1} (x^2 - 1) dx + \int_{-1}^1 (1 - x^2) dx$$

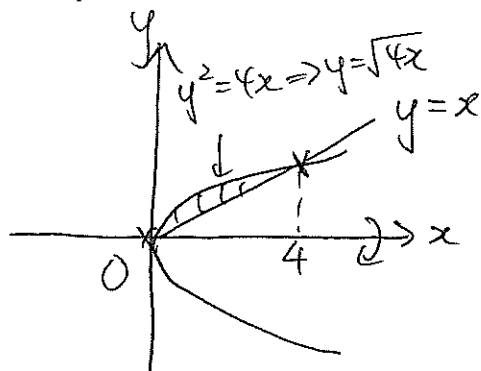
$$= \left. \frac{x^3}{3} - x \right|_{-1}^{-1} + \left. x - \frac{x^3}{3} \right|_{-1}^1$$

$$= \frac{8}{3} - 2 - \frac{1}{3} + 1 + 1 - \frac{1}{3} + 1 - \frac{1}{3}$$

$$= \frac{8}{3}.$$



- 2.(a) Find the volume of the solid generated by revolving the region bounded by the curves  $y^2 = 4x$  and  $y = x$  one complete revolution about the  $x$ -axis. (15 points)



$$\begin{cases} y^2 = 4x \\ y = x \end{cases} \Rightarrow x^2 = 4x \Rightarrow x^2 - 4x = 0 \\ \Rightarrow x(x-4) = 0 \\ \Rightarrow x = 0 \text{ or } x = 4.$$

The volume

$$\begin{aligned} &= \pi \int_0^4 (\sqrt{4x})^2 dx - \pi \int_0^4 x^2 dx \\ &= \pi \int_0^4 4x dx - \pi \frac{x^3}{3} \Big|_0^4 \\ &= \pi (2x^2) \Big|_0^4 - \frac{64\pi}{3} \\ &= \frac{32\pi}{3}. \end{aligned}$$

2.(b) Compute the arc length of the parametric curve  $x = \cos^3 t$ ,  $y = \sin^3 t$ ,  $0 \leq t \leq 2\pi$ .

(15 points)

The arc length

$$\begin{aligned}
 &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 &= \int_0^{2\pi} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2} dt \\
 &= \int_0^{2\pi} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} dt \\
 &= \int_0^{2\pi} \sqrt{9\cos^2 t \sin^2 t (\underbrace{\cos^2 t + \sin^2 t}_{=1})} dt
 \end{aligned}$$

$$(*) = 3 \int_0^{2\pi} |\cos t \sin t| dt$$

$$\begin{aligned}
 &= \frac{3}{2} \int_0^{2\pi} |\sin 2t| dt \\
 &= \frac{3}{2} \left[ \int_0^{\frac{\pi}{2}} \sin 2t dt - \int_{\frac{\pi}{2}}^{\pi} \sin 2t dt + \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \sin 2t dt - \int_{\frac{3\pi}{2}}^{2\pi} \sin 2t dt \right]
 \end{aligned}$$

$\because |\sin 2t| = \begin{cases} \sin 2t & \text{if } \sin 2t \geq 0 \\ -\sin 2t & \text{if } -\sin 2t < 0 \end{cases}$   
 $= \begin{cases} \sin 2t & \text{if } 0 \leq t \leq \frac{\pi}{2} \\ -\sin 2t & \text{if } \frac{\pi}{2} < t < \pi \\ \sin 2t & \text{if } \frac{3\pi}{2} < t < 2\pi \\ -\sin 2t & \text{if } \pi \leq t \leq \frac{3\pi}{2} \end{cases}$

$$\begin{aligned}
 &= \frac{3}{2} \left[ -\frac{1}{2} \cos 2t \Big|_0^{\frac{\pi}{2}} + \frac{1}{2} \cos 2t \Big|_{\frac{\pi}{2}}^{\pi} \right. \\
 &\quad \left. - \frac{1}{2} \cos 2t \Big|_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} + \frac{1}{2} \cos 2t \Big|_{\frac{3\pi}{2}}^{2\pi} \right]
 \end{aligned}$$

$$= \frac{3}{2} \left[ \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right]$$

$$= 6.$$

(\*) Since  $\sqrt{9\cos^2 t \sin^2 t}$  is always positive in our case and  $3\cos t \sin t = \frac{3}{2} \sin 2t$  can be positive or negative for  $0 \leq t \leq 2\pi$ , so we need to include the absolute sign on  $3\cos t \sin t$

