

MA2001 Problem Set 1 Eigenvalues and Eigenvectors

1. Find the eigenvalues and eigenvectors of the following matrices:

(a) $\begin{pmatrix} 3 & 2 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & \sqrt{2} \\ 0 & 2 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}.$

2. For the matrix in question 1(b), find a matrix P such that $P^{-1}AP = D$, a diagonal matrix with the eigenvalues of A as its elements. Check your solution by evaluating $P^{-1}AP$.

3. Find a 2×2 matrix A which has eigenvalue 1 with corresponding eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and eigenvalue 3 with corresponding eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

4. (a) Using Gaussian elimination, find a matrix X such that $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

(b) With the help of the result of (a), find a 4×4 matrix A which has eigenvalues $-1, 0, 0, 1$ with

corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, which are rows of $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}.$

5. Given that $\xi = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 2 & -1 & 2 \\ 5 & a & 3 \\ -1 & b & -2 \end{pmatrix}.$

(a) Find a and b .

(b) Find the eigenvalues and eigenvectors of A .

(c) Is A diagonalizable? Please give reasons.

6. (a) Find the eigenvalues and corresponding eigenvectors of the symmetric matrix $A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

and verify that the eigenvectors are mutually orthogonal.

(b) Let $B = A^5 - 5(A + 3I)^{-1} + 3A^T$. Find the eigenvalues of B .

7. Consider $A = \begin{pmatrix} 8 & 2 & 4 & 12 & 1 \\ 4 & 1 & 2 & 6 & 0.5 \\ 12 & 3 & 6 & 18 & 1.5 \\ 6 & 1.5 & 3 & 9 & 0.75 \\ 18 & 4.5 & 9 & 27 & 2.25 \end{pmatrix}$

(a) Find rank A .

(b) Find a column vector $\vec{x}$ and a row vector $\vec{y}^T$, where $\vec{x}, \vec{y} \in R^5$ such that $A = \vec{x}\vec{y}^T$.

- (c) Show that 0 is an eigenvalue of A and find its corresponding independent eigenvectors.
- (d) Does A have eigenvalues other than 0? If yes, find those eigenvalues and the corresponding independent eigenvectors.
- (e) Find an invertible matrix P and a diagonal matrix D such that $AP = PD$.
- (f) Find all eigenvalues of $A^2 + 3I_5$, where I_5 is the 5×5 unit matrix.

8. Suppose $AP = PD$, where A is 3×3 , $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} x & -4/5 & 0 \\ y & 3/5 & 0 \\ z & 0 & 1 \end{pmatrix}$ and P is invertible.

- (a) Find the characteristic polynomial, $\det(A - \lambda I)$, of A .
- (b) Find all eigenvalues of A .
- (c) Suppose the first column of P , $(x \ y \ z)^T$ with $x \geq 0$ is a unit vector and orthogonal to both

$$\begin{pmatrix} -4/5 \\ 3/5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{ find } \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

- (d) Compute $P^T P$ and then show that $P^T = P^{-1}$.
- (e) Find A^n , $n \geq 0$.

9. Let A be a 3×3 matrix with eigenvalues 1, 2, 3.

- (a) (i) Find the characteristic polynomial $|A - \lambda I|$ of A .
- (ii) Determine $|A|$.
- (iii) Is A invertible, why?

If A^{-1} of A exists, the adjoint $\text{adj } A$ of A is defined as the matrix $\text{adj } A = |A|A^{-1}$.

Suppose $AM = M \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, where M is invertible.

- (b) Find the eigenvalues of $\text{adj } A$.

$$\text{Let } M = \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}, AM = M \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

- (c) (i) Compute MM^T and then find M^{-1} .
- (ii) Find A^{-1} .
- (iii) Find $\text{adj } A$.

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