Note

$$\begin{aligned}
 \frac{dx}{dt} &= \frac{d}{dt} \cos^3 t \\
 &= \frac{d(\cos t)^3}{d(\cos t)} \frac{d(\cos t)}{dt} \\
 &= 3\cos^2 t (-\sin t) \\
 &= -3\cos^2 t \sin t
 \end{aligned}$$

Similarly

$$\frac{dy}{dt} = 3\sin^2 t \cos t$$

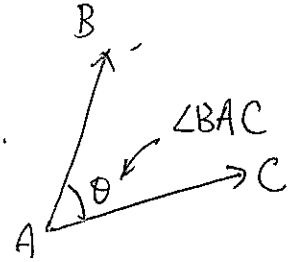
3. Given  $A(1, 2, -3)$ ,  $B(-1, 3, 0)$ ,  $C(4, -1, 2)$  and  $D(5, 0, -1)$  in  $R^3$ ,

(a) evaluate the angle  $\angle BAC$ ;

(10 points)

$$\begin{aligned}\vec{AB} &= \vec{OB} - \vec{OA} = (-\vec{i} + 3\vec{j}) - (\vec{i} + 2\vec{j} - 3\vec{k}) \\ &= -2\vec{i} + \vec{j} + 3\vec{k}\end{aligned}$$

$$\begin{aligned}\vec{AC} &= \vec{OC} - \vec{OA} = (4\vec{i} - \vec{j} + 2\vec{k}) - (\vec{i} + 2\vec{j} - 3\vec{k}) \\ &= 3\vec{i} - 3\vec{j} + 5\vec{k}\end{aligned}$$



$$\begin{aligned}\text{Thus } \cos \theta &= \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| |\vec{AC}|} = \frac{(-2 \times 3) + (1 \times (-3)) + 3 \times 5}{\sqrt{(-2)^2 + 1^2 + 3^2} \sqrt{3^2 + (-3)^2 + 5^2}} \\ &= \frac{6}{\sqrt{14} \sqrt{43}}\end{aligned}$$

$$\Rightarrow \theta = 75.8^\circ.$$

(b) determine a vector perpendicular to  $\vec{AB}$  and  $\vec{AC}$ ;

(6 points)

Using the definition of vector product, the vector  $\vec{AB} \times \vec{AC}$  is always perpendicular to both  $\vec{AB}$  and  $\vec{AC}$ .

Then the required vector is given by

$$\begin{aligned}\vec{AB} \times \vec{AC} &= (-2\vec{i} + \vec{j} + 3\vec{k}) \times (3\vec{i} - 3\vec{j} + 5\vec{k}) \\ &= -2(\vec{i} \times \vec{i}) + 6(\vec{i} \times \vec{j}) - 10(\vec{i} \times \vec{k}) \\ &\quad + 3(\vec{j} \times \vec{i}) - 3(\vec{j} \times \vec{j}) + 5(\vec{j} \times \vec{k}) \\ &\quad + 9(\vec{k} \times \vec{i}) - 9(\vec{k} \times \vec{j}) + 15(\vec{k} \times \vec{k}) \\ &= 14\vec{i} + 19\vec{j} + 3\vec{k}.\end{aligned}$$

3.(c) check whether  $A, B, C$  and  $D$  are coplanar or not.

(6 points)

We consider the scalar triple product

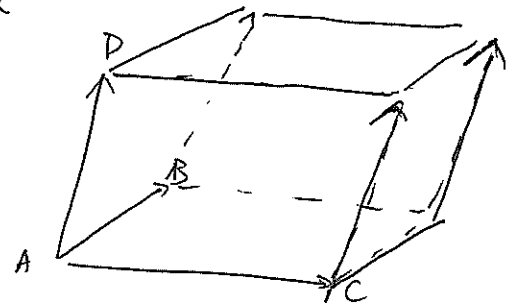
$$|\vec{AD} \cdot (\vec{AB} \times \vec{AC})| \leftarrow \text{Volume of parallelepiped}$$

$$= |(4\vec{i} - 2\vec{j} + 2\vec{k}) \cdot (14\vec{i} + 19\vec{j} + 3\vec{k})|$$

$$= |14 \times 4 - 2 \times 19 + 2 \times 3|$$

$$= 24 \neq 0$$

$\Rightarrow A, B, C, D$  are not coplanar.



$$\vec{AD} = \vec{OD} - \vec{OA}$$

$$= (5\vec{i} - \vec{k}) - (\vec{i} + 2\vec{j} - 3\vec{k})$$

$$= 4\vec{i} - 2\vec{j} + 2\vec{k}$$

-End-